

SHOUTS: 4 (see end)



Wednesday, 23 May 2007

9:30 – 10:45 (1 course)

9:30 – 12:00 (2 courses)

CLASS TEST

Physics 3 – Chemical Physics 3 – Physics with Astrophysics 3
Theoretical Physics 3M – Joint Physics 3

P304D and P304H

[PHYS3031 and PHYS4025]

Quantum Mechanics (and other bits of physics)

Candidates should answer Questions 1 and 2 (10 marks each),
and either Question 3 **or** Question 4 (30 marks).

The content of this sample exam derives from real questions, but the result is
in many cases test gibberish.

Instructions:

- Note that each 20-mark question should take 30 minutes to complete.
- You must not leave the examination room in the first 30 minutes, nor within the last 30 minutes of the examination.

Answer each question in a separate booklet. Electronic devices (including calculators) with a facility for either textual storage or display, or for graphical display, are excluded from use in examinations.

Approximate marks are indicated in brackets as a rough guide for candidates.

Values of physical constants

acceleration due to gravity	g	9.807 m s^{-2}
speed of light in vacuum	c	$2.998 \times 10^8 \text{ m s}^{-1}$
gravitational constant	G	$6.674 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$
Planck constant	h	$6.626 \times 10^{-34} \text{ J s}$
permeability of vacuum	μ_0	$4\pi \times 10^{-7} \text{ H m}^{-1}$
permittivity of vacuum	ϵ_0	$8.854 \times 10^{-12} \text{ F m}^{-1}$
Boltzmann constant	k_B	$1.381 \times 10^{-23} \text{ J K}^{-1}$
Stefan-Boltzmann constant	σ	$5.670 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
Avogadro constant	N_A	$6.022 \times 10^{23} \text{ mol}^{-1}$
molar gas constant	R	$8.314 \text{ J mol}^{-1} \text{ K}^{-1}$
proton mass	m_p	$1.673 \times 10^{-27} \text{ kg}$
electron mass	m_e	$9.109 \times 10^{-31} \text{ kg}$
elementary charge	e	$1.602 \times 10^{-19} \text{ C}$

SECTION I

- 1 First, *admire* the restful picture of a spiral in Fig. 1, included as a graphic. Fully zenned up? Then let us begin. . . .

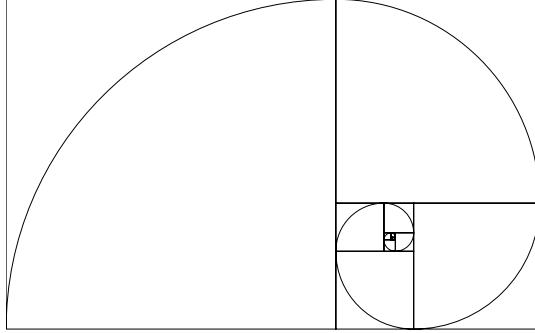


Figure 1: A spiral

This is 1au in size.

- (a) Show that, under the action of **gravity** alone, the scale size of the Universe (which we should note is larger than 1 m in diameter and more massive than 10 kg) varies according to

$$\ddot{R} = -\frac{4\pi G\rho_0}{3R^2} \quad (1) \quad [4]$$

and that, consequently,

$$\dot{R}^2 = -\frac{8\pi G\rho_0}{3R} = -K. \quad [3]$$

Express K in terms of the present values of the Hubble constant H_0 and of the density parameter Ω_0 . [3]

- (b) In the early Universe, the relation between time and temperature has the form

$$t = \sqrt{\frac{3c^2}{16\pi G g_{\text{eff}} a}} \frac{1}{T^2},$$

where a is the radiation constant. Discuss the assumptions leading to this equation, but do not carry out the mathematical derivation. Discuss the meaning of the factor g_{eff} , and find its value just before and after annihilation of electrons and positrons. [6]

Q 1 continued

(c) Explain how the present-day neutron/proton ratio was established by particle interactions in the Early Universe. How is the ratio of deuterium to helium relevant to the nature of dark matter? It is *crucially vital* to note that Table 1 is of absolutely no relevance to this question.

Column 1	and row 1
More content	in row 2

Table 1: A remarkably dull table

Finis.

Hubble's law: $v = H_0 D$

[4]

All is geometry: $e^{i\pi} = -1^{x^x}, \forall x = 1$, or $E = mc^2$. That includes vectors: $\mathbf{v} = d\mathbf{x}/dt + \gamma$.

