Astronomy – General Relativity and Gravitation I
Part 1: Introduction

Norman Gray

Public version 1.1, June 2013

Overview

Web pages:

- `http://physci.moodle.gla.ac.uk/`
- `http://www.astro.gla.ac.uk/users/norman/lectures/AH-GR/`
- `http://www.astro.gla.ac.uk/users/norman/lectures/A2SR/`

The course divides into four parts:

Part 1 – Introduction One lecture. Covers the overall motivation for the course – why Special Relativity cannot provide a complete description of gravity, and why gravity is special.

Part 2 – Vectors, tensors and functions Three lectures. Recap of linear algebra, and an introduction to tensors, vectors and one-forms. Basis transforms and components.

Part 3 – Manifolds, vectors and differentiation Four lectures. Introduces differential geometry. Definition of the tangent plane, and differentiation in flat and curved spaces. Introduces geodesics and curvature. Defines Riemann and Ricci tensors, and geodesic deviation.

Part 4 – Physics: energy, momentum and Einstein's equations Three lectures. Back to physics: introduces the energy-momentum tensor. More discussion of the equivalence principle, and a rationale for, and introduction to, Einstein's equations linking the curvature of space-time to the presence of gravitating objects. The Newtonian limit, and classical gravity as the weak-field limit of Einstein's equations.

Aims and objectives for Part 1

The point of Aims and Objectives is twofold. They help me keep on track by reminding me what things it's important I cover; and they help you follow the course, by reminding you of the motivation for the material I'm covering. The distinction between the two, as far as I'm concerned, is simple.

- The *aims* are the point of the course – why you're doing the course, and why I'm teaching it. These are the insights you'll have, and the ideas you'll understand, long after the point where you've forgotten most of the details. Unfortunately, it's easy to claim, but difficult to show, you have this understanding. So. . . .

- The *objectives* are the detailed skills, mastery of which demonstrates that you have in fact achieved the aims of the course. Hint: it is a short step from objectives to exam questions, and I regard the list of objectives as more-or-less coextensive with the set of examinable topics.

Aims You should:

1. appreciate why GR is not simply newtonian gravitation plus SR;

2. understand how the equivalence principles lead directly to GR effects such as light deflection and redshift (see also the aims in part 4).

Objectives You should be able to demonstrate that you can:

1. explain why one cannot simply add SR to newtonian gravity to obtain a 'relativised' gravitational theory;

2. quote the strong and weak equivalence principles (see also the objectives for part 4), and explain what geodesic deviation is; and

3. explain, without detailed calculation, how the equivalence principle leads to light bending in a gravitational field, or the phenomenon of gravitational redshift.

1 Three thought experiments on gravitation

What is the problem which General Relativity attempts to solve?

Put another way, why can't we get a theory of gravity by just taking SR and adding some 'relativised' newtonian gravity?

[The following discussion overlaps with the very useful discussion in Schutz §5.1, and with MTW §§7.2–7.3.]

1.1 Gravitational redshift

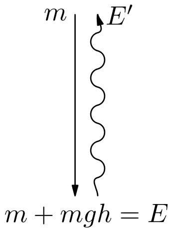

Figure 1

Imagine dropping a particle of mass m through a distance h. The particle starts off with energy m ($E = mc^2$, with $c = 1$, remember), and ends up with energy $E = m + mgh$ (see Fig. 1). Now imagine converting all of this energy into a single photon of energy E, and sending it up towards the original position. It reaches there with energy E', which we convert *back* into a particle. Now, either we have invented a perpetual motion machine, or else $E' = m$:

$$E' = m = \frac{E}{1 + gh}, \tag{1.1}$$

and we discover that a photon loses energy – is redshifted – as a necessary consequence of climbing through a gravitational field, and as a consequence of our demand that energy be conserved.

This phenomenon is termed *gravitational redshift*, and it (or rather, something very like it) has been confirmed experimentally, in the 'Pound-Rebka experiment'. It's also sometimes referred to as 'gravitational doppler shift', but inaccurately, since it is not a consequence of relative motion, and so has nothing to do with the doppler shift you are familiar with.

