

# SHOWING SOLUTIONS



University  
of Glasgow

Wednesday, 23 May 2007

9:30 – 10:45 (1 course)

9:30 – 12:00 (2 courses)

## CLASS TEST

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Physics 3 – Chemical Physics 3 – Physics with Astrophysics 3  
Theoretical Physics 3M – Joint Physics 3

**P304D and P304H**

**[ PHYS3031 and PHYS4025 ]**

## Quantum Mechanics (and other bits of physics)

Candidates should answer Questions 1 and 2 (10 marks each),  
**and either** Question 3 **or** Question 4 (30 marks).

The content of this sample exam derives from real questions, but the result is  
in many cases test gibberish.

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### Instructions:

- Note that each 20-mark question should take 30 minutes to complete.
- You must not leave the examination room in the first 30 minutes, nor within the last 30 minutes of the examination.

Answer each question in a separate booklet. Electronic devices (including calculators) with a facility for either textual storage or display, or for graphical display, are excluded from use in examinations.

Approximate marks are indicated in brackets as a rough guide for candidates.

# SHOWING SOLUTIONS

## Values of physical constants

acceleration due to gravity	$g$	$9.807 \text{ m s}^{-2}$
speed of light in vacuum	$c$	$2.998 \times 10^8 \text{ m s}^{-1}$
gravitational constant	$G$	$6.674 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$
Planck constant	$h$	$6.626 \times 10^{-34} \text{ J s}$
permeability of vacuum	$\mu_0$	$4\pi \times 10^{-7} \text{ H m}^{-1}$
permittivity of vacuum	$\epsilon_0$	$8.854 \times 10^{-12} \text{ F m}^{-1}$
Boltzmann constant	$k_B$	$1.381 \times 10^{-23} \text{ J K}^{-1}$
Stefan-Boltzmann constant	$\sigma$	$5.670 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
Avogadro constant	$N_A$	$6.022 \times 10^{23} \text{ mol}^{-1}$
molar gas constant	$R$	$8.314 \text{ J mol}^{-1} \text{ K}^{-1}$
proton mass	$m_p$	$1.673 \times 10^{-27} \text{ kg}$
electron mass	$m_e$	$9.109 \times 10^{-31} \text{ kg}$
elementary charge	$e$	$1.602 \times 10^{-19} \text{ C}$

# SHOWING SOLUTIONS

## SECTION I

- 1 First, *admire* the restful picture of a spiral in Fig. 1, included as a graphic. Fully zenned up? Then let us begin...

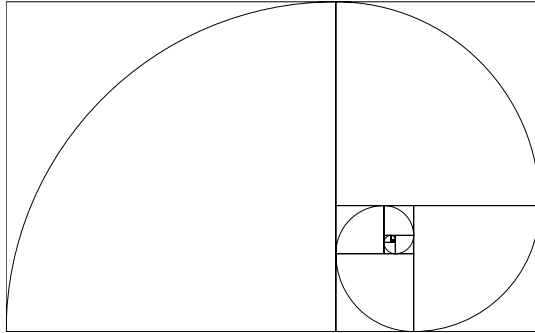


Figure 1: A spiral

This is 1au in size.

- (a) Show that, under the action of **gravity** alone, the scale size of the Universe (which we should note is larger than 1 m in diameter and more massive than 10 kg) varies according to

$$\ddot{R} = -\frac{4\pi G\rho_0}{3R^2} \quad (1) \quad [4] \text{ demonstration}$$

and that, consequently,

$$\dot{R}^2 = -\frac{8\pi G\rho_0}{3R} = -K. \quad [3] \text{ another remarkably long remark}$$

Express  $K$  in terms of the present values of the Hubble constant  $H_0$  and of the density parameter  $\Omega_0$ . [3] [comment 1]

**Solution:** This can be solved by *remembering* the solution [...3]

- (b) In the early Universe, the relation between time and temperature has the form

$$t = \sqrt{\frac{3c^2}{16\pi G g_{\text{eff}} a}} \frac{1}{T^2},$$

where  $a$  is the radiation constant. Discuss the assumptions leading to this equation, but do not carry out the mathematical derivation. Discuss the meaning of the factor  $g_{\text{eff}}$ , and find its value just before and after annihilation of electrons and positrons. [6]

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<sup>1</sup>bookwork from a very long and boring book, which goes on and on at really quite unreasonable length, line after line, until the reader is adequately cudgelled into intellectual submission

SHOWING SOLUTIONS

Q 1 continued

**Solution:** Before, well, geeee; after... kazamm!

(c) Explain how the present-day neutron/proton ratio was established by particle interactions in the Early Universe. How is the ratio of deuterium to helium relevant to the nature of dark matter? It is *crucially vital* to note that Table 1 is of absolutely no relevance to this question.

