University of Glasgow

Probing the Wider Spectrum of Gravitational Waves

Dr Martin Hendry

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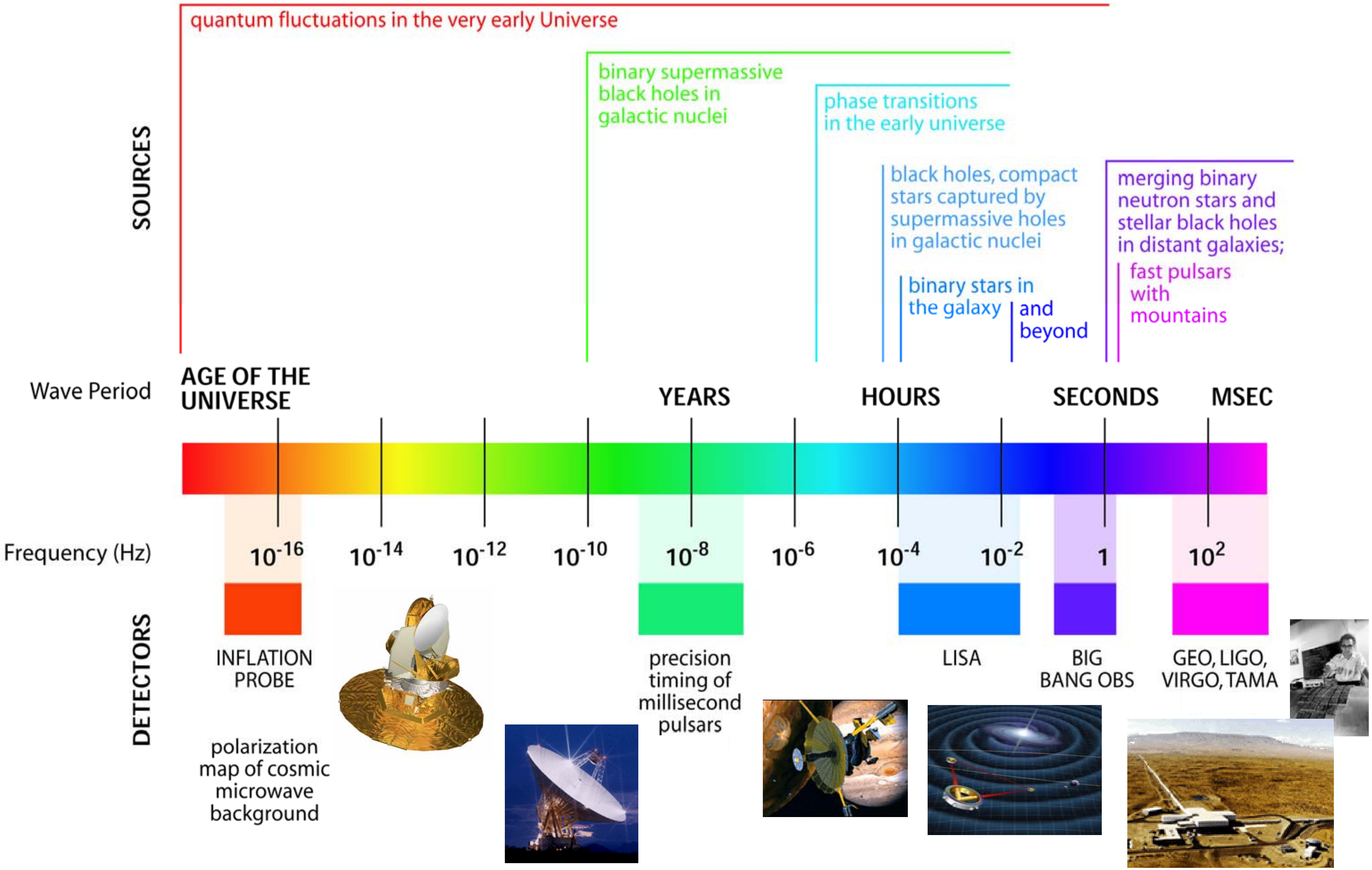
Dept of Physics and Astronomy, University of Glasgow

THE GRAVITATIONAL WAVE SPECTRUM



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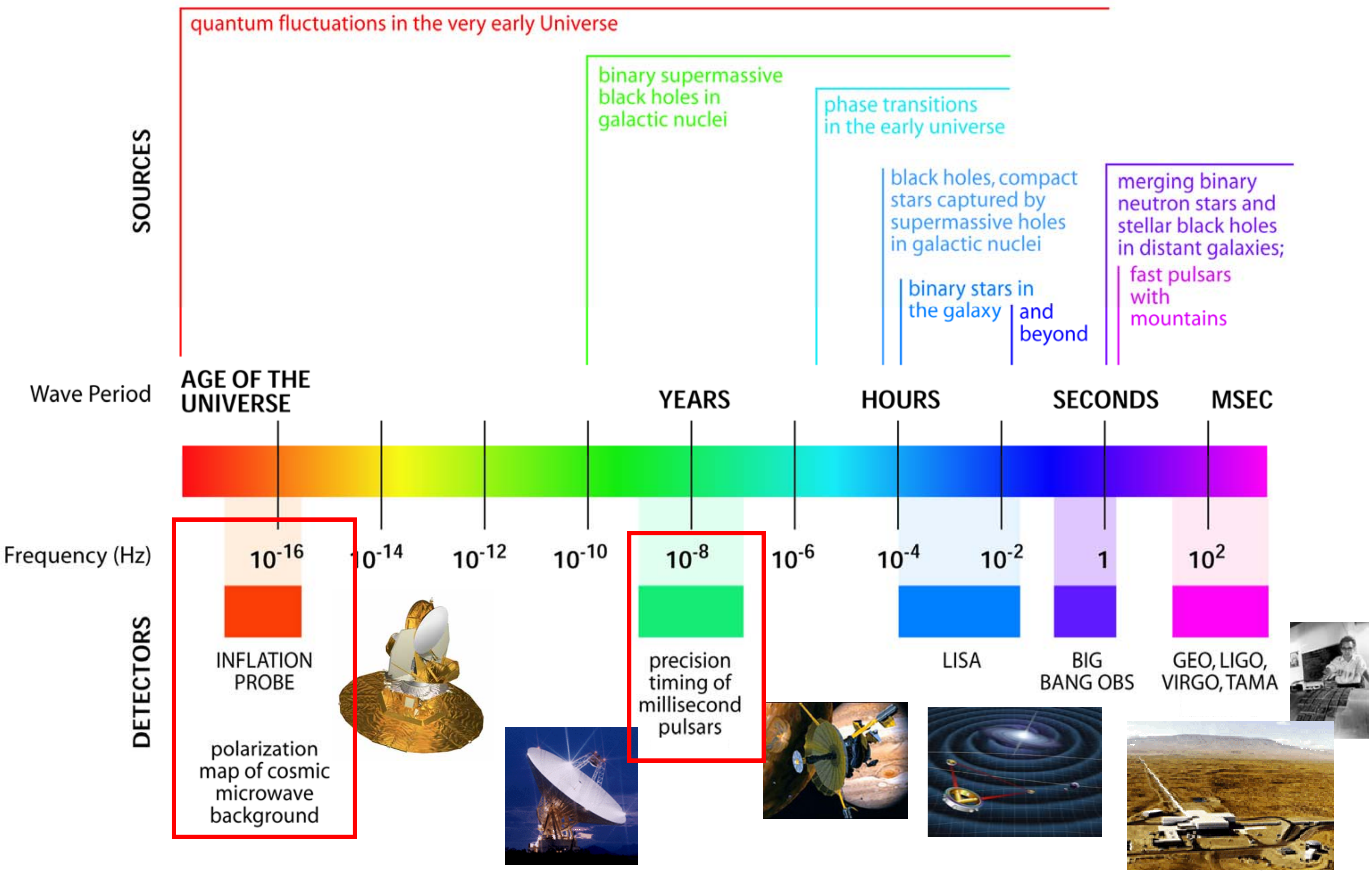

Outline of talk

- A Stochastic Background of GWs – basic concepts
- Astrophysical and primordial sources
- Probing the SB with pulsar timing arrays: methods
- Current and future GW limits with PTAs
- Probing GWs with the CMBR: methods
- Current and future GW limits with the CMBR

Astrophysical sources of GWs

..for ground-based detectors (50Hz and up):

- Pulsed: 'Chirps' and pulse-like signals
  - Compact binary coalescences (NS/NS, NS/BH, BH/BH)
  - Stellar collapse (asymmetric) to NS or BH, GRBs?
- Continuous Waves: These appear as (temporally coherent) sinusoidal signals with fixed polarisation
  - Pulsars – i.e. non-spherical neutron stars
  - Low mass X-ray binaries (e.g. SCO X1)
  - Modes and instabilities of neutron stars (?)
- Stochastic Background of GWs

Stochastic Background: Basic Concepts

We will follow closely the notation and approach adopted in Allen gr-qc/9604033 (A96; excellent reference source!)

<u>What do we mean by a stochastic background?</u>

"Random" superposition of a large number of unresolved, independent, uncorrelated events.

Later in the school you will learn about how we study the Stochastic Background of GWs using interferometric detectors.

In this lecture we consider constraints on the Stochastic Background from across the wider GW spectrum: Pulsar Timing and the CMBR.

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In this lecture we consider constraints on the Stochastic Background from across the wider GW spectrum: Pulsar Timing and the CMBR.

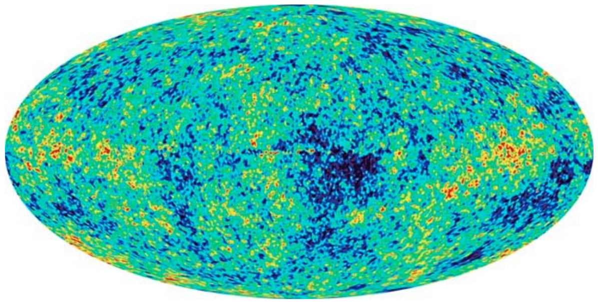
We will also introduce several important data analysis concepts.

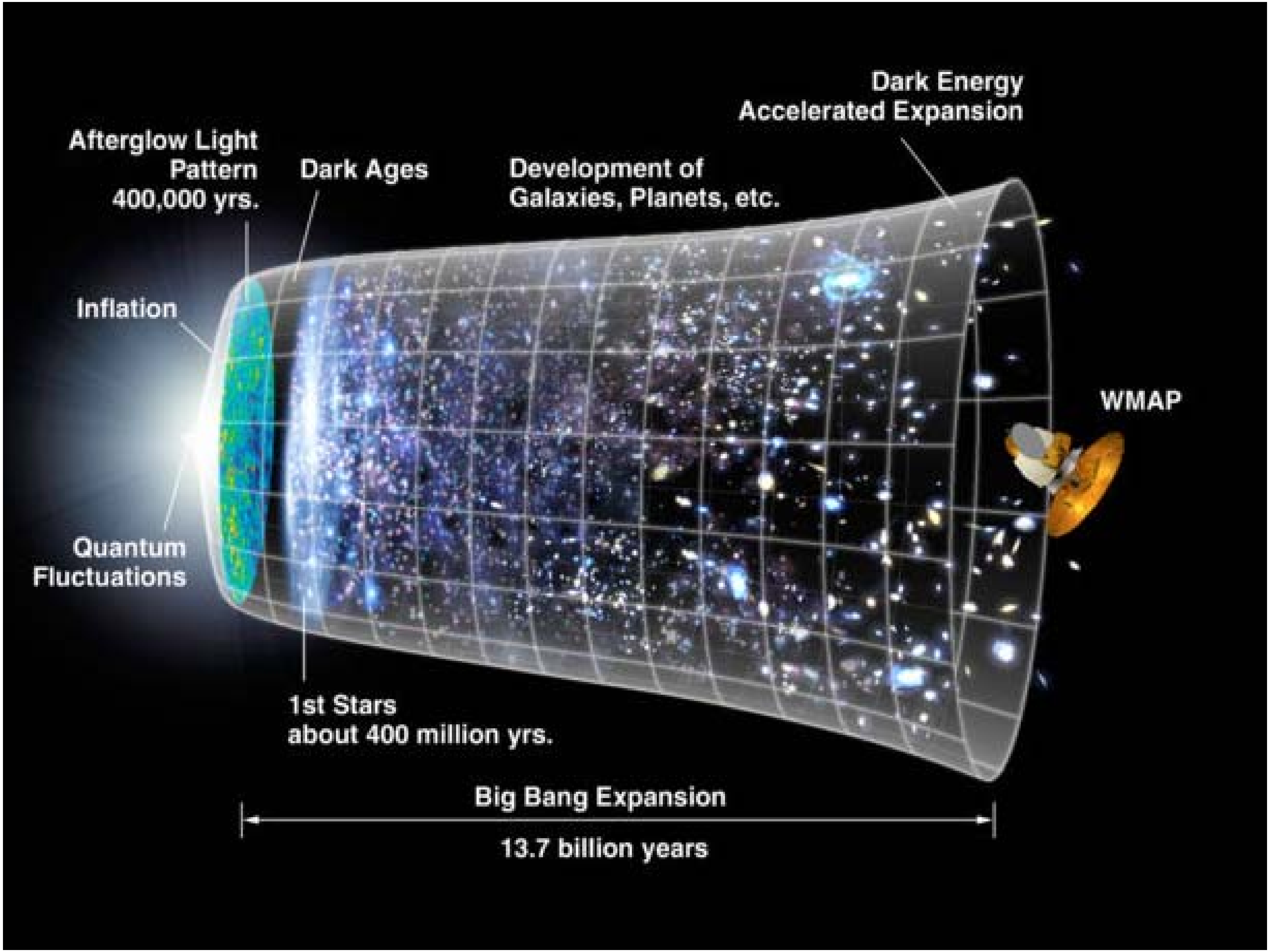
Stochastic Background: Basic Concepts

Origin of the SB?

