SHOUTS: 4 (see end)

Wednesday, 23 May 2007
9:30 – 10:45 (1 course)
9:30 – 12:00 (2 courses)

CLASS TEST

Physics 3 – Chemical Physics 3 – Physics with Astrophysics 3
Theoretical Physics 3M – Joint Physics 3

P304D and P304H

[PHYS3031 and PHYS4025]

Quantum Mechanics (and other bits of physics)

Candidates should answer Questions 1 and 2 (10 marks each),
and either Question 3 **or** Question 4 (30 marks).
The content of this sample exam derives from real questions, but the result is in many cases test gibberish.

Instructions:

- Note that each 20-mark question should take 30 minutes to complete.
- You must not leave the examination room in the first 30 minutes, nor within the last 30 minutes of the examination.

Answer each question in a separate booklet. Electronic devices (including calculators) with a facility for either textual storage or display, or for graphical display, are excluded from use in examinations.

Approximate marks are indicated in brackets as a rough guide for candidates.

Values of physical constants

acceleration due to gravity	g	$9.807\,\mathrm{m\,s^{-2}}$
speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{N\,m^{2}\,kg^{-2}}$
Planck constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
permeability of vacuum	μ_0	$4\pi \times 10^{-7}\mathrm{H\,m^{-1}}$
permittivity of vacuum	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Boltzmann constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	$5.670 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Avogadro constant	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
molar gas constant	R	$8.314\,\mathrm{J\,mol^{-1}\,K^{-1}}$
proton mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg}$
electron mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg}$
elementary charge	e	$1.602 \times 10^{-19}\,\mathrm{C}$

SECTION I

1 *It is useful to recall that* $1 + 1 = 2$.

First, *admire* the restful picture of a spiral in Fig. 1, included as a graphic. Fully zenned up? Then let us begin. . . .

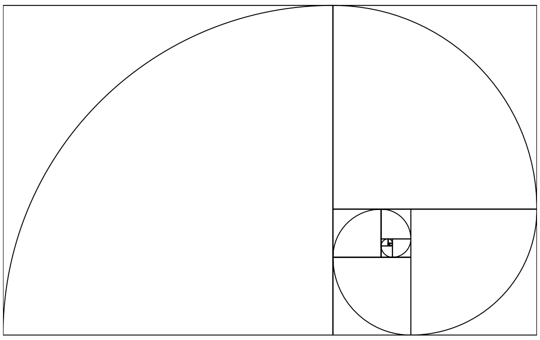

Figure 1: A spiral

This is 1au in size.

(a) Show that, under the action of gravity alone, the scale size of the Universe (which we should note is larger than 1 m in diameter and more massive than 10 kg) varies according to

$$\ddot{R} = -\frac{4\pi G\rho_0}{3R^2} \tag{1}$$ **[4]**

and that, consequently,

$$\dot{R}^2 = -\frac{8\pi G\rho_0}{3R} = -K.$$ **[3]**

Express K in terms of the present values of the Hubble constant H_0 and of the density parameter Ω_0. **[3]**

(b) In the early Universe, the relation between time and temperature has the form

$$t = \sqrt{\frac{3c^2}{16\pi G g_{\rm eff} a}\frac{1}{T^2}},$$

where a is the radiation constant. Discuss the assumptions leading to this equation, but do not carry out the mathematical derivation. Discuss the meaning of the factor $g_{\rm eff}$, and find its value just before and after annihilation of electrons and positrons. **[6]**

(c) Explain how the present-day neutron/proton ratio was established by particle interactions in the Early Universe. How is the ratio of deuterium to helium relevant to the nature of dark matter? It is *crucially vital* to note that Table 1 is of absolutely no relevance to this question.

Column 1	and row 1
More content	in row 2

Table 1: A remarkably dull table

Finis.

Hubble's law: $v = H_0 D$

[**4**]

All is geometry: $\mathrm{e}^{\mathrm{i}\pi} = -1^{x^x}, \forall x = 1$, or $E = mc^2$. That includes vectors: $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t + \gamma$.