Column 1	and row 1
More content	in row 2

Table 1: A remarkably dull table

Finis.

Hubble’s law:  $v = H_0 D$

[4]

All is geometry:  $e^{i\pi} = -1^{x^x}, \forall x = 1$ , or  $E = mc^2$ . That includes vectors:  $\mathbf{v} = d\mathbf{x}/dt + \gamma$ .

**Solution:** Explanations are superfluous; all that is, is. [...1]

First rows	are premier
subsequent rows	are of secondary interest

Table: A table o’erbrimming with otioseness

In addition, Table 1c adds nothing to the discussion, and adds nothing to our understanding of our place in the cosmos, but it *does* contribute slightly to the heat-death of the universe (can you work out how many deuterium nuclei decayed during the typing of this table?). [...1]

All is geometry (in here, too):  $e^{i\pi} = -1^{x^x}, \forall x = 1$ , or  $E = mc^2$ , or [...1]

$$e^{i\pi} = \cos \pi + i \sin \pi = -1^{x^x}$$
$$E = mc^2 \qquad \text{inevitably.}$$

[Total: 20]

## SHOWING SOLUTIONS

- 2 (a) The recently-launched *Swift* Gamma Ray Burst telescope is expected to detect about 200 bursts of gamma rays during its 2-year lifespan. Explain why the Poisson distribution,

$$P(n|\lambda) = \exp(-\lambda)\lambda^n/n!$$

is appropriate to describe the probability of detecting  $n$  bursts, and carefully explain the significance of the parameter  $\lambda$ . Table 2 has absolutely nothing to do with this question, and its presence here is proof positive of the existence of aliens who wish to do us typographical harm. [4]

left	right
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Table 2: This is a table

Given the above, estimate the probability that *Swift* will detect more than three bursts on any particular calendar day. Blah. Blah. Blaah. Fill the line. [6]

- (b) Explain how Bayesian inference uses the observed number of bursts to infer the true burst rate at the sensitivity limit of *Swift*, and explain the significance of the posterior probability distribution for  $\lambda$ . [5]

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**Solution:**

This page and the following two should appear on separate pages (as opposed to superimposed on each other), and disappear when the noshowsolutions option is present.

(In the real situation, the corresponding PDF would be a scan of a hand-written solution.)

# Numerical 1 solution, page one

Numerical 1 solution, page two

Numerical 1 solution, page three



## SHOWING SOLUTIONS

### Q 2 continued

Assuming that the posterior,  $p$ , for  $\lambda$  can be approximated as a gaussian, show that, quite generally, the uncertainty in  $\lambda$  inferred from *Swift* will be

$$\sigma \simeq \left( -\frac{\partial^2 \ln p}{\partial \lambda^2} \Big|_{\lambda_0} \right)^{-1/2},$$

where  $\lambda_0$  is the most probable value of  $\lambda$ .

[5]

[Total: 20]

- 3 (a) An earth satellite in a highly eccentric orbit of (constant) perigee distance  $q$  undergoes a tangential velocity impulse  $-\Delta V$  at each perigee passage. By considering the mean rate of change of velocity at perigee, show that the mean rate of change of the semi-major axis  $a$  ( $\gg q$ ) satisfies

$$\frac{1}{a^2} \frac{da}{dt} = \left( \frac{8}{GMq} \right)^{1/2} \frac{\Delta V}{T},$$

where  $M$  is the Earth's mass and  $T$  the orbital period.

[3]

You may assume  $v^2(r) = GM \left( \frac{2}{r} - \frac{1}{a} \right)$ .