1) Primordial – i.e. the very early Universe

(c.f. the CMBR)




Dark Energy
Accelerated Expansion
Afterglow Light
Pattern
400,000 yrs.
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
WMAP
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.7 billion years

Stochastic Background: Basic Concepts

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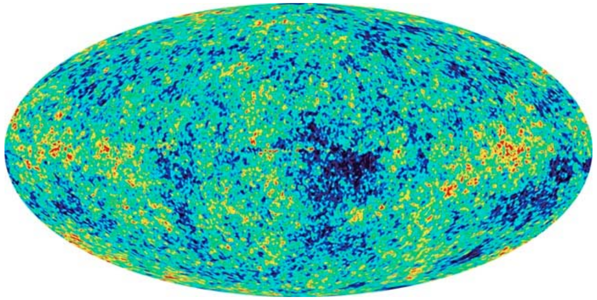
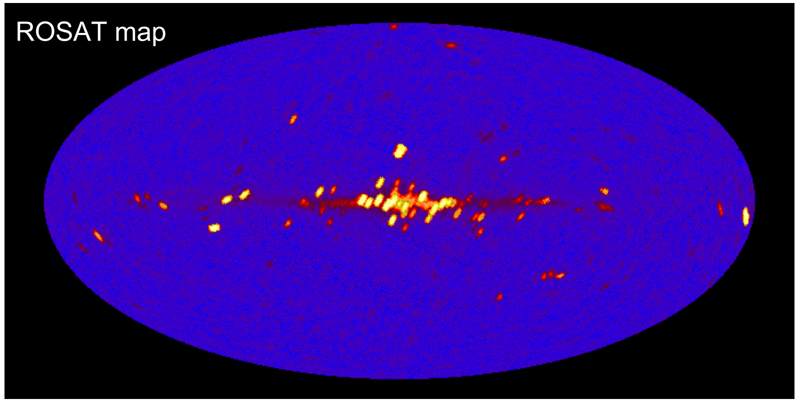
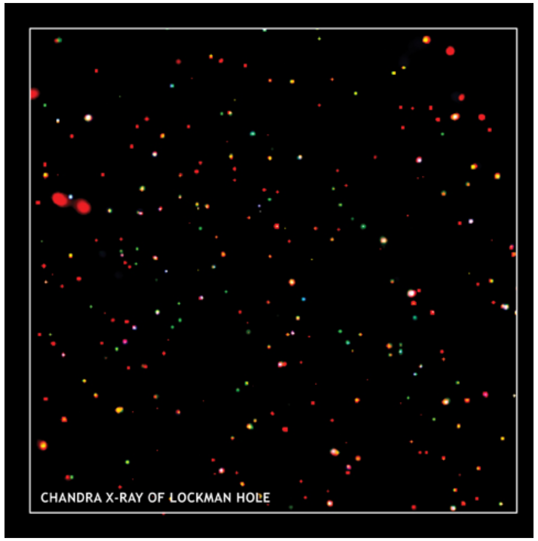


2) Recent – i.e. within a few billion years

(c.f. the diffuse X-ray background)



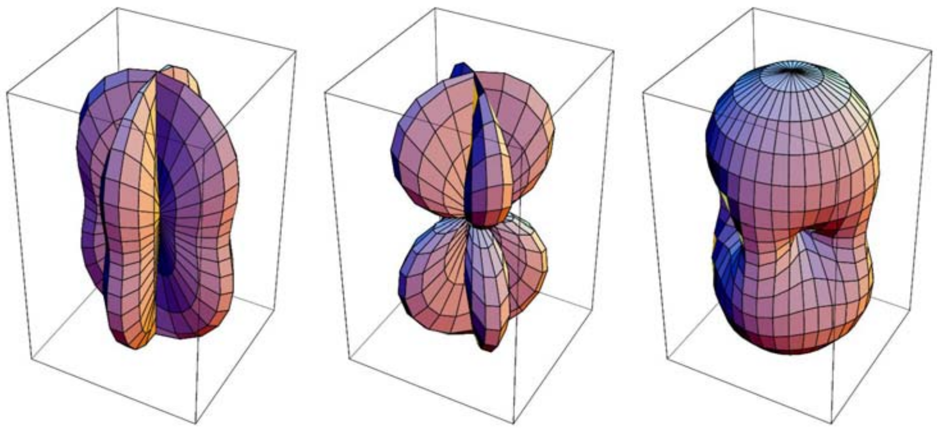


Stochastic Background: Basic Concepts

Unresolved?

In e.g. optical astronomy we can resolve a source if the angular resolution of our telescope is smaller than the angular size of the source.

In GW astronomy the antenna patterns are essentially 'all sky'. Instantaneously *any* source is unresolved.



For isolated GW sources we can 'triangulate' their sky position, but not when the SB consists of many sources distributed over the sky.

Stochastic Background: Basic Concepts

Some notation / assumptions / definitions

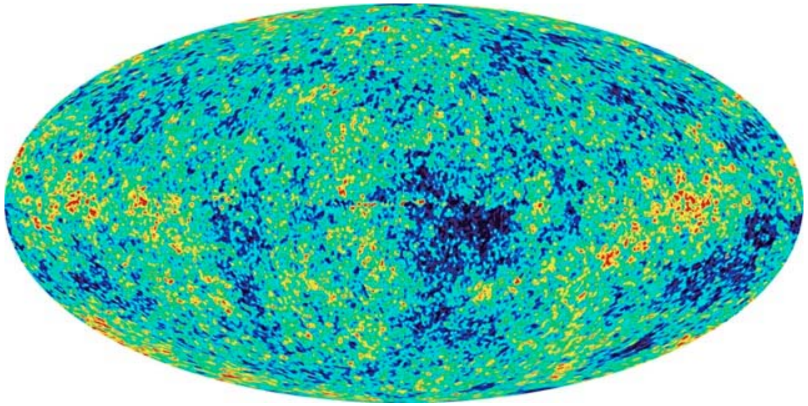
It is customary to assume that the SB is:

1) Isotropic — again, compare the CMBR…

2) Stationary — statistical properties of the GW fields do not depend on our origin of time, but only on time *differences*.

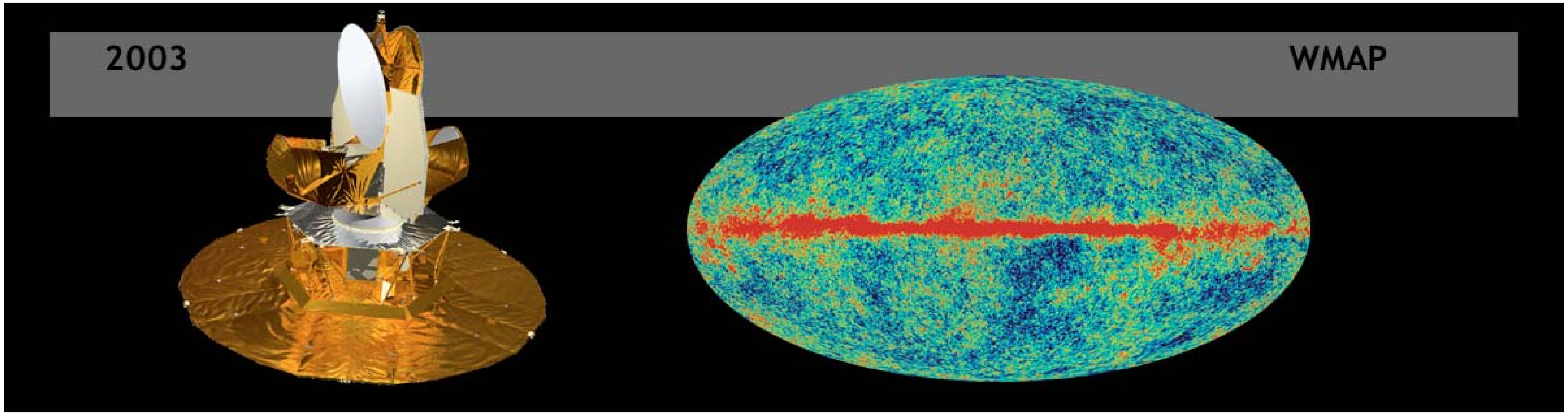
3) Gaussian — superposition of many sources + Central Limit Theorem.

Stochastic Background: Basic Concepts



The CMBR shows remarkable isotropy: temperature fluctuations are of order 10^{-5} K…

…but actually what WMAP saw was a very anisotropic distribution, contaminated by a Galactic foreground.

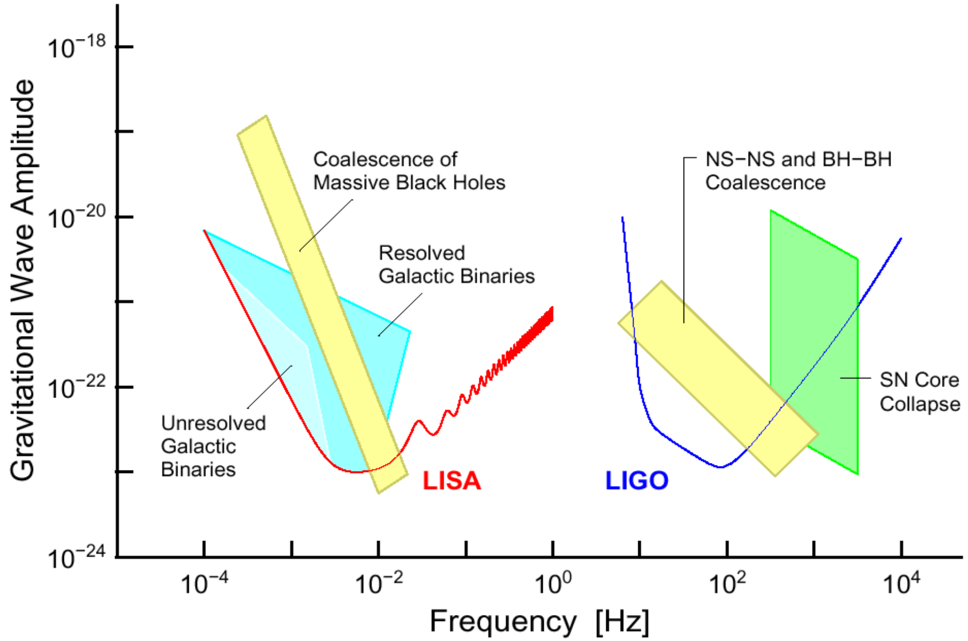


Stochastic Background: Basic Concepts

In a similar way the SB *could* be anistropic if it were dominated e.g. by unresolved Galactic sources…

LISA should see such a ‘foreground’ of WD-WD binaries.

It is harder to envisage an anisotropic SB of primordial origin, given the *isotropy* of the CMBR. (See later).



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Stochastic Background: Basic Concepts

We characterise the SB by its **spectrum**.

Following A96:

$$\Omega(f) = \Omega_{\rm gw}(f) \equiv \frac{1}{\rho_{\rm critical}} \frac{d\rho_{\rm gw}}{d\ln f}$$

Completely characterises SB if it is isotropic, stationary and Gaussian

where $\rho_{\rm critical} = \frac{3c^2 H_0^2}{8\pi G}$ and $H_0 = h_{100}\, 100 \, \frac{\rm Km}{\rm sec - Mpc}$

Energy density corresponding to a 'flat' Universe containing only matter

Present-day value of the Hubble parameter

We can relate $\Omega(f)$ to a characteristic 'chirp' amplitude (see e.g. A96):

$$h_{\rm c}(f) = 3 \times 10^{-20} h_{100} \sqrt{\Omega(f)} \frac{100\,{\rm Hz}}{f}$$

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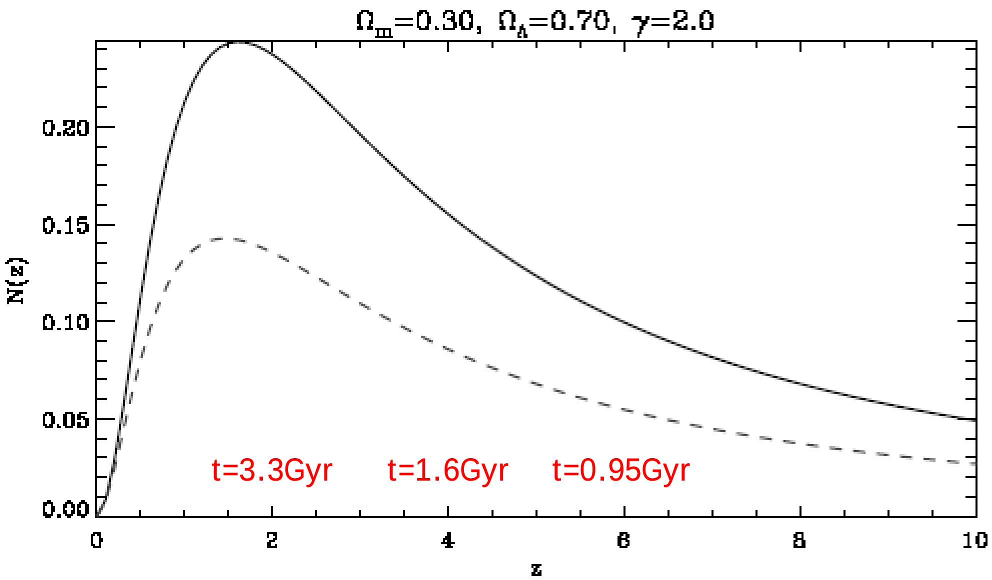
Origin of the Stochastic Background

1) Astrophysical.

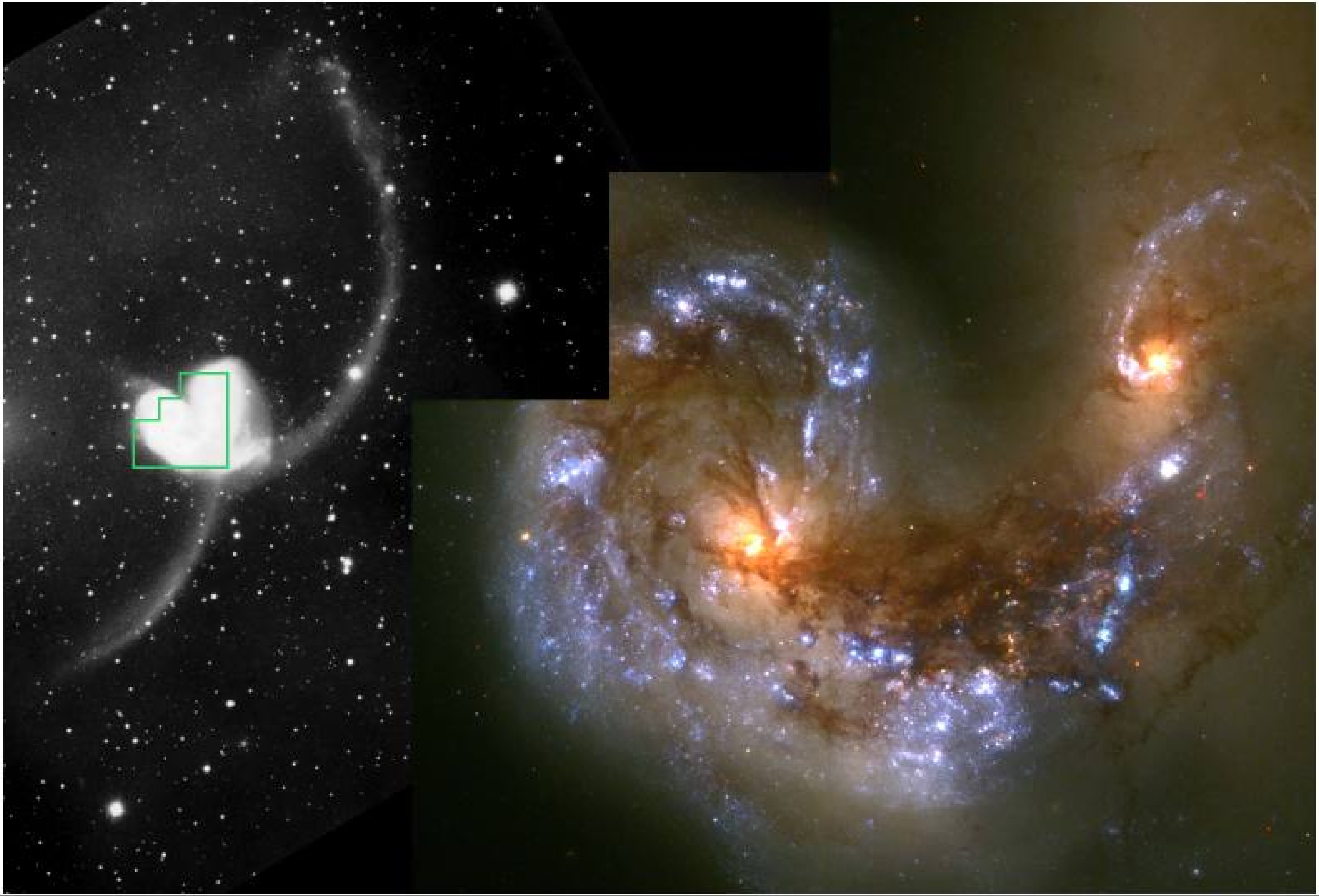
Population of 'nearby' sources of GWs – e.g. coalescing NS-NS, SMBH binaries in galaxies.

From Jaffe & Backer (2003). Predicted number of SMBH mergers as a function of redshift, for different merger models in a given cosmology.