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1.2 Schild's photons

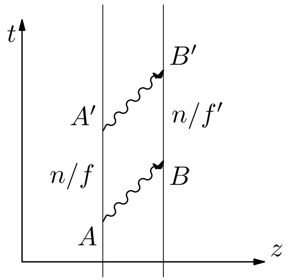

Figure 2

Imagine firing a photon, of frequency f, from a point A to a point B directly above it in a gravitational field (see Fig. 2). As we discovered in Sect. 1.1, the photon will be redshifted to a new frequency f'. After some number of periods, n, we repeat this, and send up another photon (between the points marked A' and B' on the spacetime diagram).

Photons are a kind of clock, in that the interval between 'wavecrests', $1/f$, forms a kind of 'tick'. This will be measured to have different numerical values in different frames, but it nonetheless defines two frame-independent events.

Since nothing will have changed between sending off the two photons, the intervals AB and $A'B'$ will be the same (I've drawn these as straight lines on the diagram, but the argument doesn't depend on that). However the intervals AA' and BB', as measured by local clocks, are *different*. That is, we have not constructed the parallelogram we might have expected, and have therefore discovered that the *geometry* of this spacetime is not the flat geometry we might have expected, and that this is purely as a result of the presence of the gravitational field through which we are sending the photons.

Finding out more about this geometry is what this course is about, and one of the first physical principles we will use is illustrated by a falling lift.

1.3 The falling lift

Recall from Special Relativity that we may define an *inertial frame* to be one in which Newton's laws hold, so that particles which are not acted on by an external force move in straight lines at a constant velocity. In Misner, Thorne and Wheeler's words, inertial frames are defined so that motion looks simple. This is so if we are in a box far away from any gravitational forces, and so we may identify that as a *local inertial frame* (because the previous section suggests that we cannot carelessly make claims about extended frames). Another way of removing gravitational forces, less extreme than going into deep space, is to put ourselves in free fall. Einstein asserted that these two situations are indeed fully equivalent, and defined an inertial frame as one in free fall.

Objects at rest in an inertial frame – in either of the equivalent situations of being far away from gravitating matter or freely falling in a gravitational field – will stay at rest. If we accelerate the box cum inertial frame, perhaps by attaching rockets to its 'floor', then the box will accelerate but its contents won't; they will therefore move towards the floor at an increasing speed, from the point of view of someone in the box[1]. This will happen irrespective of the mass or composition of the objects in the box; they will all appear to increase their speed at the *same* rate.

Note that we are carefully *not* using the word 'accelerate' for the objects' change in speed. We reserve that word for the physical phenomenon measured by an accelerometer, and the result of a real force, and try to *avoid* using it (not, I fear, always successfully) to refer to the second derivative of a position – depending on the coordinate system, the one does not always imply the other, as we shall see later.

This is very similar to Galileo's observation that all objects appear to fall under gravity at the same rate, irrespective of their mass or composition, and this has been verified to considerable precision in the Eötvös experiments. Einstein supposed that this was not a

[1] By 'point of view' I mean 'as measured with respect to a reference frame fixed to the box', but such circumlocution can distract from the point that this is an *observation* we're talking about – we can see this happening.

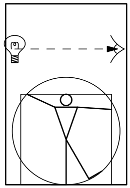

Figure 3

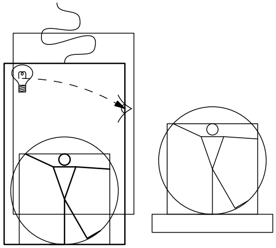

Figure 4

coincidence, and that there was a deep equivalence between acceleration and gravity (we shall see later, in part 4, that the force of gravity we feel standing in one place is the result of us being accelerated away from the path we would have if we were in free fall). He raised this to the status of a postulate:

> *The (weak) Equivalence Principle (EP):* Uniform gravitational fields are equivalent to frames that accelerate uniformly relative to inertial frames.