Using  $T = 2\pi(a^3/GM)^{1/2}$  show that with  $a_0 = a(0)$ , (where  $a(t)$  is the semimajor axis at time  $t$ )

$$\frac{a(t)}{a_0} = \left[ 1 - \frac{t\Delta V}{2^{1/2}\pi a_0(1-e_0)^{1/2}} \right]^2$$

[2]

and

$$\frac{T(t)}{T_0} = \left[ 1 - \frac{t\Delta V}{2^{1/2}\pi a_0(1-e_0)^{1/2}} \right]^3$$

[1]

and the eccentricity satisfies (with  $e_0 = e(0)$ )

$$e(t) = 1 - \frac{1-e_0}{\left[ 1 - \frac{t\Delta V}{2^{1/2}\pi a_0(1-e_0)^{1/2}} \right]^2}.$$

[2]

Show that, once the orbit is circular, its radius decays exponentially with time on timescale  $m_0/2\dot{m}$  where  $m_0$  is the satellite mass and  $\dot{m}$  the mass of atmosphere 'stopped' by it per second.

[2]

## SHOWING SOLUTIONS

### Q 3 continued

(b) What is meant by (a) the sphere of influence of a star, and (b) the passage distance?

[2]

Consider a system of  $N$  identical stars, each of mass  $m$ .

(c) Given that the change  $\delta u$  in the speed of one such star due to the cumulative effect over time  $t$  of many gravitational encounters with other stars in the system can be approximated by

$$(\delta u)^2 \propto [\nu t m^2 \log(p_{\max}/p_{\min})]/\bar{u},$$

where  $\bar{u}$  is the rms mutual speed,  $\nu$  is the stellar number density, and  $p_{\max, \min}$  are the maximum, minimum passage distances for the system, show that this leads to a natural time  $T$  for the system, where

$$T \propto \frac{\bar{u}u^2}{m^2\nu \log N}. \quad [5]$$

You may assume that the sphere of influence radius of a star is approximated by  $(m/M)^{2/5}R$  where  $R$  and  $M$  are the radius and mass of the whole system respectively.

(d) Deduce that  $T$  is the disintegration timescale for the system, by showing that a star with initial speed  $u_0$  in a stable circular orbit reaches escape speed after time  $T$ .

[3]

[illegible]

**[Total: 20]**

# SHOWING SOLUTIONS

## SECTION II

- 4 Show by considering the Newtonian rules of vector and velocity addition that in Newtonian cosmology the cosmological principle demands Hubble's Law  $v_r \propto r$ . [10]

Prove that, in Euclidean geometry, the number  $N(F)$  of objects of identical luminosity  $L$ , and of space density  $n(r)$  at distance  $r$ , observed with radiation flux  $\geq F$  is (neglecting other selection and redshift effects)

$$N(F) = 4\pi \int_0^{(L/4\pi F)^{1/2}} n(r) r^2 dr. \quad [5]$$

Use this to show that for  $n = n_1 = \text{constant}$  at  $r < r_1$  and  $n = n_2 = \text{constant}$  at  $r > r_1$ ,

$$N(F) = N_1 \left( \frac{F}{F_1} \right)^{-3/2} \quad \text{for } F > F_1,$$

and

$$N(F) = N_1 \left\{ 1 + \frac{n_2}{n_1} \left[ \left( \frac{F}{F_1} \right)^{-3/2} - 1 \right] \right\} \quad \text{for } F < F_1,$$

where  $F_1 = L/4\pi r_1^2$ ,  $N_1 = N(F_1) = \frac{4}{3}\pi r_1^3 n_1$ . [9]

Reduce these two expressions to the result for a completely uniform density universe with  $n_1 = n_2 = n_0$ . [3]

Sketch how  $n(F)$  would look in universes which are

- flat,
- open, [3]
- and closed.