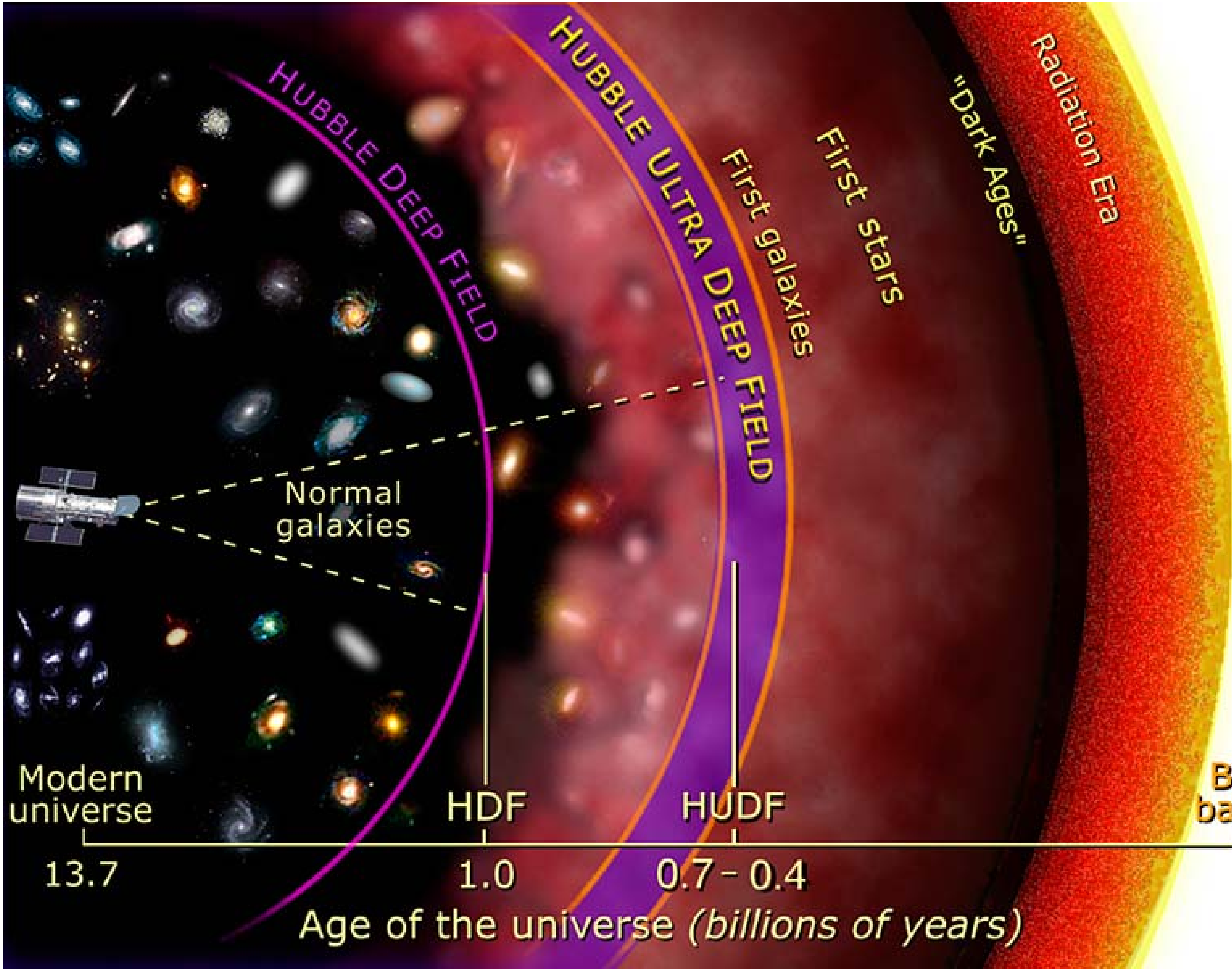
A long time ago,

in a galaxy far, far away…

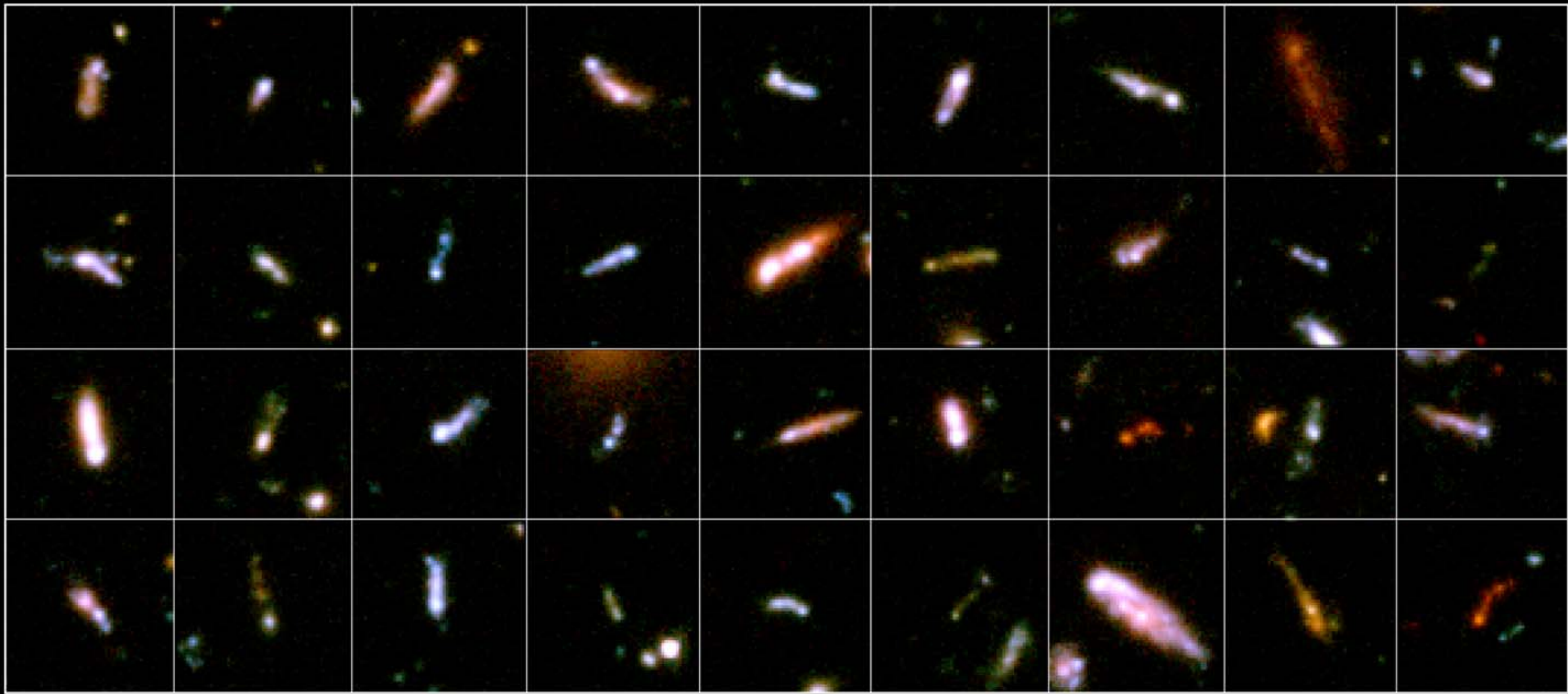


Colliding Galaxies NGC 4038 and NGC 4039 HST • WFPC2

PRC97-34a • ST ScI OPO • October 21, 1997 • B, Whitmore (ST ScI) and NASA



HUBBLE DEEP FIELD
HUBBLE ULTRA DEEP FIELD
First galaxies
First stars
"Dark Ages"
Radiation Era
Normal galaxies
Modern universe
HDF
HUDF
Big bang
13.7
1.0
0.7 – 0.4
Age of the universe (billions of years)



“Tadpole” Galaxies in the Hubble Ultra Deep Field
Hubble Space Telescope ▪ ACS/WFC

NASA, ESA, A. Straughn, S. Cohen and R. Windhorst (Arizona State University), and the HUDF team (STScI) STScI-PRC06-04

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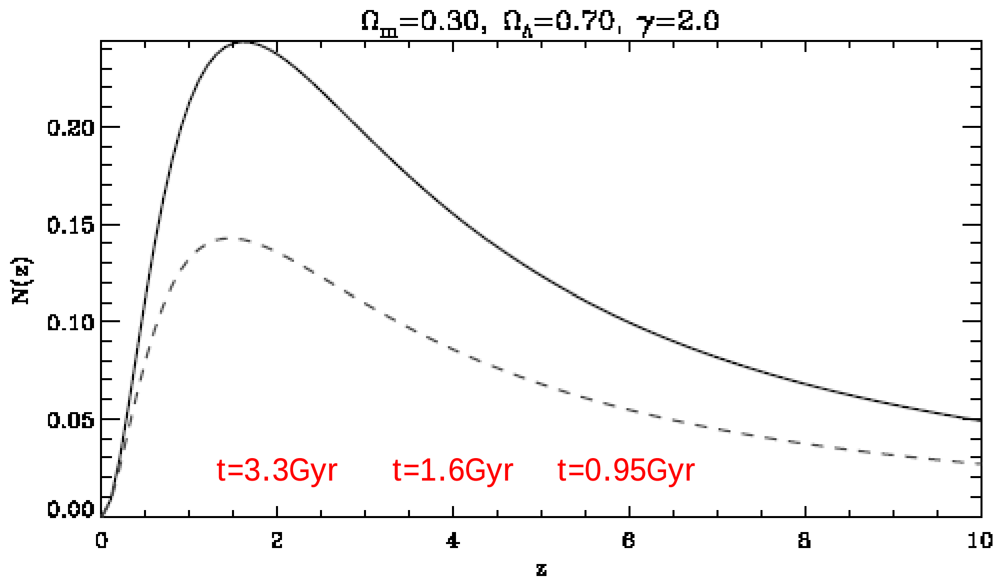


Subsequent lectures will consider in more detail the constraints on the number and event rate of GW sources

Origin of the Stochastic Background

2) Primordial.

GWs generated by processes in the very early Universe.

Three illustrative examples (but perhaps other exotic possibilities?...)

- Network of cosmic strings
- Early-universe phase transitions
- Inflation

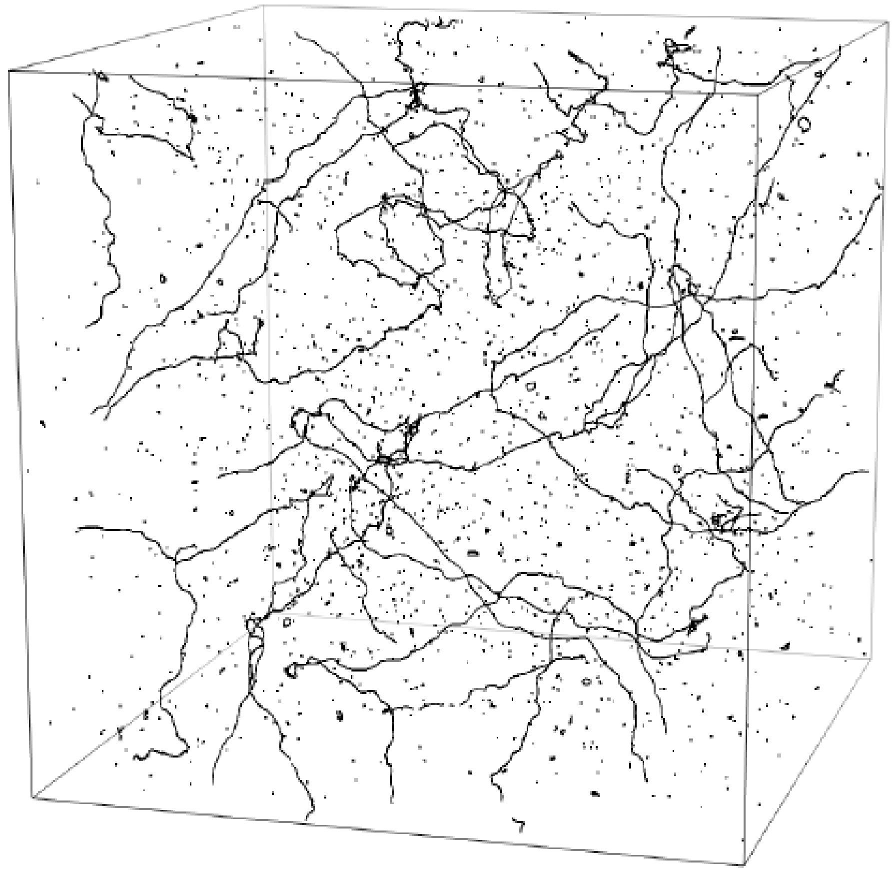
Origin of the Stochastic Background

<u>Cosmic strings</u>

- Proposed as 'seeds' of large scale structure in the Universe.
- 1-d topological defects (analogous to phase transitions in crystals).
- Very high tension $\Rightarrow$ oscillate relativistically, radiating GWs and shrinking in size.
- e.g. GUT scale: $\mu = 10^{23}\,\mathrm{kg\,m^{-1}}$

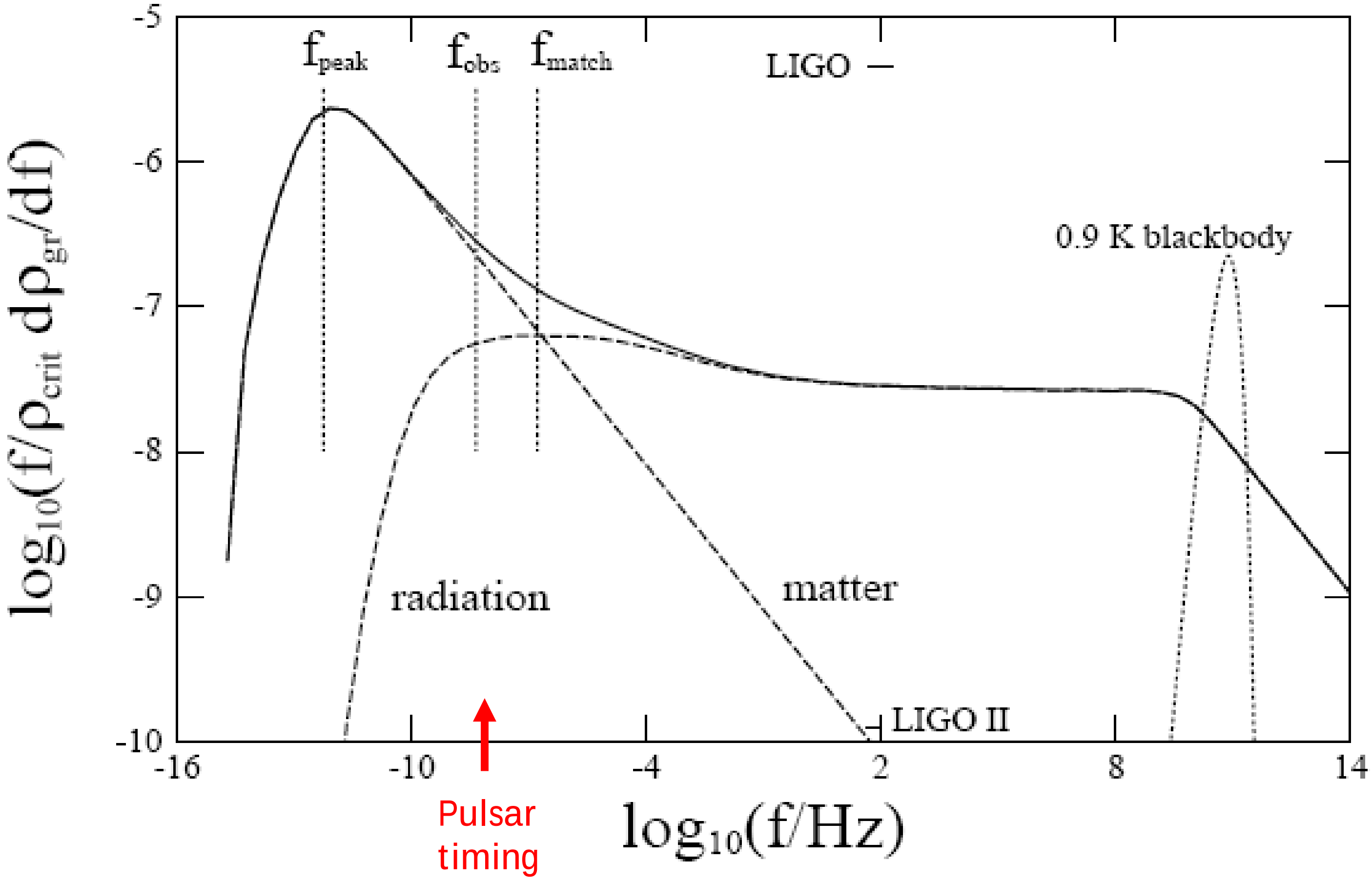
  Mass per unit length

- Predicted to have flat spectrum, across a wide frequency range.