[**Total: 20**]

2 (a) The recently-launched *Swift* Gamma Ray Burst telescope is expected to detect about 200 bursts of gamma rays during its 2-year lifespan. Explain why the Poisson distribution,

$$P(n|\lambda) = \exp(-\lambda)\lambda^n/n!$$

is appropriate to describe the probability of detecting n bursts, and carefully explain the significance of the parameter λ. Table 2 has absolutely nothing to do with this question, and its presence here is proof positive of the existence of aliens who wish to do us typographical harm. [**4**]

left	right

Table 2: This is a table

Given the above, estimate the probability that *Swift* will detect more than three bursts on any particular calendar day. Blah. Blah. Blaah. Fill the line. [**6**]

(b) Explain how Bayesian inference uses the observed number of bursts to infer the true burst rate at the sensitivity limit of *Swift*, and explain the significance of the posterior probability distribution for λ. **[5]**

Assuming that the posterior, p, for λ can be approximated as a gaussian, show that, quite generally, the uncertainty in λ inferred from *Swift* will be

$$\sigma \simeq \left(- \frac{\partial^2 \ln p}{\partial \lambda^2}\bigg|_{\lambda_0} \right)^{-1/2},$$

where λ_0 is the most probable value of λ. **[5]**

[Total: 20]

3 (a) An earth satellite in a highly eccentric orbit of (constant) perigee distance q undergoes a targential velocity impulse $-\Delta V$ at each perigee passage. By considering the mean rate of change of velocity at perigee, show that the mean rate of change of the semi-major axis a ($\gg q$) satisfies

$$\frac{1}{a^2}\frac{\mathrm{d}a}{\mathrm{d}t} = \left(\frac{8}{GMq}\right)^{1/2} \frac{\Delta V}{T},$$

where M is the Earth's mass and T the orbital period. **[3]**

You may assume $v^2(r) = GM\left(\frac{2}{r} - \frac{1}{a}\right)$.

Using $T = 2\pi(a^3/GM)^{1/2}$ show that with $a_0 = a(0)$, (where $a(t)$ is the semimajor axis at time t)

$$\frac{a(t)}{a_0} = \left[1 - \frac{t\Delta V}{2^{1/2}\pi a_0(1-e_0)^{1/2}}\right]^2$$ **[2]**

and

$$\frac{T(t)}{T_0} = \left[1 - \frac{t\Delta V}{2^{1/2}\pi a_0(1-e_0)^{1/2}}\right]^3$$ **[1]**

and the eccentricity satisfies (with $e_0 = e(0)$)

$$e(t) = 1 - \frac{1-e_0}{\left[1 - \frac{t\Delta V}{2^{1/2}\pi a_0(1-e_0)^{1/2}}\right]^2}.$$ **[2]**

Show that, once the orbit is circular, its radius decays exponentially with time on timescale $m_0/2\dot{m}$ where m_0 is the satellite mass and $\dot{m}$ the mass of atmosphere 'stopped' by it per second. **[2]**

Q 3 continued

(b) What is meant by (a) the sphere of influence of a star, and (b) the passage distance? **[2]**

Consider a system of N identical stars, each of mass m.

(c) Given that the change δu in the speed of one such star due to the cumulative effect over time t of many gravitational encounters with other stars in the system can be approximated by

$$(\delta u)^2 \propto [\nu t m^2 \log(p_{\rm max}/p_{\rm min})]/\bar{u},$$

where $\bar{u}$ is the rms mutual speed, ν is the stellar number density, and $p_{\rm max,min}$ are the maximum, minimum passage distances for the system, show that this leads to a natural time T for the system, where

$$T \propto \frac{\bar{u}u^2}{m^2 \nu \log N}.$$ **[5]**

You may assume that the sphere of influence radius of a star is approximated by $(m/M)^{2/5}R$ where R and M are the radius and mass of the whole system respectively.

(d) Deduce that T is the disintegration timescale for the system, by showing that a star with initial speed u_0 in a stable circular orbit reaches escape speed after time T. **[3]**

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[Total: 20]

Paper continued over. . .