Imagine a box floating freely in space, and imagine shining a torch horizontally across it.

Where will the beam end up? Obviously, the beam will end up at a point on the wall directly opposite the torch (Fig. 3). There's nothing exotic about this. The weak equivalence principle tells us that the *same* must happen for a box in free fall. That is, a person inside a falling lift would observe the torch beam to end up level with the point at which it was emitted, in the (inertial) frame of the lift. This is a straightforward and unsurprising use of the EP. How would this appear to someone watching the lift fall?

Since the light takes a finite time to cross the lift cabin, the spot on the wall where it strikes will have dropped some finite (though small) distance, and so will be lower than the point of emission, in the frame of someone watching this from a position of safety (Fig. 4). That is, this non-free-fall observer will measure the light's path as being curved in the gravitational field. Even massless light is affected by gravity.

2 Relativity and gravitation

2.1 Tides and geodesic deviation

Consider two particles, A and B, falling towards the earth (Fig. 5). They start off level with each other, at a height $z(t)$ from the centre of the earth, and separated by a horizontal distance $\xi(t)$.

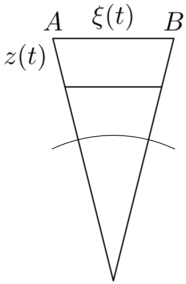

Figure 5

From the diagram, the separation $\xi(t)$ is proportional to $z(t)$, so that $\xi(t) = kz(t)$, for some constant k. The gravitational force on a particle of mass m, at altitude z is $F = GMm/z^2$, thus

$$\frac{d^2\xi}{dt^2} = k\frac{d^2 z}{dt^2} = -k\frac{F}{m} = -k\frac{GM}{z^2} = -\xi\frac{GM}{z^3}.$$

This tells us that the inertial frames attached to these freely falling particles approach each other at an increasing speed (that is, they ‘accelerate’ towards each other in the sense that the second derivative of their separation is non-zero, but since they are in free fall, there is no physical acceleration).

2.2 There is no universal inertial frame

A lot of Special Relativity depended on inertial frames having infinite extent: if I am an inertial observer, then any other inertial observer must be moving at a constant velocity with respect to me.

Consider observers plummeting down liftshafts, in free-fall, on opposite sides of the earth. These are inertial observers, but the second derivative of their spatial separation is not zero – they are accelerating with respect to one another. This means that, if I am one of these inertial observers, then (presuming I do not have more pressing things to worry about) I cannot use SR to calculate what the *other* inertial observer would measure in their frame, nor calculate what I would measure if I observed a bit of physics that I understand, which is happening in the other inertial observer's frame.

But this is precisely what I do want to do, supposing that the bit of physics in question is happening in free fall in the accretion disk surrounding a black hole, and I want to interpret what I am seeing through my telescope. Gravitational redshift of spectral lines is just the beginning of it!

It is General Relativity which tells us how we must patch together such disparate inertial frames.

2.3 What does GR plan to do about it?

Newton's second law is

$$\overline{F} = m\overline{a}.$$

That makes the geometrical statement that when you apply to an object a force acting in a certain direction, that object accelerates in the *same direction* as the force, with an acceleration which is proportional to it. If we want to give numerical values to this statement, then we need a coordinate system – where is the origin, what scale are the axes, and so on – but the physical law is true irrespective of which system we pick, and it remains true if we change our mind.

This is not just a peculiar property of Newton's laws. We (and Einstein) can elevate this to another principle:

> *The principle of general covariance:* All physical laws must be invariant under all coordinate transformations.

That is, only geometrical objects matter – to be a physical law, an equation must be expressible in a form which is purely geometrical, and thus independent of the choice of coordinate system used to represent it.

That is what GR does: it describes the physics of gravity in a purely geometrical way, avoiding giving fundamental importance to any particular set of coordinates. It describes gravity, not as the rather mysterious, instantly-acting, force which Newton described in his 'law of universal gravitation', but instead as the inevitable consequence of our movement through a curved spacetime.