Origin of the Stochastic Background

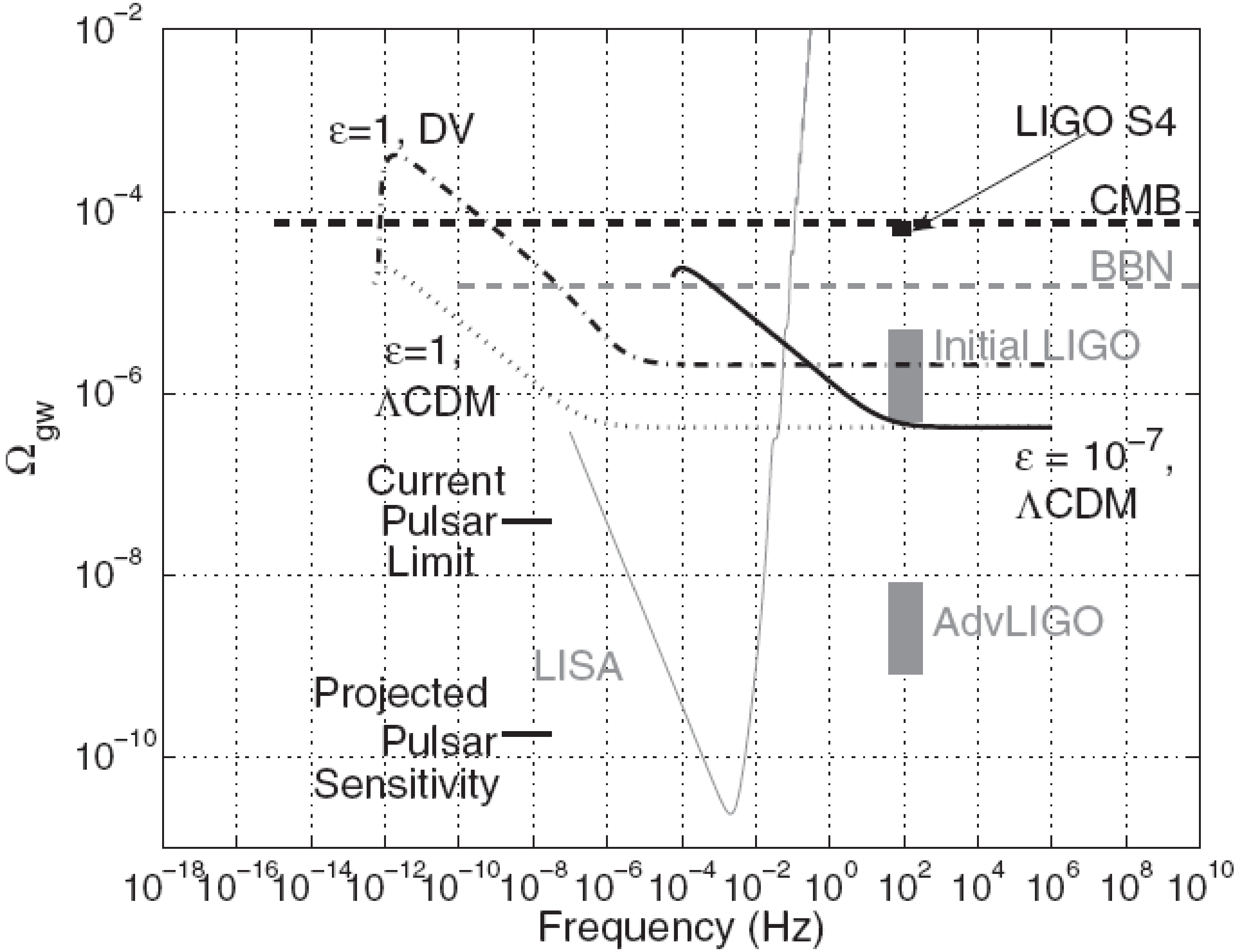
Predictions from A96



Origin of the Stochastic Background

Siemens et al (2007)

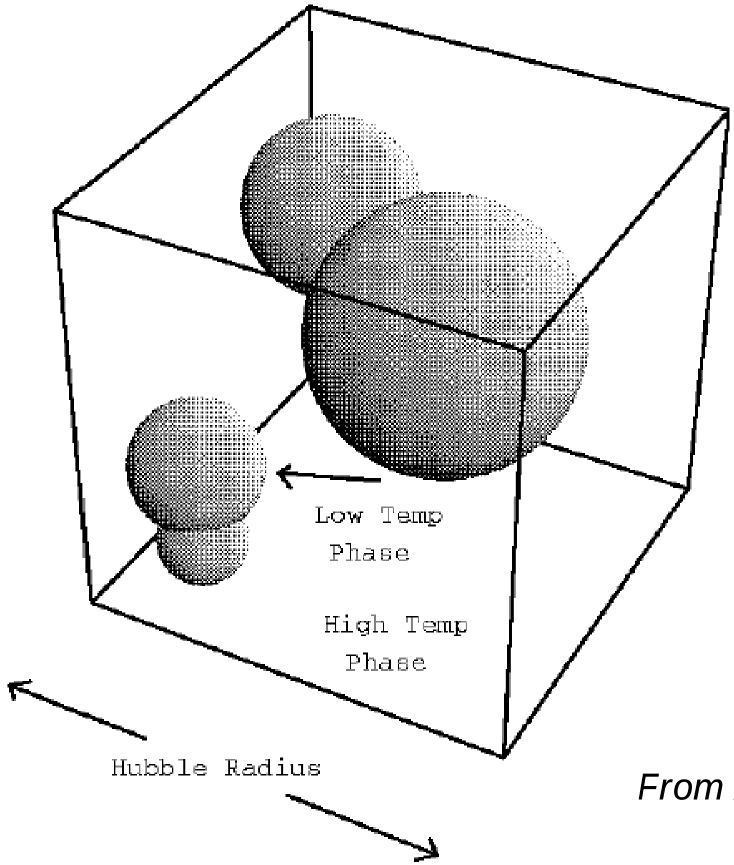
Depending on the String loop length (parameter ε), cosmic strings *could* be an interesting target for Advanced LIGO, LISA and Pulsar Timing Arrays.



Origin of the Stochastic Background

Phase transitions

- As the Universe expands and cools, a first-order PT takes place in some regions.
- Bubbles of new (low-energy) phase created – expand rapidly and convert ΔE into K.E. of the walls.
- Wall collisions → GWs



From A96

Origin of the Stochastic Background

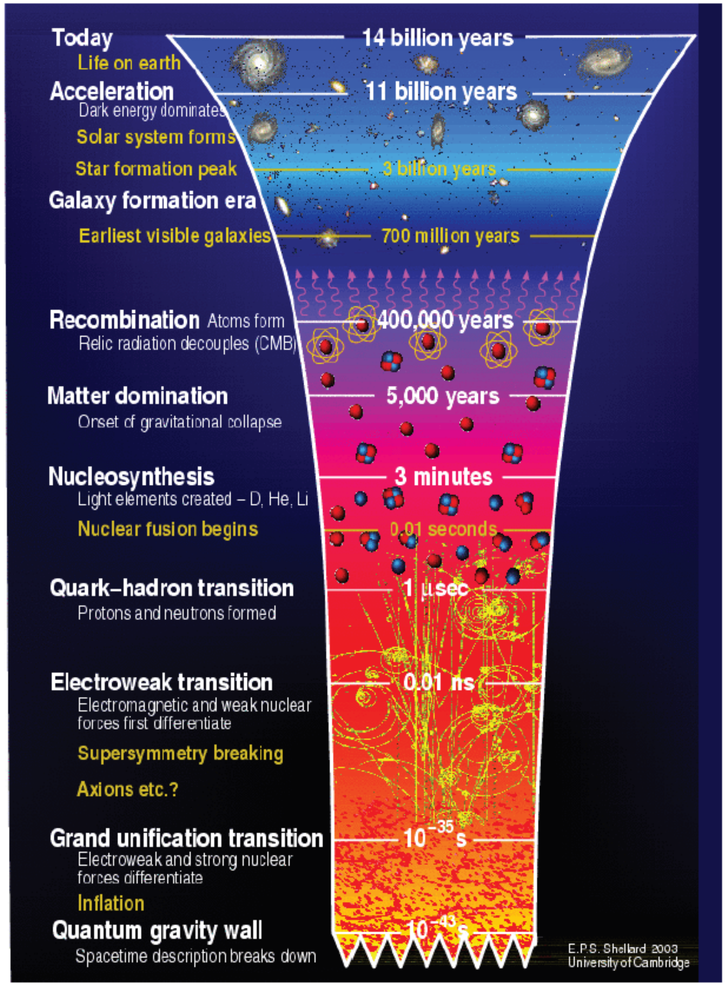
Phase transitions

- Predicted spectrum peaks at frequency characteristic of expansion rate when bubbles collided.
- When might this happen?

  Electro-weak PT:

  $$f_{\mathrm{max}} \approx 4.1 \times 10^{-3}\ \mathrm{Hz}$$

  Lots of recent (and older) literature predicting the spectrum, and how it depends on PT parameters.



Origin of the Stochastic Background

Example: Grojean and Servant, 2006

Peak could be a target for LISA and ground-based intrferometers.

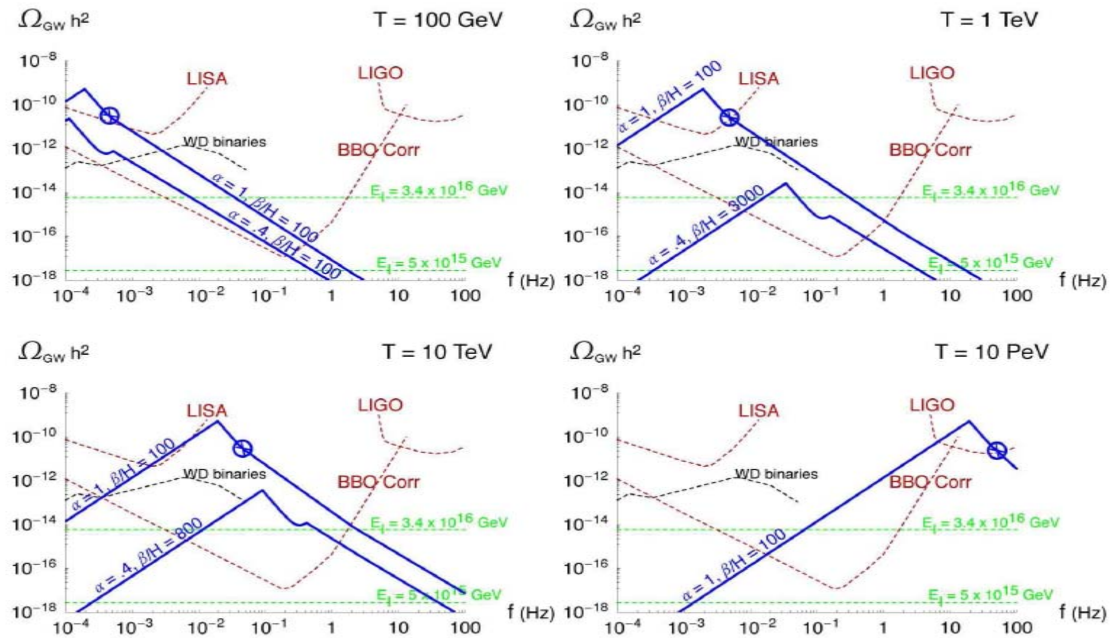
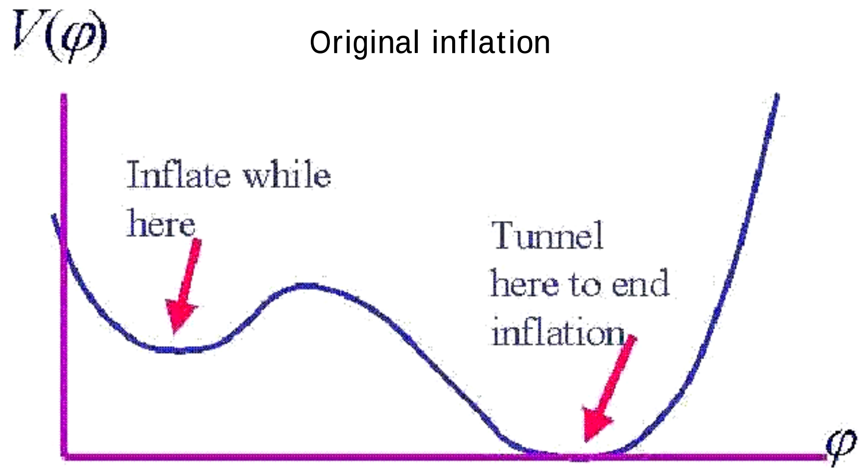


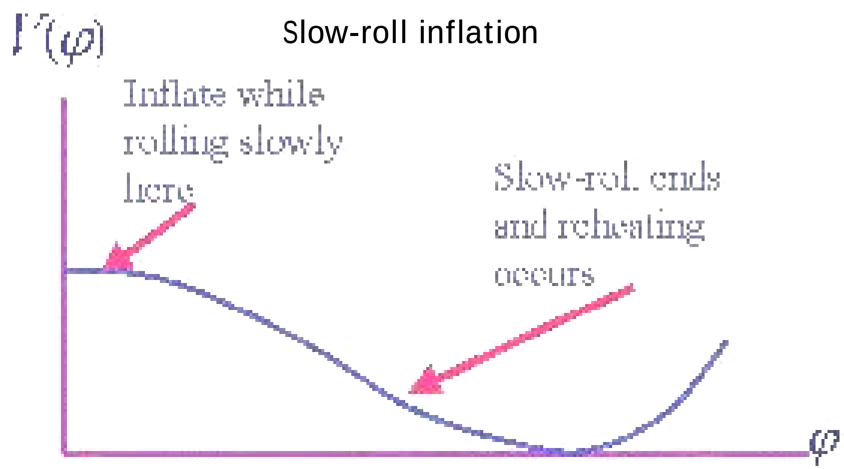
Figure 1: Spectrum of gravitational waves expected from a first order phase transition (solid blue line) for four temperatures and for some choices of $(\alpha, \beta/H)$ values. The dashed red lines are the (approximate) predicted sensitivities of LISA, BBO, LIGO-III. The horizontal dashed green lines are the gravitational spectra expected from inflation, for two scales of inflation, for comparison. The black dashed curve is the estimate for the irreducible foreground due to white dwarf binaries (from [17]). At large α, only the peak from turbulence can be seen as well as a change of slope (shown as a circled cross) corresponding to the high frequency tail of the bubble collision spectrum. For low α, it is possible to see the collision peak as well.