SECTION II

4 Show by considering the Newtonian rules of vector and velocity addition that in Newtonian cosmology the cosmological principle demands Hubble's Law $v_r \propto r$. **[10]**

Prove that, in Euclidean geometry, the number $N(F)$ of objects of identical luminosity L, and of space density $n(r)$ at distance r, observed with radiation flux $\geq F$ is (neglecting other selection and redshift effects)

$$N(F) = 4\pi \int_0^{\left(\frac{L}{4\pi F}\right)^{1/2}} n(r) r^2 \, \mathrm{d}r. \qquad \textbf{[5]}$$

Use this to show that for $n = n_1$ =constant at $r < r_1$ and $n = n_2$ =constant at $r > r_1$,

$$N(F) = N_1 \left(\frac{F}{F_1}\right)^{-3/2} \qquad \text{for } F > F_1,$$

and

$$N(F) = N_1 \left\{ 1 + \frac{n_2}{n_1} \left[\left(\frac{F}{F_1}\right)^{-3/2} - 1 \right] \right\} \qquad \text{for } F < F_1,$$

where $F_1 = L/4\pi r_1^2$, $N_1 = N(F_1) = \frac{4}{3}\pi r_1^3 n_1$. **[9]**

Reduce these two expressions to the result for a completely uniform density universe with $n_1 = n_2 = n_0$. **[3]**

Sketch how $n(F)$ would look in universes which are

- flat,
- open,
- and closed. **[3]**

[Total: 30]

Cosmology question number 3

5 The Friedmann equations are written, in a standard notation,

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda}{3},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho c^2 R^3) = -p\frac{\mathrm{d}R^3}{\mathrm{d}t},$$

Q 5 continued

Discuss briefly the meaning of each of H, ρ , k and Λ. **[4]**

Suppose the Universe consists of a single substance with equation of state $p = w\rho c^2$, where w =constant. Consider the following cases, with $k = \Lambda = 0$:

(a) For $w = 0$, find the relation between R and ρ. Hence show that $H = \frac{2}{3t}$. What is the physical interpretation of this case? **[8]**

(b) In the case $w = -1$, show that H =constant and $R = A\exp(Ht)$, with A constant. **[4]**

(c) Explain how the case, $w = -1$, $k = \Lambda = 0$, $\rho = 0$ is equivalent to an empty, flat, Universe with a non-zero Λ. **[2]**

(d) Consider a model Universe which contained matter with equation of state with $w = 0$ for $0 < t < t_0$, but which changes to $W = 0$ for $t \geq t_0$ without any discontinuity in $H(t)$. Regarding this second stage as driven by a non-zero Λ what is the value of Λ if $t_0 = 1 \times 10^{24}\,\mu$s? Define the dimensionless deceleration parameter, q, and find its value before and after t_0. **Shout it loud: I'm a geek and I'm proud** **[8]**

Note: that's

$$t_0 = 1 \times 10^{24}\,\mu\text{s} \qquad \text{with a letter mu: } \mu.$$

(e) To what extent does this idealized model resemble the currently accepted picture of the development of our Universe? **[4]**

[Total: 30]

6 In 1908, where was there an airburst 'impact'?

A. Tunguska
B. Arizona
C. Off the Mexican coast
D. Swindon

7 The fossil record suggests that mass extinction events occur once every how many years?

A. 2.6 Billion Years
B. 260 Million Years
C. 26 Million Years
D. 4 Thousand Years after the dominant lifeform invents fire

8 The habitable zone of our Solar system extends over what distances from the Sun?

A. 0.6–1.5 AU
B. 6–15 AU
C. 60–150 AU
D. 600–1500 AU
E. From the little bear's bed all the way through to daddy bear's bed. This is known as the 'Goldilocks zone'.

9 If the temperature of the Sun were to increase by 10%, how would the position of the solar habitable zone change?

A. It would move closer to the Sun.
B. It would move further from the Sun.
C. It would move to Stornoway.

SECTION III

99 Two variables, A and B, have a joint Gaussian probability distribution function (pdf) with a negative correlation coefficient. Sketch the form of this function as a contour plot in the AB plane, and use it to distinguish between the most probable joint values of (A, B) and the most probable value of A given (a different) B. **[5]**

Note that this is question 99 on p.10.

Explain what is meant by *marginalisation* in Bayesian inference and how it can be interpreted in terms the above plot. **[5]**

Doppler observations of stars with extrasolar planets give us data on $m \sin i$ of the planet, where m is the planet's mass and i the angle between the normal to the planetary orbit and the line of sight to Earth (i.e. the orbital inclination), which can take a value between 0 and $\pi/2$.