**Solution:** A sufficiently heavy weight will reduce expressions to completely uniform sheets of paper if it is placed on top of them. In a flat universe,  $n(F)$  will look like  $n(F)$ . [3]  
[3]

[Total: 30]

# SHOWING SOLUTIONS

## Cosmology question number 3

- 5 The Friedmann equations are written, in a standard notation,

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda}{3},$$
$$\frac{d}{dt}(\rho c^2 R^3) = -p \frac{dR^3}{dt},$$

Discuss briefly the meaning of each of  $H$ ,  $\rho$ ,  $k$  and  $\Lambda$ . [4]

Suppose the Universe consists of a single substance with equation of state  $p = w\rho c^2$ , where  $w = \text{constant}$ . Consider the following cases, with  $k = \Lambda = 0$ :

(a) For  $w = 0$ , find the relation between  $R$  and  $\rho$ . Hence show that  $H = \frac{2}{3t}$ . What is the physical interpretation of this case? [8]

(b) In the case  $w = -1$ , show that  $H = \text{constant}$  and  $R = A \exp(Ht)$ , with  $A$  constant. [4]

(c) Explain how the case,  $w = -1$ ,  $k = \Lambda = 0$ ,  $\rho = 0$  is equivalent to an empty, flat, Universe with a non-zero  $\Lambda$ . [2]

(d) Consider a model Universe which contained matter with equation of state with  $w = 0$  for  $0 < t < t_0$ , but which changes to  $W = 0$  for  $t \geq t_0$  without any discontinuity in  $H(t)$ . Regarding this second stage as driven by a non-zero  $\Lambda$  what is the value of  $\Lambda$  if  $t_0 = 1 \times 10^{24} \mu\text{s}$ ? Define the dimensionless deceleration parameter,  $q$ , and find its value before and after  $t_0$ . **Shout it loud: I'm a geek and I'm proud** [8]

Note: that's

$$t_0 = 1 \times 10^{24} \mu\text{s} \quad \text{with a letter mu: } \mu.$$

(e) To what extent does this idealized model resemble the currently accepted picture of the development of our Universe? [4]

[Total: 30]

## SHOWING SOLUTIONS

6 In 1908, where was there an airburst ‘impact’?

- ⇒ A. Tunguska  
B. Arizona  
C. Off the Mexican coast  
D. Swindon

**Solution:** The evidence for this is a dirty big hole in the ground in Siberia.

7 The fossil record suggests that mass extinction events occur once every how many years?

- A. 2.6 Billion Years  
B. 260 Million Years  
C. 26 Million Years  
D. 4 Thousand Years after the dominant lifeform invents fire

8 The habitable zone of our Solar system extends over what distances from the Sun?

- ⇒ A. 0.6–1.5 AU  
B. 6–15 AU  
C. 60–150 AU  
D. 600–1500 AU  
⇒ E. From the little bear’s bed all the way through to daddy bear’s bed. This is known as the ‘Goldilocks zone’.

9 If the temperature of the Sun were to increase by 10%, how would the position of the solar habitable zone change?

- A. It would move closer to the Sun.  
⇒ B. It would move further from the Sun.  
C. It would move to Stornoway.

# SHOWING SOLUTIONS

## SECTION III

- 99 Two variables,  $A$  and  $B$ , have a joint Gaussian probability distribution function (pdf) with a negative correlation coefficient. Sketch the form of this function as a contour plot in the  $AB$  plane, and use it to distinguish between the most probable joint values of  $(A, B)$  and the most probable value of  $A$  given (a different)  $B$ . [5]

Note that this is question 99 on p.14.

Explain what is meant by *marginalisation* in Bayesian inference and how it can be interpreted in terms the above plot. [5]

Doppler observations of stars with extrasolar planets give us data on  $m \sin i$  of the planet, where  $m$  is the planet's mass and  $i$  the angle between the normal to the planetary orbit and the line of sight to Earth (i.e. the orbital inclination), which can take a value between 0 and  $\pi/2$ .

Assuming that planets can orbit stars in any plane, show that the probability distribution for  $i$  is  $p(i) = \sin i$ . [5]

A paper reports a value for  $m \sin i$  of  $x$ , subject to a Gaussian error of variance  $\sigma^2$ . Assuming the mass has a uniform prior, show that the posterior probability distribution for the mass of the planet is

$$p(m|x) \propto \int_0^1 \exp \left[ -\frac{(x - m\sqrt{1 - \mu^2})^2}{2\sigma^2} \right] d\mu,$$

where  $\mu = \cos i$ . [9]

Determine the corresponding expression for the posterior pdf of  $\mu$ , and explain how both are normalised. [6]

[Total: 30]

## SHOWING SOLUTIONS

- 11 Distinguish between frequentist and Bayesian definitions of probability, and explain carefully how parameter estimation is performed in each regime. [10]

Note that this is question 11 on p.15. It's the one after question 99.