Origin of the Stochastic Background

Cosmological Inflation

- Period of accelerated expansion in the very early Universe.
- First proposed as a mechanism to explain several 'strange' observed characteristics of the Universe today (see more later).
- Basic idea: as Universe cooled it became trapped in a false vacuum state – acquired negative pressure which drove exponential expansion.
- Original model had problems with reheating. Later solved by 'slow roll' of potential.

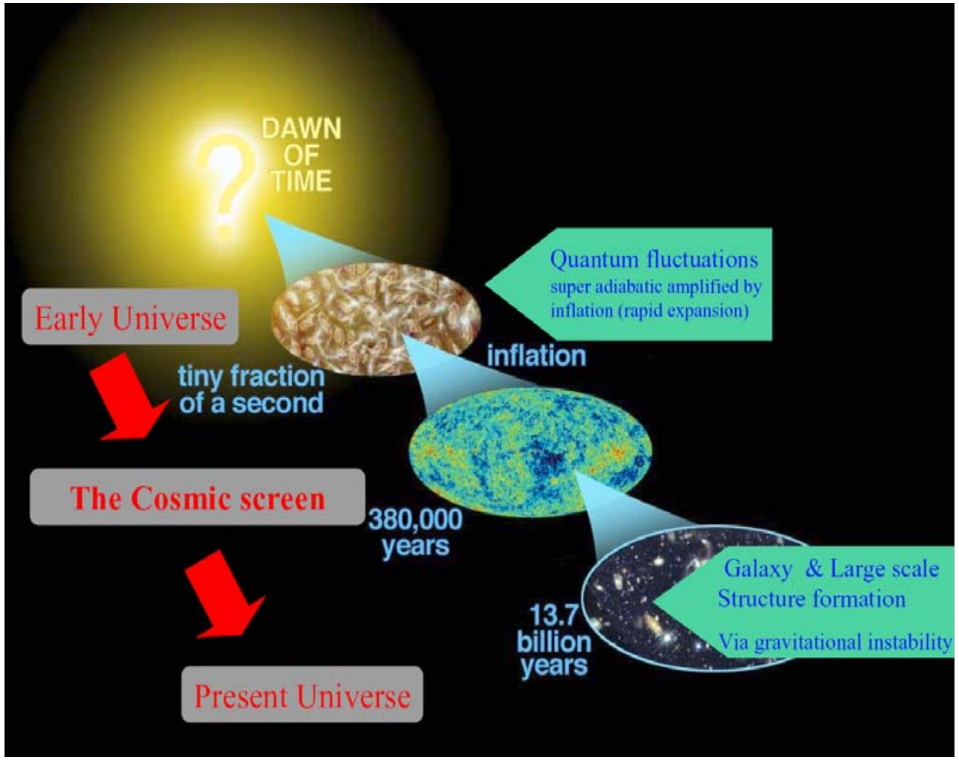




Origin of the Stochastic Background

Cosmological Inflation

- Inflation also provides a mechanism for generating large scale structure in the Universe.
- Primordial quantum fluctuations become the ‘seeds’ of structure that we see in the CMBR.
- These fluctuations are both **scalar** (density perturbations) and **tensor** (gravitational waves).
- We can hope to measure the latter directly, and by the imprint they leave on the temperature distribution of the CMBR (see later).



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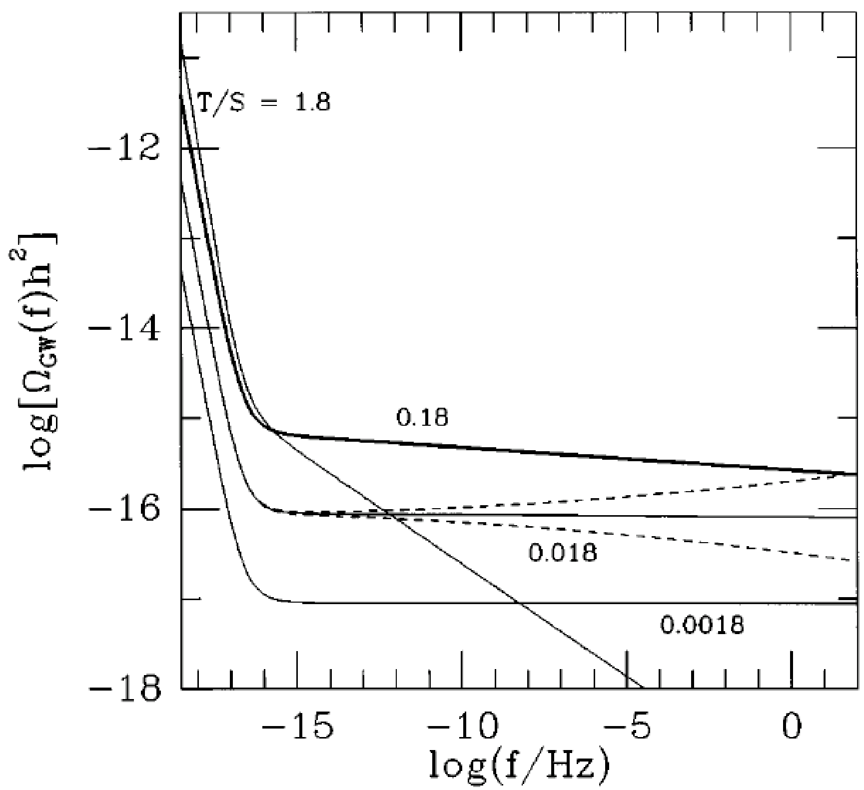
Turner (1997)



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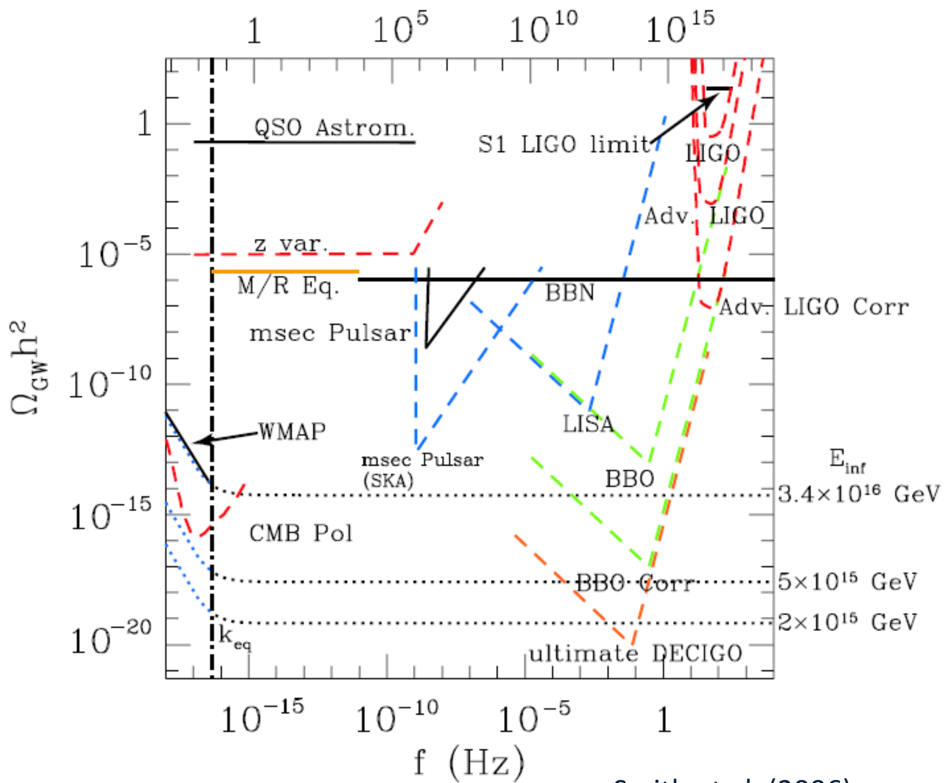

Smith et al. (2006)

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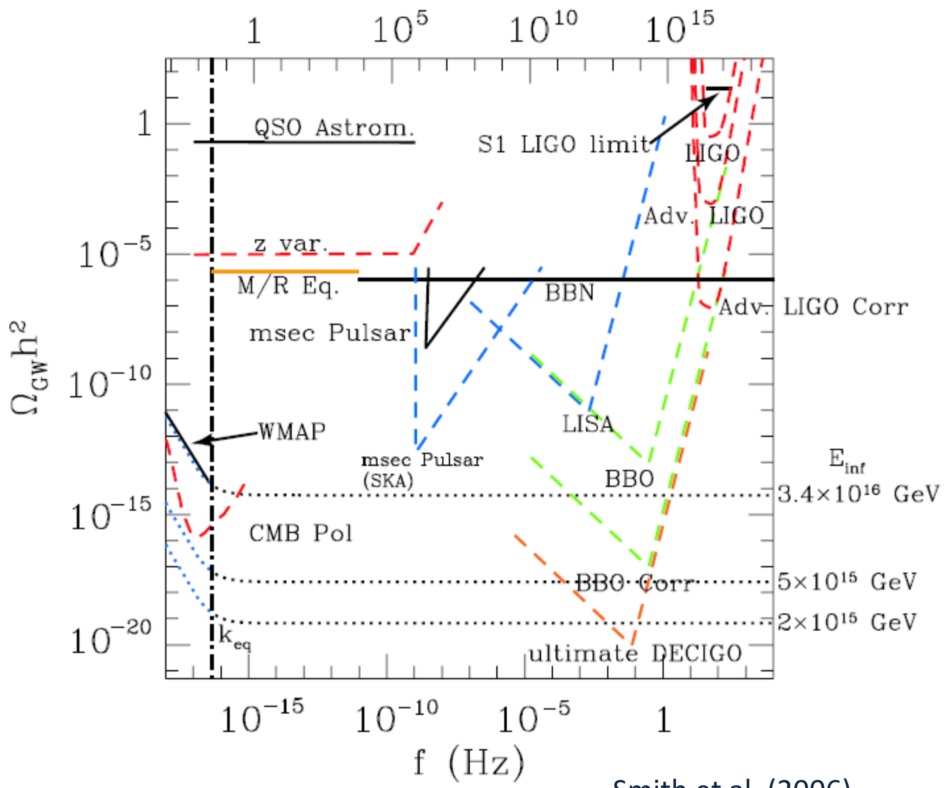

Smith et al. (2006)

So we have some plausible candidates for SB sources

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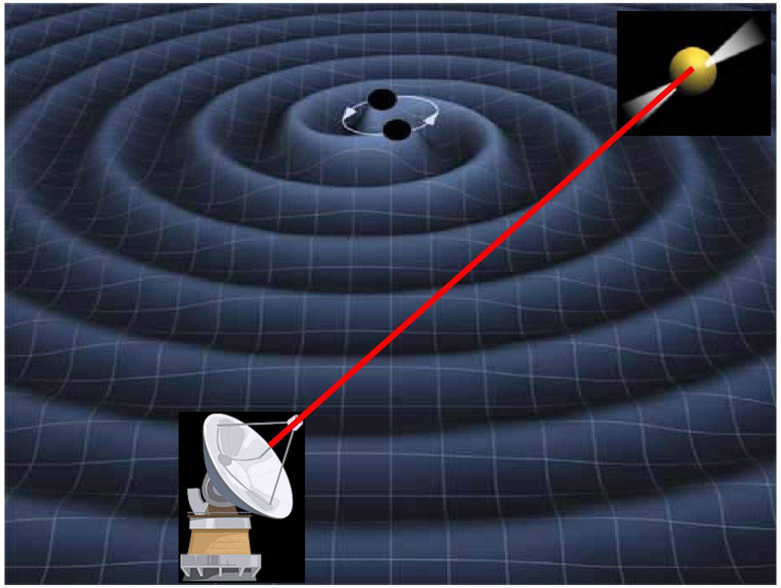
Pulsar timing arrays as a probe of GWs

- Gravitational waves distort spacetime as they propagate.
- A periodic gravitational wave passing across the line of sight to a pulsar will produce a periodic variation in the time of arrival (TOA) of pulses.

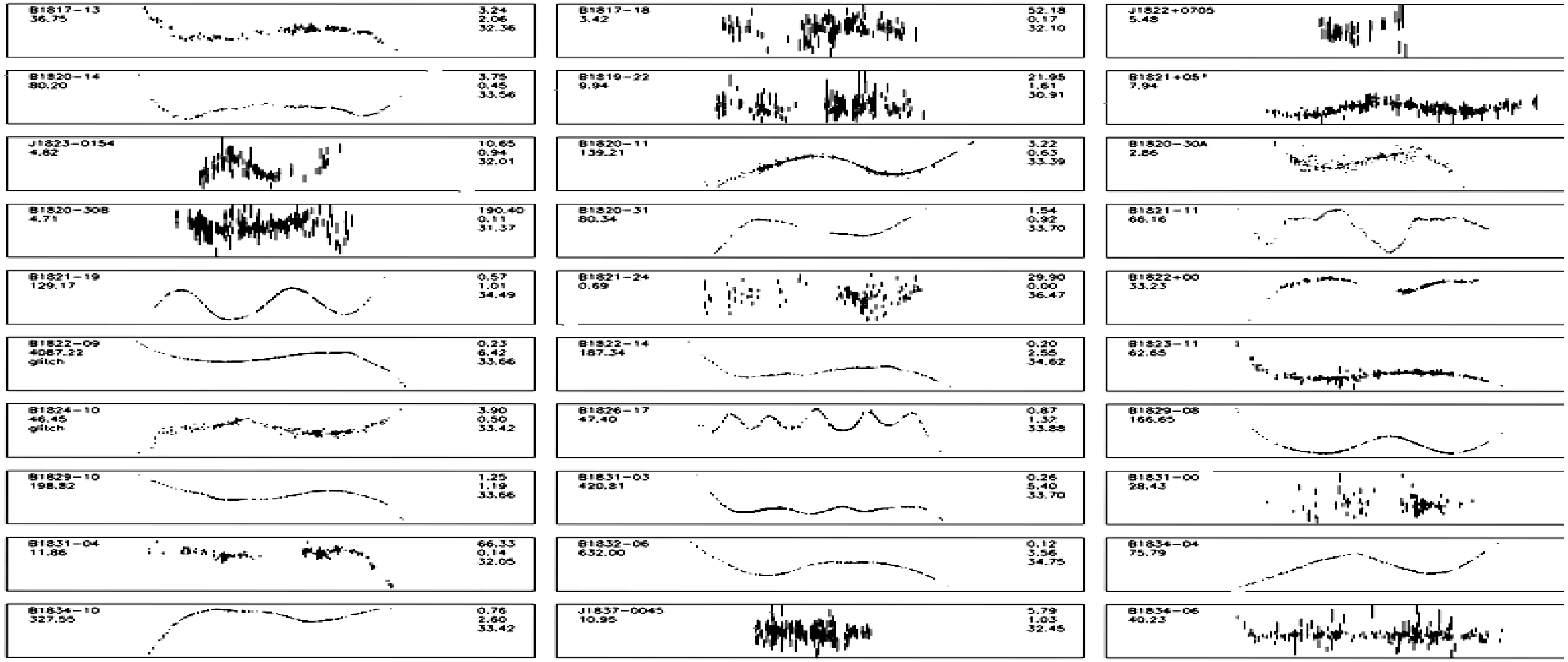


If the strain along the line-of-sight is *h*, then the fractional change in the pulse arrival rate due to the gravitational wave just depends on the strain at emission and reception.