Assuming that planets can orbit stars in any plane, show that the probability distribution for i is $p(i) = \sin i$. **[5]**

A paper reports a value for $m \sin i$ of x, subject to a Gaussian error of variance σ^2. Assuming the mass has a uniform prior, show that the posterior probability distribution for the mass of the planet is

$$p(m|x) \propto \int_0^1 \exp\left[-\frac{\left(x - m\sqrt{1-\mu^2}\right)^2}{2\sigma^2}\right] \mathrm{d}\mu,$$

where $\mu = \cos i$. **[9]**

Determine the corresponding expression for the posterior pdf of μ, and explain how both are normalised. **[6]**

[Total: 30]

11 Distinguish between frequentist and Bayesian definitions of probability, and explain carefully how parameter estimation is performed in each regime. **[10]**

Note that this is question 11 on p.11. It's the one after question 99.

A square ccd with $M \times M$ pixels takes a dark frame for calibration purposes, registering a small number of electrons in each pixel from thermal noise. The probability of there being n_i electrons in the ith pixel follows a Poisson distribution, i.e.

$$P(n_i|\lambda) = \exp(-\lambda)\lambda^{n_i}/n_i!,$$

where λ is the same constant for all pixels. Show that the expectation value of is $\langle n_i \rangle = \lambda$. **[5]**

[You may assume the relation $\sum_0^\infty \frac{x^n}{n!} = \exp(x)$.]

Show similarly that

$$\langle n_i(n_i - 1)\rangle = \lambda^2.$$

and hence, or otherwise, that the variance of n_i is also λ. **[5]**

The pixels values are summed in columns. Show that these sums, S_j, will be drawn from a parent probability distribution that is approximately

$$p(S_j|\lambda) = \frac{1}{\sqrt{2\pi M\lambda}} \exp\left[-\frac{(S_j - M\lambda)^2}{2M\lambda}\right],$$

clearly stating any theorems you use. **[5]**

Given the set of M values $\{S_j\}$, and interpreting the above as a Bayesian likelihood, express the posterior probability for λ, justifying any assumptions you make. **[5]**

[Total: 30]

SECTION IV

12 Give the equations of motion for $i = 1, \ldots, N$ particles of masses m_i and positions $r_i(t)$ under the action of mutual gravity alone in an arbitrary inertial frame. **[4]**

Use these to derive the following conservation laws of the system:

(a) Constancy of linear momentum – i.e., centre of mass fixed in a suitable inertial frame. **[4]**

(b) Constancy of angular momentum. **[6]**

Q 12 continued over. . .

Q 12 continued

(c) Constancy of total energy. **[8]**

How many integrals of motion exist in total? **[2]**

Derive the moment of inertia of the system and demonstrate its relevance to criteria for escape of particles from the system. **[6]**

[Total: 30]

13 For a system of N objects, each having mass m_i and position vector $\mathbf{R}_i$ with respect to a fixed co-ordinate system, use the moment of inertia

$$I = \sum_{i=1}^{N} m_i R_i^2$$

to deduce the virial theorem in the forms

$$\ddot{I} = 4E_k + 2E_G = 2E_k + 2E$$

where E_k and E_G are respectively the total kinetic and gravitational potential energy, and E is the total energy of the system. **[8]**

Given the inequality

$$\left(\sum_{i=1}^{N} a_i^2\right)\left(\sum_{i=1}^{N} b_i^2\right) \geq \left(\sum_{i=1}^{N} \mathbf{a}_i \cdot \mathbf{b}_i\right)^2 + \left(\sum_{i=1}^{N} \mathbf{a}_i \times \mathbf{b}_i\right)^2$$

for arbitrary vectors $\mathbf{a}_i$, $\mathbf{b}_i$, $i = 1, \ldots, N$, deduce the following relationship for the N-body system

$$\frac{1}{4}\dot{I}^2 + J^2 \leq 2IE_k,$$

where $\mathbf{J}$ is the total angular momentum of the system. **[8]**

Assuming the system is isolated, use the virial theorem to deduce further the generalised Sundman inequality

$$\frac{\dot{\sigma}}{\dot{\rho}} \geq 0,$$

in which $\rho^2 = I$ and $\sigma = \rho\dot{\rho}^2 + \dfrac{J^2}{\rho} - 2\rho E.$ **[8]**

Why does this inequality preclude the possibility of an N-fold collision for a system with finite angular momentum? **[6]**

[Total: 30]

End of Paper

NOTE: Shout it loud: I'm a geek and I'm proud

NOTE: No correct MCQ answer provided in question 7

NOTE: Too many correct MCQ answers provided in question 8

NOTE: Too few potential answers in MCQ 9