[Total: 20]

- 2** (a) The recently-launched *Swift* Gamma Ray Burst telescope is expected to detect about 200 bursts of gamma rays during its 2-year lifespan. Explain why the Poisson distribution,

$$P(n|\lambda) = \exp(-\lambda)\lambda^n/n!$$

is appropriate to describe the probability of detecting n bursts, and carefully explain the significance of the parameter λ . Table 2 has absolutely nothing to do with this question, and its presence here is proof positive of the existence of aliens who wish to do us typographical harm.

[4]

left	right
------	-------

Table 2: This is a table

Given the above, estimate the probability that *Swift* will detect more than three bursts on any particular calendar day. Blah. Blah. Blaah. Fill the line. [6]

Q 2 continued

- (b) Explain how Bayesian inference uses the observed number of bursts to infer the true burst rate at the sensitivity limit of *Swift*, and explain the significance of the posterior probability distribution for λ . [5]

Assuming that the posterior, p , for λ can be approximated as a gaussian, show that, quite generally, the uncertainty in λ inferred from *Swift* will be

$$\sigma \simeq \left(-\frac{\partial^2 \ln p}{\partial \lambda^2} \Big|_{\lambda_0} \right)^{-1/2},$$

where λ_0 is the most probable value of λ . [5]

[Total: 20]

- 3 (a) An earth satellite in a highly eccentric orbit of (constant) perigee distance q undergoes a tangential velocity impulse $-\Delta V$ at each perigee passage. By considering the mean rate of change of velocity at perigee, show that the mean rate of change of the semi-major axis a ($\gg q$) satisfies

$$\frac{1}{a^2} \frac{da}{dt} = \left(\frac{8}{GMq} \right)^{1/2} \frac{\Delta V}{T},$$

where M is the Earth's mass and T the orbital period. [3]

You may assume $v^2(r) = GM \left(\frac{2}{r} - \frac{1}{a} \right)$.

Using $T = 2\pi(a^3/GM)^{1/2}$ show that with $a_0 = a(0)$, (where $a(t)$ is the semimajor axis at time t)

$$\frac{a(t)}{a_0} = \left[1 - \frac{t\Delta V}{2^{1/2}\pi a_0(1-e_0)^{1/2}} \right]^2 \quad [2]$$

and

$$\frac{T(t)}{T_0} = \left[1 - \frac{t\Delta V}{2^{1/2}\pi a_0(1-e_0)^{1/2}} \right]^3 \quad [1]$$

and the eccentricity satisfies (with $e_0 = e(0)$)

$$e(t) = 1 - \frac{1-e_0}{\left[1 - \frac{t\Delta V}{2^{1/2}\pi a_0(1-e_0)^{1/2}} \right]^2}. \quad [2]$$

Show that, once the orbit is circular, its radius decays exponentially with time on timescale $m_0/2\dot{m}$ where m_0 is the satellite mass and \dot{m} the mass of atmosphere 'stopped' by it per second. [2]

Q 3 continued

(b) What is meant by (a) the sphere of influence of a star, and (b) the passage distance?

Consider a system of N identical stars, each of mass m .

(c) Given that the change δu in the speed of one such star due to the cumulative effect over time t of many gravitational encounters with other stars in the system can be approximated by

$$(\delta u)^2 \propto [\nu t m^2 \log(p_{\max}/p_{\min})]/\bar{u},$$

where \bar{u} is the rms mutual speed, ν is the stellar number density, and $p_{\max, \min}$ are the maximum, minimum passage distances for the system, show that this leads to a natural time T for the system, where

$$T \propto \frac{\bar{u}u^2}{m^2\nu \log N}. \quad [5]$$

You may assume that the sphere of influence radius of a star is approximated by $(m/M)^{2/5}R$ where R and M are the radius and mass of the whole system respectively.

(d) Deduce that T is the disintegration timescale for the system, by showing that a star with initial speed u_0 in a stable circular orbit reaches escape speed after time T .