$$\frac{\delta\nu}{\nu} = -\mathcal{H}^{ij}(h_{ij}(t_e, x_e^i) - h_{ij}(t_e - d, x_p^i))$$

Pulsar timing arrays as a probe of GWs

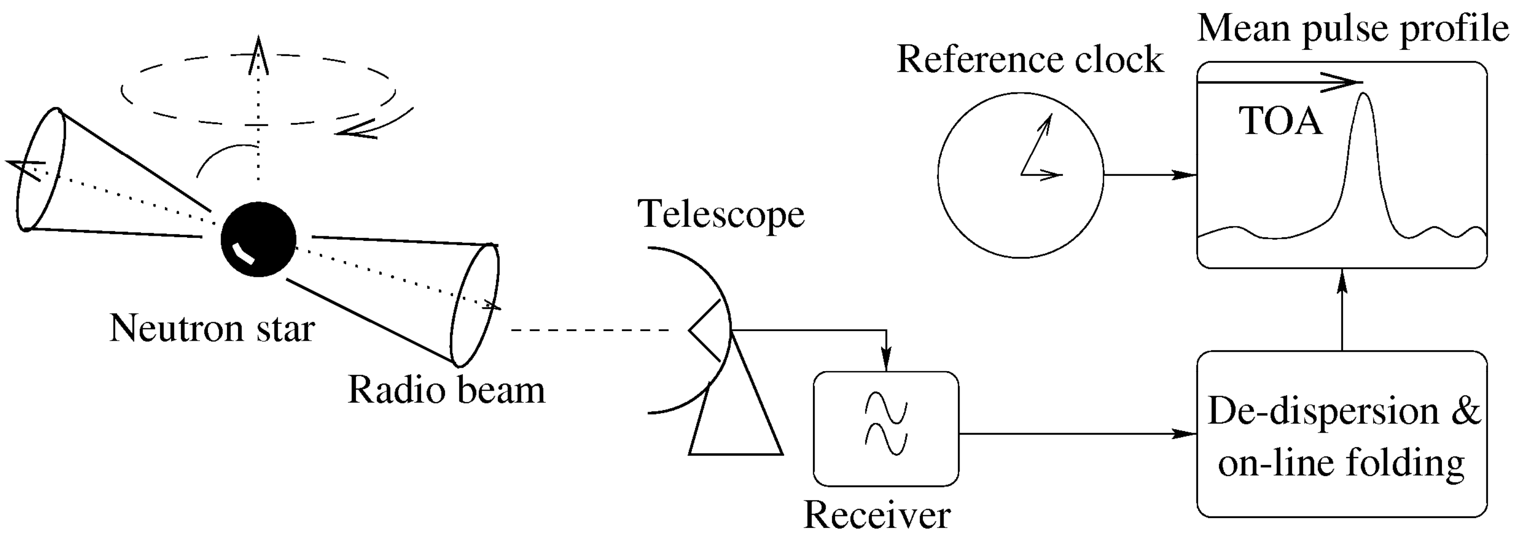
At some level all pulsars show timing noise, some of which may be the result of interaction with gravitational waves along the propagation path.



Timing residuals (i.e., the difference between observed and expected pulse arrival times) for a selection of pulsars over several years -- George Hobbs

Pulsar timing arrays as a probe of GWs

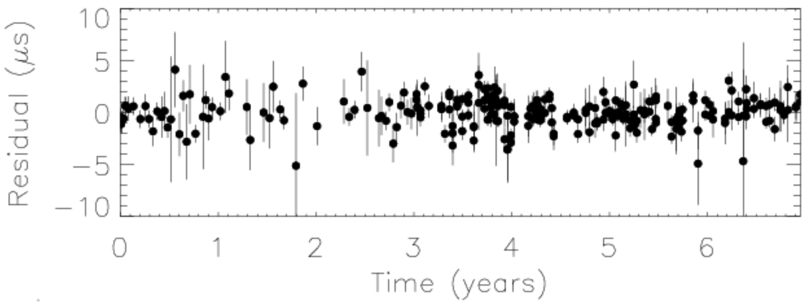
- TOA is determined by matching the pulse profile to a **template** - the best available representation of the pulsar's profile
- Template may be a high signal-to-noise profile, or a fit to noisier data composed of a sum of Gaussian components



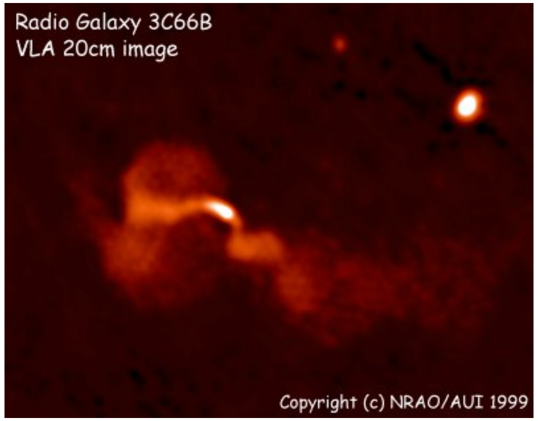
Duncan Lorimer and Michael Kramer, Handbook of Pulsar Astronomy

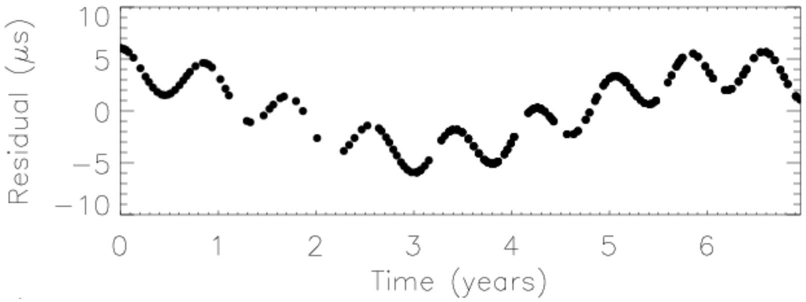
Pulsar timing arrays as a probe of GWs

- Correlating data from an array of pulsars, we can hope to disentangle the signal from background source(s) – either a SB or individually.
- See e.g. seminal work by Jenet et al. (2004, 2005, 2006)



Observed timing residuals for PSR B1855+09.





Simulated timing residuals induced from a putative black hole binary in 3C66B. (Jenet et al. 2004)

Pulsar timing arrays as a probe of GWs

- Parkes Pulsar Timing Array (PPTA)
  - Data from Parkes 64 m radio telescope in Australia
  - High-quality (rms residual < 2.5 μs) data for 20 millisecond-pulsars
- North American NanoHertz Observatory for Gravitational waves (NANOGrav)
  - Data from Arecibo and Green Bank Telescope
  - High-quality data for 17 millisecond pulsars
- European Pulsar Timing Array (EPTA)
  - Radio telescopes at Westerbork, Effelsberg, Nancay, Jodrell Bank, (Cagliari)
  - Normally used separately, but can be combined for more sensitivity
  - High-quality data for 9 millisecond pulsars

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So how does it work?…

Probing the SB with PTAs

The measured pulsar timing residuals contain:

- deceleration of the pulsar spin
- imperfect knowledge of the pulsar's sky position
- ephemeris variations due to the planets
- equipment change 'jumps'

Deterministic

- receiver noise
- clock noise
- changes in the ISM refractive index
- intrinsic timing noise
- GW background

Stochastic

Probing the SB with PTAs

The measured pulsar timing residuals contain:

Deterministic:

- **deceleration of the pulsar spin**
- imperfect knowledge of the pulsar's sky position
- ephemeris variations due to the planets
- equipment change 'jumps'

Stochastic:

- receiver noise
- clock noise
- changes in the ISM refractive index
- **intrinsic timing noise**
- **GW background**

How do we extract information on the GWB from pulsar data?

van Haasteren (2009) provides an elegant, *Bayesian*, formulation

van Haasteren (2009) Formalism

Model for the i^{th} timing residual (TR) of the a^{th} observed pulsar:

$$\delta t_{ai} = \delta t_{ai}^{\text{GW}} + \delta t_{ai}^{\text{PN}} + Q(t_{ai})$$

GW background

Pulsar timing noise

Quadratic model for the pulsar spin-down

$$Q_a(t_{ai}) = A_{a1} + A_{a2} t_{ai} + A_{a3} t_{ai}^2$$

Assume that the GW background and pulsar timing noise are **Gaussian random processes**, each with mean zero $\Rightarrow$ they can be described by an $(n \times n)$ "coherence" (covariance) matrix.

$$\left\langle \delta t_{ai}^{\text{GW}} \delta t_{bj}^{\text{GW}} \right\rangle = \mathbf{C}_{(ai)(bj)}^{\text{GW}} \qquad \left\langle \delta t_{ai}^{\text{PN}} \delta t_{bj}^{\text{PN}} \right\rangle = \mathbf{C}_{(ai)(bj)}^{\text{PN}}$$

van Haasteren (2009) Formalism

We expect that the GWB and Pulsar timing noise will be uncorrelated, so the covariance matrices add together, to give a total covariance matrix:

$$\mathbf{C}_{(ai)(bj)} = \mathbf{C}^{\mathrm{GW}}_{(ai)(bj)} + \mathbf{C}^{\mathrm{PN}}_{(ai)(bj)}$$

Our model for the stochastic part of the TRs is, then, a **multivariate Gaussian** probability distribution:

$$P(\delta \boldsymbol{t}) = \frac{1}{\sqrt{(2\pi)^n \det \mathbf{C}}} \exp \left\{ -\frac{1}{2} \sum_{(ai)(bj)} [\delta \boldsymbol{t}_{(ai)} - Q_a(t_{ai})] \, \mathbf{C}^{-1}_{(ai)(bj)} [\delta \boldsymbol{t}_{(bj)} - Q_b(t_{bj})] \right\}$$

The bivariate normal distribution

Let x and y be RVs with the following joint pdf

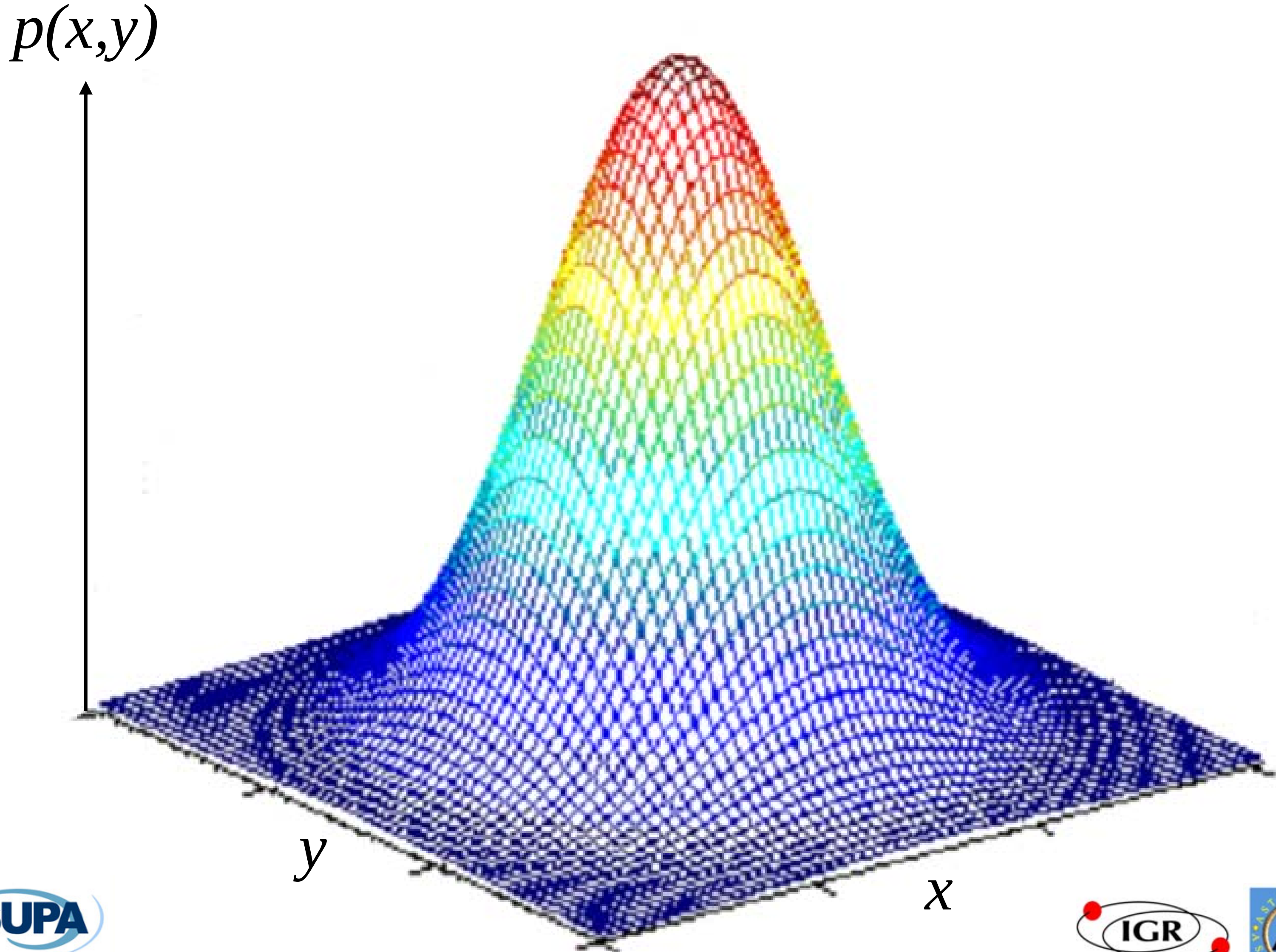
$$p(x,y) = \frac{1}{2\pi\sigma_{\rm x}\sigma_{\rm y}\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}Q(x,y)\right]$$

where the quadratic form, $Q(x,y)$ is given by

$$Q(x,y) = (\frac{x-\mu_{\rm x}}{\sigma_{\rm x}})^2 - 2\rho(\frac{x-\mu_{\rm x}}{\sigma_{\rm x}})(\frac{y-\mu_{\rm y}}{\sigma_{\rm y}}) + (\frac{y-\mu_{\rm y}}{\sigma_{\rm y}})^2$$

Then $p(x,y)$ is known as the **bivariate normal pdf** and is specified by the 5 parameters $\mu_{\rm x}$, $\mu_{\rm y}$, $\sigma_{\rm x}$, $\sigma_{\rm y}$ and ρ. This pdf is used often in the physical sciences to model the joint pdf of two random variables.

The bivariate normal distribution



The bivariate normal distribution

The first 4 parameters of the bivariate normal pdf are, in fact, equal to the following expectation values:-

1. $E(x) = \mu_{\mathrm{x}}$
2. $E(y) = \mu_{\mathrm{y}}$
3. $\mathrm{var}(x) = \sigma_{\mathrm{x}}^2$
4. $\mathrm{var}(y) = \sigma_{\mathrm{y}}^2$

The bivariate normal distribution

The parameter ρ is known as the **correlation coefficient** and satisfies

$$E[(x - \mu_{\mathrm{x}})(y - \mu_{\mathrm{y}})] \quad = \quad \rho\sigma_{\mathrm{x}}\sigma_{\mathrm{y}}$$

Note that if $\rho = 0$ then x and y are independent.

$E[(x - \mu_{\mathrm{x}})(y - \mu_{\mathrm{y}})]$ is known as the **covariance** of x and y and is often denoted by $\mathrm{cov}(x, y)$.

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$E[(x-\mu_{\mathrm{x}})(y-\mu_{\mathrm{y}})]$ is known as the **covariance** of x and y and is often denoted by $\mathrm{cov}(x,y)$.