The problem is, that doing geometry on a curved space is tricky. . . .

In the following parts, there are various passages, and a couple of complete sections, marked with dangerous bend signs, like this one. They indicate supplementary detail, or material beyond the scope of the course which I think may be nonetheless interesting, or extra discussion of concepts or techniques which students have found confusing or misunderstandable in the past. These (and especially the last category) are passages you will probably want to skip on a first reading; none of it will be examinable.

A Natural units

In Special Relativity, we normally use *natural units* (also *geometrical units*), in which we use the same units, metres, to measure both distance and time, with the result that we measure distance in these two directions in spacetime using the same units (because of the high speed of light, metres and seconds are otherwise absurdly mismatched). We extend this in General Relativity, but now measuring mass in metres also. First, a recap of natural units in SR.

It is straightforward to measure distances in seconds, and we do this naturally when we talk of the Earth being 8 light-minutes from the sun, or the nearest star being a little more than 4 light-years away, or Edinburgh being 50 minutes from Glasgow (ScotRail permitting). In fact, since 1981 or so, the International Standard definition of the metre is that it is the distance light travels in 1/299792458 seconds; that is, the speed of light is precisely $299\,792\,458\,\mathrm{m\,s^{-1}}$ *by definition*, and so c is therefore demoted to being merely a conversion factor between two different units of time. In the same sense, the inch is defined to be precisely $2.54\,\mathrm{cm}$ long, and this figure of 2.54 is merely a conversion factor between two different, and only historically distinct, units of length. We write this as $1\,\mathrm{in} = 2.54\,\mathrm{cm}$, or $1 = 2.54\,\mathrm{cm\,in^{-1}}$.

There are several advantages to this. (i) In relativity, space and time are not really distinct, but having different units for the two 'directions' can obscure this. (ii) If we measure time in metres, then we no longer need the conversion factor c in our equations, which are consequently simpler. (iii) In these units, light travels a distance of one metre in a time of one metre, giving the speed of light as an easy-to-remember, and dimensionless, $c = 1$. We also quote other speeds in these units of metres per metre, so that all speeds are dimensionless and less than one. That means that $1 = c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$ so that, just as with the expression $1 = 2.54\,\mathrm{cm\,in^{-1}}$ above, we are using the figure 3×10^8 as a conversion factor between the alternative length units of metres and seconds.

For example, to convert $10\,\mathrm{J} = 10\,\mathrm{kg\,m^2\,s^{-2}}$ to natural units, we could proceed in two ways. Since $c = 1$, we have $1\,\mathrm{s} = 3 \times 10^8\,\mathrm{m}$ (this looks very bizarre, but compare the closely analogous statement $1\,\mathrm{in} = 2.54\,\mathrm{cm}$), and so $1\,\mathrm{s^{-2}} = (9 \times 10^{16})^{-1}\,\mathrm{m^{-2}}$. So

$10\,\mathrm{kg\,m^2\,s^{-2}} = 10\,\mathrm{kg\,m^2} \times (9\times 10^{16})^{-1}\,\mathrm{m^{-2}} = 1.1\times 10^{-16}\,\mathrm{kg}$.

Alternatively, we can write $1 = 3\times 10^8\,\mathrm{m\,s^{-1}}$ (compare $1 = 2.54\,\mathrm{cm\,in^{-1}}$), or $1 = (3\times 10^8)^{-1}\,\mathrm{s\,m^{-1}}$. Thus

$$\begin{aligned} 10\,\mathrm{J} &= 10\,\mathrm{kg\,m^2\,s^{-2}} \times (1)^2 \\ &= 10\,\mathrm{kg\,m^2\,s^{-2}} \times (3\times 10^8)^{-2}\,\mathrm{s^2\,m^{-2}} \\ &= 1.1\times 10^{-16}\,\mathrm{kg}. \end{aligned}$$

In GR we must additionally deal with the masses of objects, and we measure masses in metres also, with the conversion factor between kilogrammes and metres fixed by the demand that the gravitational constant have the easy-to-remember value $G = 1$. That means that the expression $1 = G = 6.673\times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ becomes a conversion factor between kilogrammes and the other units. In the same way, and for the same general reasons of convenience, it is common in relativistic quantum mechanics – high energy particle physics – to choose units so that $\hbar = c = 1$.