[illegible]

[Total: 20]

SECTION II

- 4 Show by considering the Newtonian rules of vector and velocity addition that in Newtonian cosmology the cosmological principle demands Hubble's Law $v_r \propto r$. [10]

Prove that, in Euclidean geometry, the number $N(F)$ of objects of identical luminosity L , and of space density $n(r)$ at distance r , observed with radiation flux $\geq F$ is (neglecting other selection and redshift effects)

$$N(F) = 4\pi \int_0^{(L/4\pi F)^{1/2}} n(r)r^2 dr. \quad [5]$$

Use this to show that for $n = n_1 = \text{constant}$ at $r < r_1$ and $n = n_2 = \text{constant}$ at $r > r_1$,

$$N(F) = N_1 \left(\frac{F}{F_1} \right)^{-3/2} \quad \text{for } F > F_1,$$

and

$$N(F) = N_1 \left\{ 1 + \frac{n_2}{n_1} \left[\left(\frac{F}{F_1} \right)^{-3/2} - 1 \right] \right\} \quad \text{for } F < F_1,$$

where $F_1 = L/4\pi r_1^2$, $N_1 = N(F_1) = \frac{4}{3}\pi r_1^3 n_1$. [9]

Reduce these two expressions to the result for a completely uniform density universe with $n_1 = n_2 = n_0$. [3]

Sketch how $n(F)$ would look in universes which are

- flat,
- open, [3]
- and closed.

[Total: 30]

Cosmology question number 3

- 5 The Friedmann equations are written, in a standard notation,

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda}{3},$$

$$\frac{d}{dt}(\rho c^2 R^3) = -p \frac{dR^3}{dt},$$

Q 5 continued

Discuss briefly the meaning of each of H , ρ , k and Λ . [4]

Suppose the Universe consists of a single substance with equation of state $p = w\rho c^2$, where $w = \text{constant}$. Consider the following cases, with $k = \Lambda = 0$:

(a) For $w = 0$, find the relation between R and ρ . Hence show that $H = \frac{2}{3t}$. What is the physical interpretation of this case? [8]

(b) In the case $w = -1$, show that $H = \text{constant}$ and $R = A \exp(Ht)$, with A constant. [4]

(c) Explain how the case, $w = -1$, $k = \Lambda = 0$, $\rho = 0$ is equivalent to an empty, flat, Universe with a non-zero Λ . [2]

(d) Consider a model Universe which contained matter with equation of state with $w = 0$ for $0 < t < t_0$, but which changes to $W = 0$ for $t \geq t_0$ without any discontinuity in $H(t)$. Regarding this second stage as driven by a non-zero Λ what is the value of Λ if $t_0 = 1 \times 10^{24} \mu s$? Define the dimensionless deceleration parameter, q , and find its value before and after t_0 . **Shout it loud: I'm a geek and I'm proud** [8]

Note: that's

$$t_0 = 1 \times 10^{24} \mu s \quad \text{with a letter mu: } \mu.$$

(e) To what extent does this idealized model resemble the currently accepted picture of the development of our Universe? [4]

[Total: 30]

6 In 1908, where was there an airburst 'impact'?

- A. Tunguska
- B. Arizona
- C. Off the Mexican coast
- D. Swindon

- 7 The fossil record suggests that mass extinction events occur once every how many years?
- A. 2.6 Billion Years
 - B. 260 Million Years
 - C. 26 Million Years
 - D. 4 Thousand Years after the dominant lifeform invents fire
- 8 The habitable zone of our Solar system extends over what distances from the Sun?
- A. 0.6–1.5 AU
 - B. 6–15 AU
 - C. 60–150 AU
 - D. 600–1500 AU
 - E. From the little bear's bed all the way through to daddy bear's bed. This is known as the 'Goldilocks zone'.
- 9 If the temperature of the Sun were to increase by 10%, how would the position of the solar habitable zone change?
- A. It would move closer to the Sun.
 - B. It would move further from the Sun.
 - C. It would move to Stornoway.

SECTION III

- 99** Two variables, A and B , have a joint Gaussian probability distribution function (pdf) with a negative correlation coefficient. Sketch the form of this function as a contour plot in the AB plane, and use it to distinguish between the most probable joint values of (A, B) and the most probable value of A given (a different) B . [5]

Note that this is question 99 on p.10.

- Explain what is meant by *marginalisation* in Bayesian inference and how it can be interpreted in terms the above plot. [5]

Doppler observations of stars with extrasolar planets give us data on $m \sin i$ of the planet, where m is the planet's mass and i the angle between the normal to the planetary orbit and the line of sight to Earth (i.e. the orbital inclination), which can take a value between 0 and $\pi/2$.