In fact, for *any* two variables x and y, we define

$$\mathrm{cov}(x,y) \quad = \quad E\big[(x-E(x))(y-E(y))\big]$$

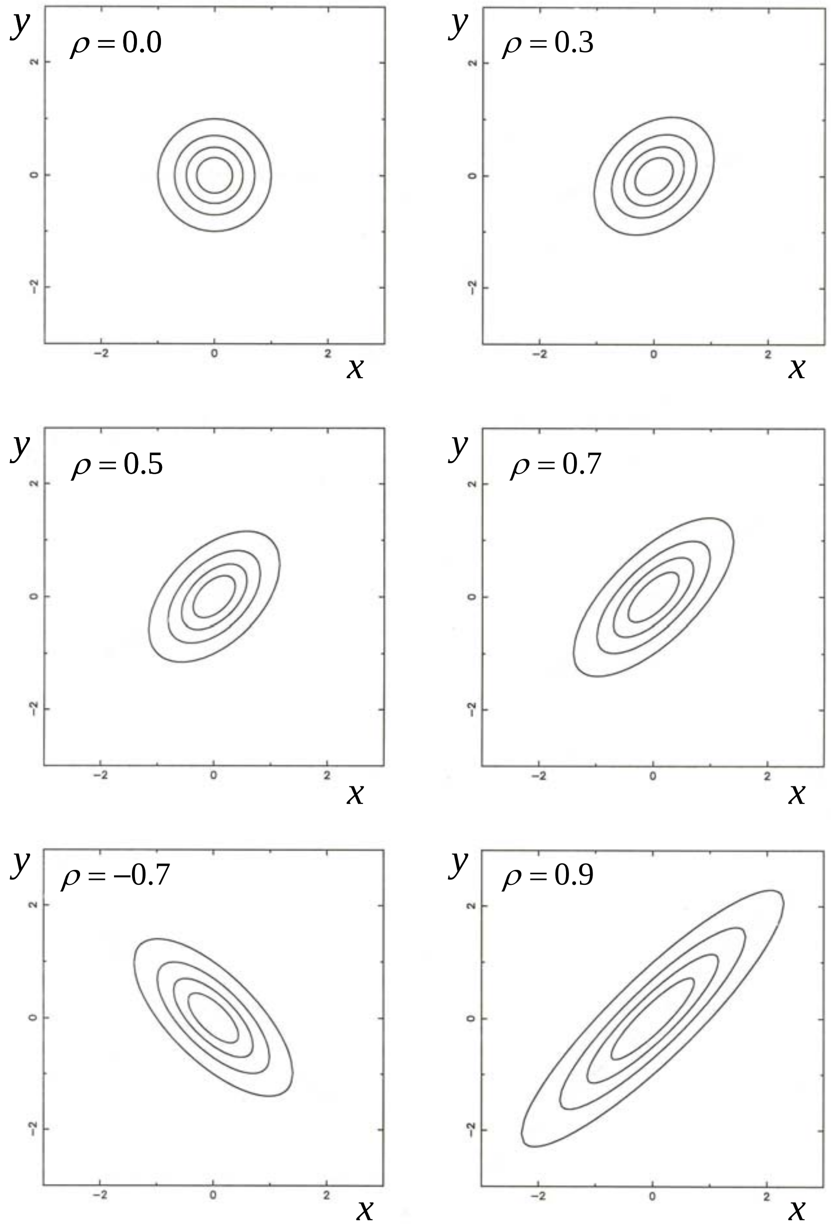
Isoprobability contours for the bivariate normal pdf

$\rho > 0$: positive correlation

y tends to increase as x increases

$\rho < 0$: negative correlation

y tends to decrease as x increases



Isoprobability contours for the bivariate normal pdf

$\rho > 0$:   positive correlation

y tends to increase as x increases

$\rho < 0$:   negative correlation

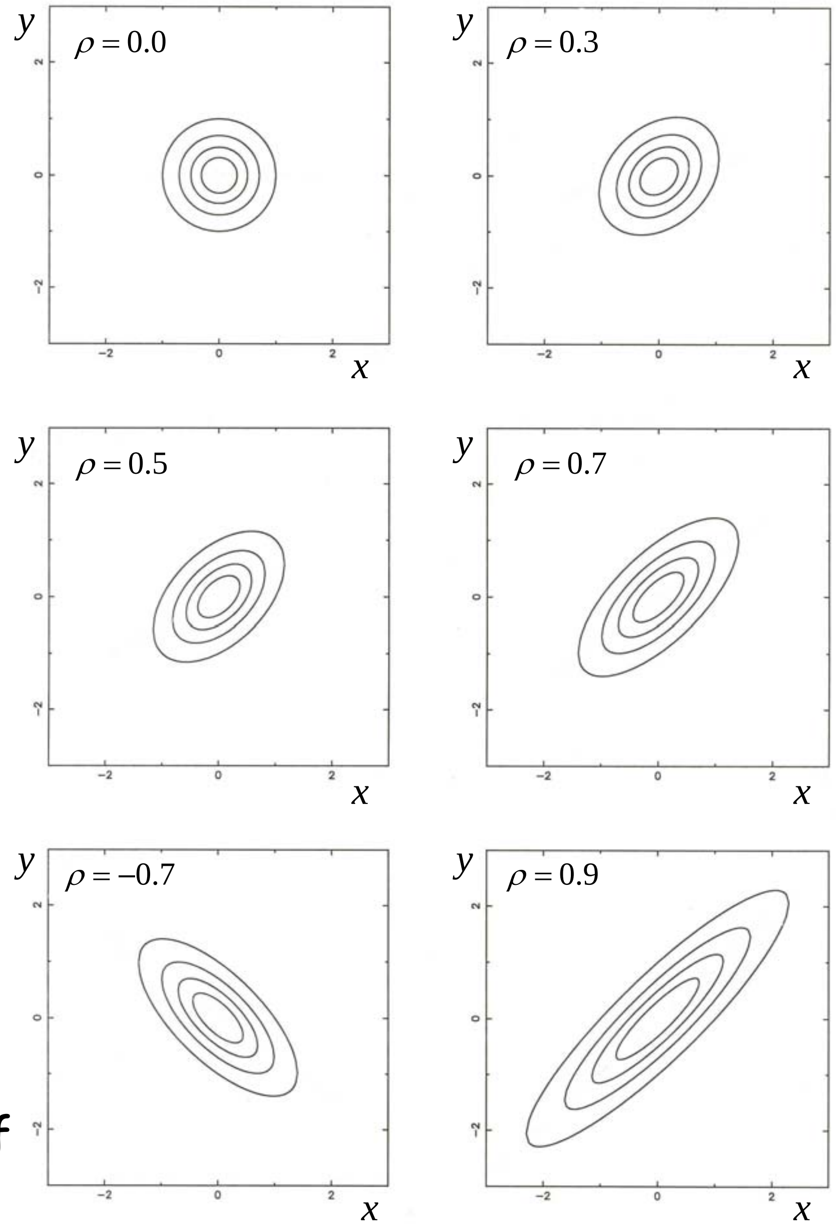
y tends to decrease as x increases

Contours become narrower and steeper as

$$|\rho| \to 1$$

$\Rightarrow$ stronger (anti) correlation between x and y.

i.e. *Given* value of x, value of y is tightly constrained.



van Haasteren (2009) Formalism

$P(\delta t)$ depends on a lot of parameters, which:

1) characterise the spin-down model $Q_a(t_{ai}) = \underline{A_{a1}} + \underline{A_{a2}} t_{ai} + \underline{A_{a3}} t_{ai}^2$

2) **characterise the GW covariance matrix** $\mathbf{C}^{\mathrm{GW}}_{(ai)(bj)}$

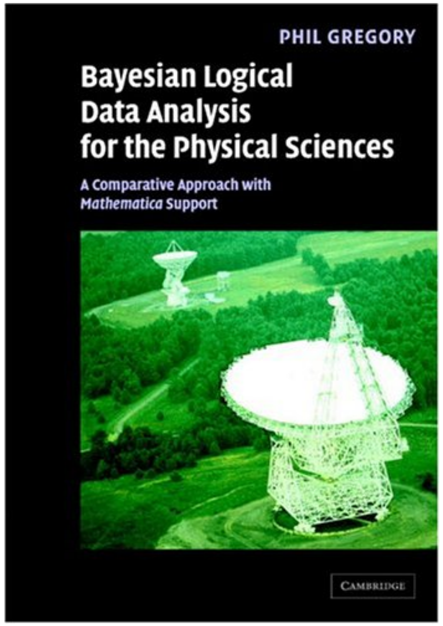
3) characterise the PN covariance matrix $\mathbf{C}^{\mathrm{PN}}_{(ai)(bj)}$

We are only really interested in (2); the parameters associated with (1) and (3) are 'nuisance' parameters.

Bayesian Inference provides a natural framework in which to constrain these parameters, making optimal use of the information contained in the observed data – together with our model for the other sources of noise.

Aside: a quick primer on Bayesian inference

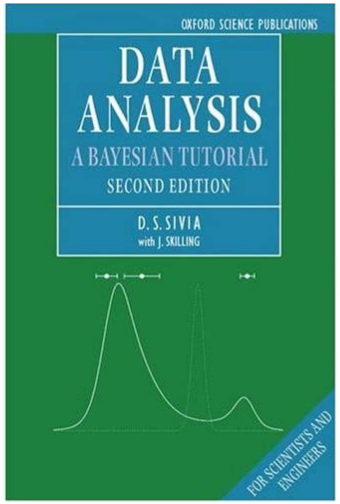
Reasonable thinking?...



PREFACE

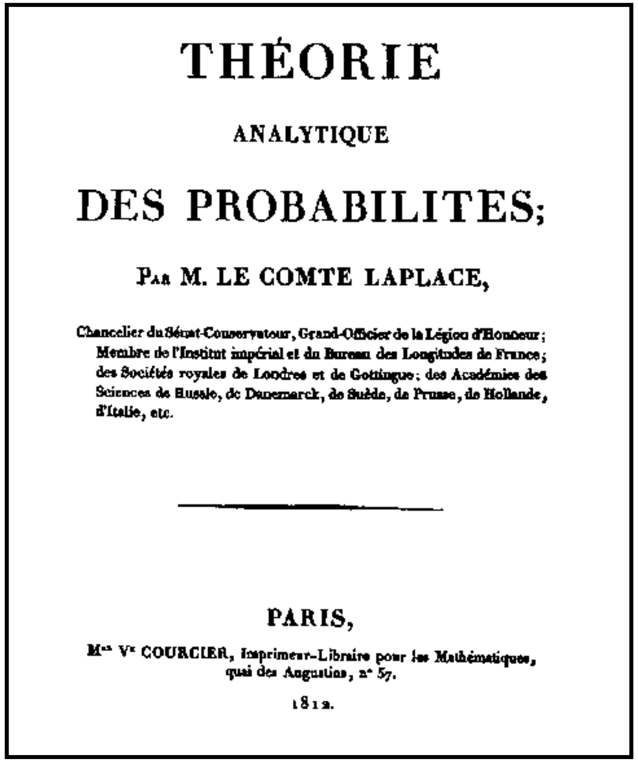
The goal of science is to unlock nature's secrets…Our understanding comes through the development of theoretical models capable of explaining the existing observations as well as making testable predictions…Statistical inference provides a means for assessing the plausibility of one or more competing models, and estimating the model parameters and their uncertainities. These topics are commonly referred to as "data analysis".

The most we can hope to do is to make the best inference based on the experimental data and any prior knowledge that we have available.



Pierre-Simon Laplace
(1749 – 1827)

"Probability theory is nothing but common sense reduced to calculation"



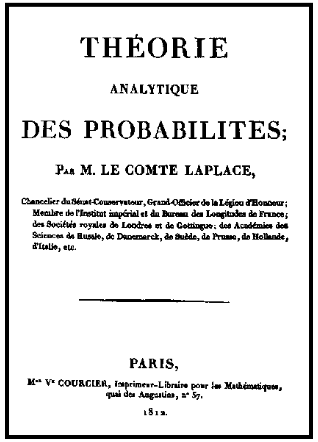
THÉORIE
ANALYTIQUE
DES PROBABILITES;

PAR M. LE COMTE LAPLACE,

Chancelier du Sénat-Conservateur, Grand-Officier de la Légion d'Honneur; Membre de l'Institut impérial et du Bureau des Longitudes de France; des Sociétés royales de Londres et de Gottingue; des Académies des Sciences de Russie, de Danemarck, de Suède, de Prusse, de Hollande, d'Italie, etc.

PARIS,
Mme Ve COURCIER, Imprimeur-Libraire pour les Mathématiques, quai des Augustins, n° 57.
1812.

Laplace (1812)



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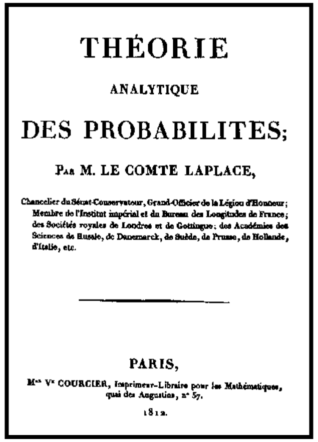
Mathematical framework for probability as a basis for plausible reasoning:

Probability measures our degree of belief that something is true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Mathematical framework for probability as a basis for plausible reasoning:

> Probability measures our degree of belief that something is true

$\text{Prob}(X) = 1 \quad \Rightarrow$ we are *certain* that X is true

$\text{Prob}(X) = 0 \quad \Rightarrow$ we are *certain* that X is false

Our degree of belief always depends on the available background information:

We write

$$\text{Prob}(\,X \mid I\,)$$

"Probability that X is true, given I"

Background information

Vertical line denotes conditional probability:

our state of knowledge about X is *conditioned* by background info, I

Rules for combining probabilities

$$p(X,Y \mid I) \quad = \quad p(X \mid Y,I) \times p(Y \mid I)$$

X,Y denotes the proposition that X **and** Y are true

Rules for combining probabilities

$$p(X,Y\,|\,I) \quad = \quad p(X\,|\,Y,I) \times p(Y\,|\,I)$$

X,Y denotes the proposition that X **and** Y are true

$p(X\,|\,Y,I)$ = Prob(X is true, given Y is true)

$p(Y\,|\,I)$ = Prob(Y is true, irrespective of X)

Also

$$p(Y, X \mid I) = p(Y \mid X, I) \times p(X \mid I)$$

but

$$p(Y, X \mid I) = p(X, Y \mid I)$$

Hence

$$p(Y \mid X, I) = \frac{p(X \mid Y, I) \times p(Y \mid I)}{p(X \mid I)}$$

Bayes' theorem:

$$p(Y \mid X, I) = \frac{p(X \mid Y, I) \times p(Y \mid I)}{p(X \mid I)}$$

Laplace rediscovered work of Rev. Thomas Bayes (1763)

Bayesian Inference

Thomas Bayes
(1702 – 1761 AD)

Bayes' theorem:

$$p(Y \mid X, I) = \frac{p(X \mid Y, I) \times p(Y \mid I)}{p(X \mid I)}$$

$$p(\text{model} \mid \text{data}, I) = \frac{p(\text{data} \mid \text{model}, I) \times p(\text{model} \mid I)}{p(\text{data} \mid I)}$$

Bayes' theorem:

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Posterior — Likelihood — Prior — Evidence

$$p(\text{model} \mid \text{data}, I) = \frac{p(\text{data} \mid \text{model}, I) \times p(\text{model} \mid I)}{p(\text{data} \mid I)}$$

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Posterior

Likelihood

Prior

$$p(\text{model} \mid \text{data}, I) = \frac{p(\text{data} \mid \text{model}, I) \times p(\text{model} \mid I)}{p(\text{data} \mid I)}$$

Evidence

We can calculate these terms

Bayes' theorem:

$$p(Y \mid X, I) = \frac{p(X \mid Y, I) \times p(Y \mid I)}{p(X \mid I)}$$

Posterior — Likelihood — Prior

$$p(\text{model} \mid \text{data}, I) \propto p(\text{data} \mid \text{model}, I) \times p(\text{model} \mid I)$$

What we know now — Influence of our observations — What we knew before

Marginal Distributions

The **marginal pdf**, $p_1(x_1)$ of x_1 is defined by

$$p_1(x_1) \quad = \quad \int_{-\infty}^{\infty} p(x_1, x_2)\, dx_2$$

and is a pdf in the usual sense that

1. $p_1(x_1) \geq 0$, for all x_1

2. $\text{Prob}(a < x_1 < b) \quad = \quad \int_a^b p_1(x_1) dx_1$

3. $\int_{-\infty}^{\infty} p_1(x_1) dx_1 \quad = \quad 1$

Marginal Distributions

Similarly, the marginal pdf of x_2 is

$$p_2(x_2) = \int_{-\infty}^{\infty} p(x_1, x_2)\, dx_1$$

In general, given any multivariate pdf, we may find the marginal pdf of any subset of the $x_1, ..., x_n$ by integrating over all other variables.

e.g.

$$p_{13}(x_1, x_3) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} p(x_1, ..., x_n)\, dx_2 dx_4 dx_5 ... dx_n$$

van Haasteren (2009) Formalism

Model for $\mathbf{C}^{\mathrm{GW}}_{(ai)(bj)}$?