It is easy, once you have a little practice, to convert values and equations between the different systems of units. Throughout the rest of this course, I will quote equations in units where $c = 1$, and, when we come to that, $G = 1$, so that the factors c and G disappear from the equations.

B Further reading

When learning relativity, even more than with other subjects, you benefit from hearing or reading things multiple times, from different authors, and from different points of view. I mention a couple of good introductions below, but there is really no substitute for going to section ‘Physics C25’ in the library, looking through the books there, and finding one which makes sense to *you*.

The course book is [1] (herafter simply ‘Schutz’). Though possession of this book is not compulsory, it is very heavily recommended: of the books below, it is the one closest in style to these lectures; also, I will occasionally direct you to particular sections of it. Although the later parts of the first edition [2] are now somewhat out of date, the earlier parts, which are the main overlap with this course, have not much changed, and this edition may still be available second hand.

Other textbooks you might want to look at are below.

- Carroll [3] is good, but I don’t necessarily recommend buying it. Although it’s mathematically similar, the order of the material, and the things it stresses, are sufficiently different from this course and Schutz that it might be confusing. However, that difference is also a virtue: the book introduces topics clearly, and in a way which usefully contrasts with my way. Unfortunately, this book isn’t in the University library, but Sean Carroll’s relativity lecture notes, from a few years ago, are easily findable on the web.

- Rindler [4] always explains the physics clearly, particularly the differences between the strong and weak equivalence principles, and the motivation for GR. However it’s now rather old-fashioned in many respects, in particular in its treatment of differential geometry.

- Similarly, again, Narlikar [5] is worthwhile looking at, to see if it suits you. The mathematical approach is one which introduces vectors and tensors via components (like

Rindler), rather than the more functional approach we'll use here. I think that Narlikar is good at transmitting mathematical and physical insights.

- Wald [6] is comprehensive and well thought-of.
- Misner, Thorne and Wheeler [7] is a glorious, comprehensive, doorstop of a book. Its distinctive prose style and typographical oddities have fans and enemies in roughly equal numbers. If you liked Taylor & Wheeler's *Spacetime Physics*, there's a good chance you'll like this one. There's much, much, more in here than you need for the course. Chapter 1 in particular is worth reading for an overview of the subject.

This is a pretty mathematical course, but it is supposed to be a *physics* course, so we're looking for they physical insights which can easily become buried beneath the maths.

- Another Schutz book, *Gravity from the Ground Up* [8] aims to cover all of gravitational physics from falling apples to black holes using the minimum of maths. It won't help with the differential geometry, but it'll supply lots of insight.
- Longair's book [9] is excellent. The section on GR (only a smallish part of the book) is concerned with motivating the subject rather than doing a lot of maths, and is in a seat-of-the-pants style that might be to your taste.

There are also many more advanced texts. The following are graduate-level texts, and so reach well beyond the level of this course. They are mathematically very sophisticated. If, however, your tastes and experience run that way, then the introductory chapters of these books might be instructive, and give you a taste of the vast wonderland of beautiful maths that can be found in this subject.

- Chapter 1 of Stewart [10] covers more than the content of this course in just 60 pages.
- *Geometrical Methods of Mathematical Physics* [11] is by the same author as [1] above. It's a lovely book, which explains the differential geometry clearly and sparsely, including

applications beyond relativity and cosmology. However, it appeals only to those with a strong mathematical background, and horrifies everyone else.

- Hawking and Ellis [12], chapter 2, covers more than all the differential geometry of this course.