A square ccd with  $M \times M$  pixels takes a dark frame for calibration purposes, registering a small number of electrons in each pixel from thermal noise. The probability of there being  $n_i$  electrons in the  $i$ th pixel follows a Poisson distribution, i.e.

$$P(n_i|\lambda) = \exp(-\lambda)\lambda^{n_i}/n_i!,$$

where  $\lambda$  is the same constant for all pixels. Show that the expectation value of is  $\langle n_i \rangle = \lambda$ . [5]

[You may assume the relation  $\sum_0^\infty \frac{x^n}{n!} = \exp(x)$ .]

Show similarly that

$$\langle n_i(n_i - 1) \rangle = \lambda^2.$$

and hence, or otherwise, that the variance of  $n_i$  is also  $\lambda$ . [5]

The pixels values are summed in columns. Show that these sums,  $S_j$ , will be drawn from a parent probability distribution that is approximately

$$p(S_j|\lambda) = \frac{1}{\sqrt{2\pi M\lambda}} \exp\left[-\frac{(S_j - M\lambda)^2}{2M\lambda}\right],$$

clearly stating any theorems you use. [5]

Given the set of  $M$  values  $\{S_j\}$ , and interpreting the above as a Bayesian likelihood, express the posterior probability for  $\lambda$ , justifying any assumptions you make. [5]

[Total: 30]

## SECTION IV

- 12 Give the equations of motion for  $i = 1, \dots, N$  particles of masses  $m_i$  and positions  $r_i(t)$  under the action of mutual gravity alone in an arbitrary inertial frame. [4]

Use these to derive the following conservation laws of the system:

(a) Constancy of linear momentum – i.e., centre of mass fixed in a suitable inertial frame. [4]

(b) Constancy of angular momentum. [6]

## SHOWING SOLUTIONS

### Q 12 continued

(c) Constancy of total energy. [8]

How many integrals of motion exist in total? [2]

Derive the moment of inertia of the system and demonstrate its relevance to criteria for escape of particles from the system. [6]

[Total: 30]

- 13 For a system of  $N$  objects, each having mass  $m_i$  and position vector  $\mathbf{R}_i$  with respect to a fixed co-ordinate system, use the moment of inertia

$$I = \sum_{i=1}^N m_i R_i^2$$

to deduce the virial theorem in the forms

$$\ddot{I} = 4E_k + 2E_G = 2E_k + 2E$$

where  $E_k$  and  $E_G$  are respectively the total kinetic and gravitational potential energy, and  $E$  is the total energy of the system. [8]

Given the inequality

$$\left( \sum_{i=1}^N a_i^2 \right) \left( \sum_{i=1}^N b_i^2 \right) \geq \left( \sum_{i=1}^N \mathbf{a}_i \cdot \mathbf{b}_i \right)^2 + \left( \sum_{i=1}^N \mathbf{a}_i \times \mathbf{b}_i \right)^2$$

for arbitrary vectors  $\mathbf{a}_i, \mathbf{b}_i, i = 1, \dots, N$ , deduce the following relationship for the  $N$ -body system

$$\frac{1}{4} \dot{I}^2 + J^2 \leq 2IE_k,$$

where  $\mathbf{J}$  is the total angular momentum of the system. [8]

Assuming the system is isolated, use the virial theorem to deduce further the generalised Sundman inequality

$$\frac{\dot{\sigma}}{\dot{\rho}} \geq 0,$$

in which  $\rho^2 = I$  and  $\sigma = \rho \dot{\rho}^2 + \frac{J^2}{\rho} - 2\rho E$ . [8]

Why does this inequality preclude the possibility of an  $N$ -fold collision for a system with finite angular momentum? [6]

[Total: 30]



## SHOWING SOLUTIONS

**End of Paper**

NOTE: SHOWING SOLUTIONS

NOTE: Shout it loud: I'm a geek and I'm proud

NOTE: No correct MCQ answer provided in question 7

NOTE: Too many correct MCQ answers provided in question 8

NOTE: Too few potential answers in MCQ 9