- Assuming that planets can orbit stars in any plane, show that the probability distribution for i is $p(i) = \sin i$. [5]

A paper reports a value for $m \sin i$ of x , subject to a Gaussian error of variance σ^2 . Assuming the mass has a uniform prior, show that the posterior probability distribution for the mass of the planet is

$$p(m|x) \propto \int_0^1 \exp \left[-\frac{(x - m\sqrt{1 - \mu^2})^2}{2\sigma^2} \right] d\mu,$$

- where $\mu = \cos i$. [9]

- Determine the corresponding expression for the posterior pdf of μ , and explain how both are normalised. [6]

[Total: 30]

- 11 Distinguish between frequentist and Bayesian definitions of probability, and explain carefully how parameter estimation is performed in each regime. [10]

Note that this is question 11 on p.11. It's the one after question 99.

A square ccd with $M \times M$ pixels takes a dark frame for calibration purposes, registering a small number of electrons in each pixel from thermal noise. The probability of there being n_i electrons in the i th pixel follows a Poisson distribution, i.e.

$$P(n_i|\lambda) = \exp(-\lambda)\lambda^{n_i}/n_i!,$$

where λ is the same constant for all pixels. Show that the expectation value of is $\langle n_i \rangle = \lambda$. [5]

[You may assume the relation $\sum_0^\infty \frac{x^n}{n!} = \exp(x)$.]

Show similarly that

$$\langle n_i(n_i - 1) \rangle = \lambda^2.$$

and hence, or otherwise, that the variance of n_i is also λ . [5]

The pixels values are summed in columns. Show that these sums, S_j , will be drawn from a parent probability distribution that is approximately

$$p(S_j|\lambda) = \frac{1}{\sqrt{2\pi M\lambda}} \exp \left[-\frac{(S_j - M\lambda)^2}{2M\lambda} \right],$$

clearly stating any theorems you use. [5]

Given the set of M values $\{S_j\}$, and interpreting the above as a Bayesian likelihood, express the posterior probability for λ , justifying any assumptions you make. [5]

[Total: 30]

SECTION IV

- 12 Give the equations of motion for $i = 1, \dots, N$ particles of masses m_i and positions $r_i(t)$ under the action of mutual gravity alone in an arbitrary inertial frame. [4]

Use these to derive the following conservation laws of the system:

- (a) Constancy of linear momentum – i.e., centre of mass fixed in a suitable inertial frame. [4]
- (b) Constancy of angular momentum. [6]

Q 12 continued

(c) Constancy of total energy. [8]

How many integrals of motion exist in total? [2]

Derive the moment of inertia of the system and demonstrate its relevance to criteria for escape of particles from the system. [6]

[Total: 30]

- 13 For a system of N objects, each having mass m_i and position vector \mathbf{R}_i with respect to a fixed co-ordinate system, use the moment of inertia

$$I = \sum_{i=1}^N m_i R_i^2$$

to deduce the virial theorem in the forms

$$\ddot{I} = 4E_k + 2E_G = 2E_k + 2E$$

where E_k and E_G are respectively the total kinetic and gravitational potential energy, and E is the total energy of the system. [8]

Given the inequality

$$\left(\sum_{i=1}^N a_i^2 \right) \left(\sum_{i=1}^N b_i^2 \right) \geq \left(\sum_{i=1}^N \mathbf{a}_i \cdot \mathbf{b}_i \right)^2 + \left(\sum_{i=1}^N \mathbf{a}_i \times \mathbf{b}_i \right)^2$$

for arbitrary vectors $\mathbf{a}_i, \mathbf{b}_i, i = 1, \dots, N$, deduce the following relationship for the N -body system

$$\frac{1}{4} \dot{I}^2 + J^2 \leq 2IE_k,$$

where \mathbf{J} is the total angular momentum of the system. [8]

Assuming the system is isolated, use the virial theorem to deduce further the generalised Sundman inequality

$$\frac{\dot{\sigma}}{\dot{\rho}} \geq 0,$$

in which $\rho^2 = I$ and $\sigma = \rho \dot{\rho}^2 + \frac{J^2}{\rho} - 2\rho E$. [8]

Why does this inequality preclude the possibility of an N -fold collision for a system with finite angular momentum? [6]

[Total: 30]

End of Paper

NOTE: Shout it loud: I'm a geek and I'm proud

NOTE: No correct MCQ answer provided in question 7

NOTE: Too many correct MCQ answers provided in question 8

NOTE: Too few potential answers in MCQ 9