Notation conventions There are a number of different *sign conventions* in use in relativity books. The sign conventions used in this course match those in Schutz, [7], [11], and [12]. Stewart [10] has the opposite signs for g, R and G; and Rindler [4] has opposite signs for g and G.

References

[1] Bernard F Schutz. *A First Course in General Relativity*. Cambridge University Press, second edition, 2009. ISBN 978-0-521-88705-2.

[2] Bernard F Schutz. *A First Course in General Relativity*. Cambridge University Press, first edition, 1985. ISBN 0-521-27703-5.

[3] Sean M Carroll. *Spacetime and Geometry*. Pearson Education, 2004. ISBN 0-8053-8732-3. See also `http://spacetimeandgeometry.net`.

[4] Wolfgang Rindler. *Essential Relativity: Special, General and Cosmological*. Springer-Verlag, 2nd edition, 1977. ISBN 0-387-10090-3.

[5] Jayant V Narlikar. *An Introduction to Relativity*. Cambridge, 2010. ISBN 978-0-521-73561-2.

[6] Robert M Wald. *General Relativity*. University of Chicago Press, 1984. ISBN 0-226-87033-2.

[7] Charles W Misner, Kip S Thorne, and John Archibald Wheeler. *Gravitation*. Freeman, 1973. ISBN 0-7167-0334-3.

[8] Bernard F Schutz. *Gravity from the Ground Up*. Cambridge, 2003. ISBN 0-521-45506-5.

[9] Malcolm S Longair. *Theoretical Concepts in Physics: An Alternative View of Theoretical Reasoning in Physics*. Cambridge, second edition, 2003. ISBN 978-0521528788.

[10] John Stewart. *Advanced General Relativity*. Cambridge, 1991. ISBN 0-521-44946-4.

[11] Bernard F Schutz. *Geometrical Methods of Mathematical Physics*. Cambridge University Press, 1980. ISBN 0-521-29887-3.

[12] Stephen W Hawking and G F R Ellis. *The Large Scale Structure of Space-Time*. Cambridge, 1973. ISBN 0-521-09906-4.

Examples

Example 1.1 (section 1.3)

A photon of frequency ν is emitted vertically upwards from the floor of the box while a rocket, firing beneath the box, is accelerating it upwards at $1g$. What is the frequency (or energy) of the photon when it is absorbed by a detector fixed in the box at a height h above the floor? Use the Doppler redshift formula $\nu_{\text{em}}/\nu_{\text{obs}} = 1 + v$ (in units where $c = 1$). How does this link to other remarks in this section?

Example 1.2 (section 2.1)

If two 1kg balls, 1m apart, fall down a liftshaft near the surface of the earth, how much is their tidal acceleration towards each other? How much is their acceleration towards each other as a result of their mutual gravitational attraction?

Example 1.3 (section A)

Convert the following to units in which $c = 1$: (a) 10 J; (b) lightbulb power,100 W; (c) Planck's constant, $\hbar = 1.05 \times 10^{-34}\,\mathrm{J\,s}$; (d) velocity of a car, $v = 30\,\mathrm{m\,s^{-1}}$; (e) momentum of a car, $3 \times 10^4\,\mathrm{kg\,m\,s^{-1}}$; (f) pressure of 1 atmosphere, $10^5\,\mathrm{N\,m^{-2}}$; (g) density of water, $10^3\,\mathrm{kg\,m^{-3}}$; (h) luminosity flux, $10^6\,\mathrm{J\,s^{-1}\,cm^{-2}}$.

Convert the following to physical units (SI): (i) velocity, $v = 10^{-2}$; (j) pressure $10^{19}\,\mathrm{kg\,m^{-3}}$; (k) time 10^{18} m; (l) energy density $u = 1\,\mathrm{kg\,m^{-3}}$; (m) acceleration $10\,\mathrm{m^{-1}}$; (n) the Lorentz transformation, $t' = \gamma(t - vx)$; (o) the 'mass-shell' equation $E^2 = p^2 + m^2$. (Example slightly adapted from Schutz [1, ch.1])