Take the **spectral density** of the SB to be a power law $S_h = A^2 \left(\frac{f}{\mathrm{yr}^{-1}} \right)^{-\gamma}$

This implies

Gamma function

$$\tau = 2\pi(t_{ai} - t_{bj})$$

Low cut-off frequency

$$\mathbf{C}^{\mathrm{GW}}_{(ai)(bj)} = \frac{A^2 \alpha_{ab}}{(2\pi)^2 f_L^{1+\gamma}} \left\{ \Gamma(-1-\gamma) \sin\left(\frac{-\pi\gamma}{2} \right) (f_L \tau)^{\gamma+1} - \sum_{n=0}^{\infty} (-1)^n \frac{(f_L \tau)^{2n}}{(2n)!(2n-1-\gamma)} \right\}$$

Here α_{ab} is a geometrical factor that takes account of the angle θ_{ab} between each pair of pulsars – which determines how they are correlated.

$$\alpha_{ab} = \frac{3}{2} \frac{1-\cos\theta_{ab}}{2} \ln\left(\frac{1-\cos\theta_{ab}}{2} \right) - \frac{1}{4} \frac{1-\cos\theta_{ab}}{2} + \frac{1}{2} + \frac{1}{2}\delta_{ab}$$

van Haasteren (2009) Formalism

Model for $\mathbf{C}^{\mathrm{PN}}_{(ai)(bj)}$? Three alternatives considered.

1) White noise: $\mathbf{C}^{\mathrm{PN-white}}_{(ai)(bj)} = \underline{N_a^2}\,\delta_{ab}\delta_{ij}$

2) 'Lorentzian' spectrum: $S_a(f) = \dfrac{N_a^2}{f_0\left[1 + \left(\frac{f}{f_0}\right)^2\right]}$

$$\mathbf{C}^{\mathrm{PN-lor}}_{(ai)(bj)} = \underline{N_a^2}\,\delta_{ab}\exp(-\underline{f_0}\,\tau)$$

3) Power-law spectrum: equivalent expressions to those for $\mathbf{C}^{\mathrm{GW}}_{(ai)(bj)}$ with parameters N_a and γ_a .

van Haasteren (2009) Formalism

So we have the GW parameters of interest (A, γ) and nuisance parameters Θ_{PN} and Θ_Q .

It then follows from Bayes' theorem that:

$$p(A, \gamma, \Theta_{\mathrm{PN}}, \Theta_Q \mid \mathrm{data}) \propto p(\mathrm{data} \mid A, \gamma, \Theta_{\mathrm{PN}}, \Theta_Q)\, p(A, \gamma, \Theta_{\mathrm{PN}}, \Theta_Q)$$

posterior likelihood prior

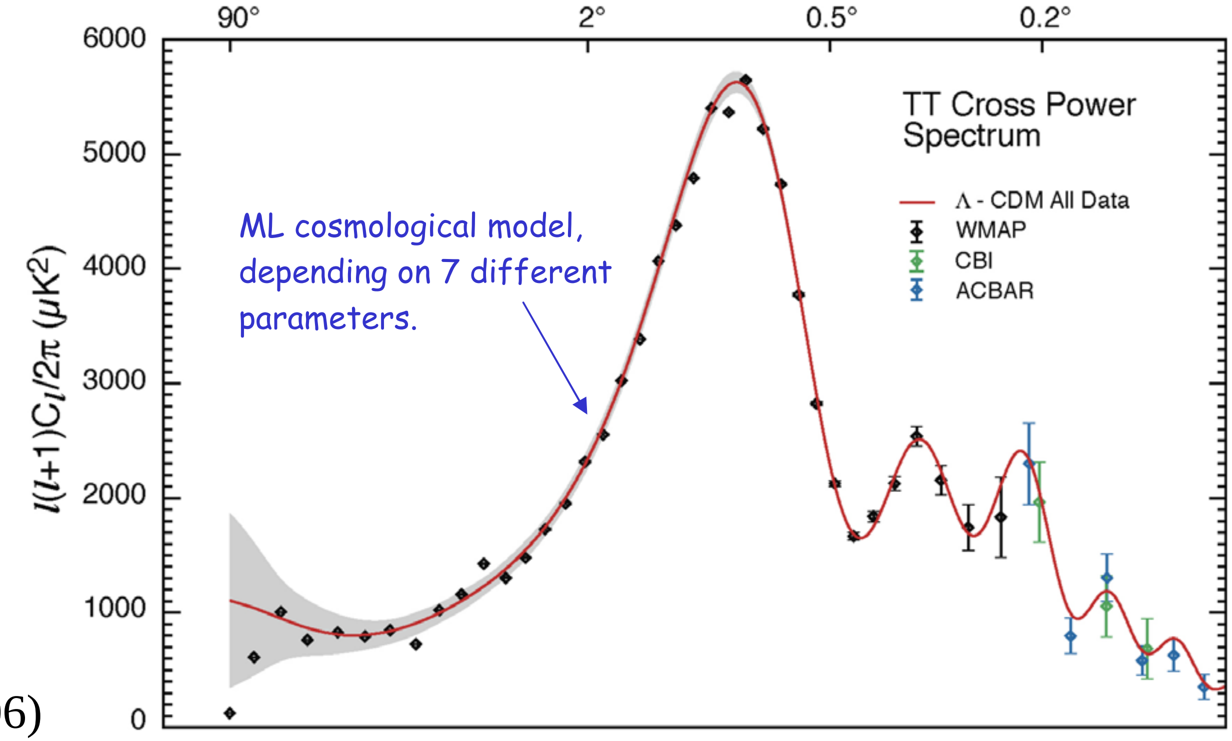
And to obtain the posterior for the GW parameters, we must integrate, or **marginalise**, with respect to the nuisance parameters. For Θ_Q this can be done analytically. For the other parameters we can use **MCMC**.

An Introduction to Markov Chain Monte Carlo

This is a very powerful, new (at least in astronomy!) method for sampling from pdfs. (These can be complicated and/or of high dimension).

MCMC widely used e.g. in cosmology to determine 'maximum likelihood' model to CMBR data.

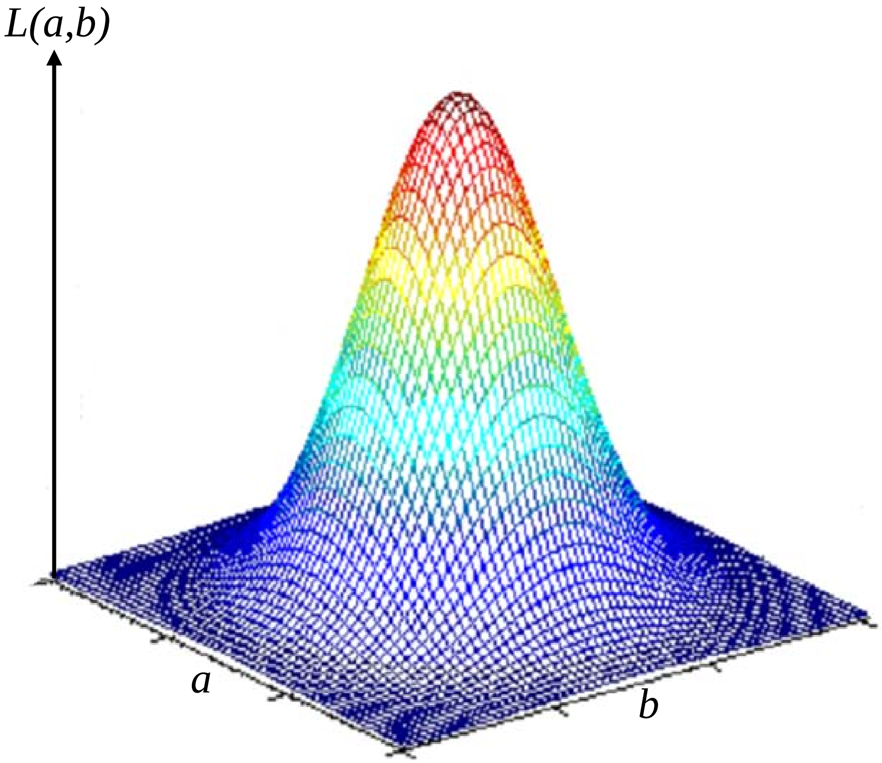
Angular power spectrum of CMBR temperature fluctuations



(Hinshaw et al 2006)

Consider a 2-D example (e.g. bivariate normal distribution); Likelihood function depends on parameters a and b.

Suppose we are trying to find the maximum of $L(a,b)$



1) Start off at some randomly chosen value (a_1, b_1)

2) Compute $L(a_1, b_1)$ and gradient $\left(\frac{\partial L}{\partial a}, \frac{\partial L}{\partial b}\right)_{(a_1, b_1)}$

3) Move in direction of steepest +ve gradient – i.e. $L(a_1, b_1)$ is increasing fastest

4) Repeat from step 2 until (a_n, b_n) converges on maximum of likelihood

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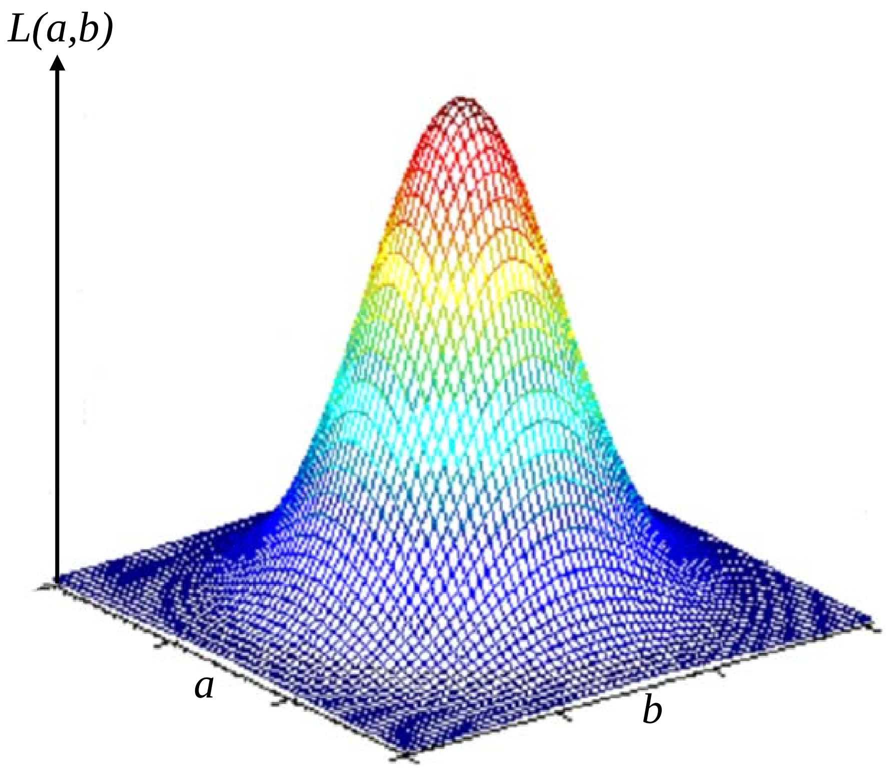

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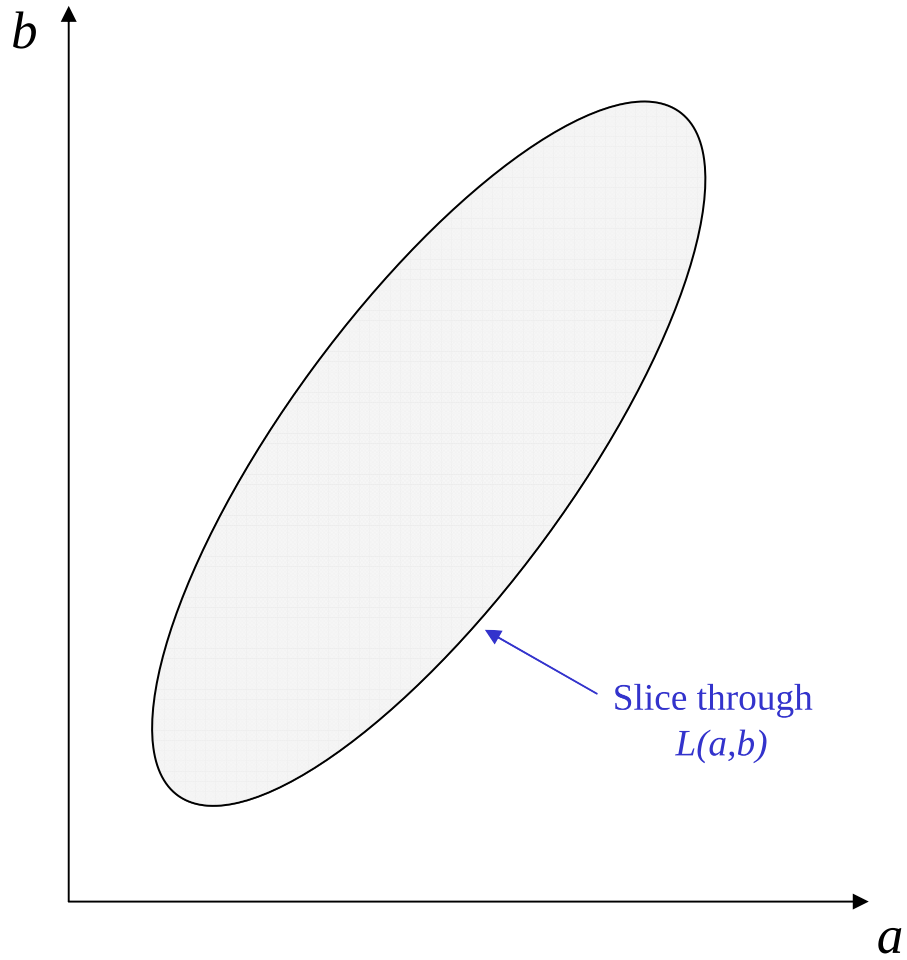
OK for finding maximum, but not for generating a sample from $L(a,b)$
or for determining errors on the the ML parameter estimates.

MCMC provides a simple **Metropolis algorithm** for generating random samples of points from $L(a,b)$



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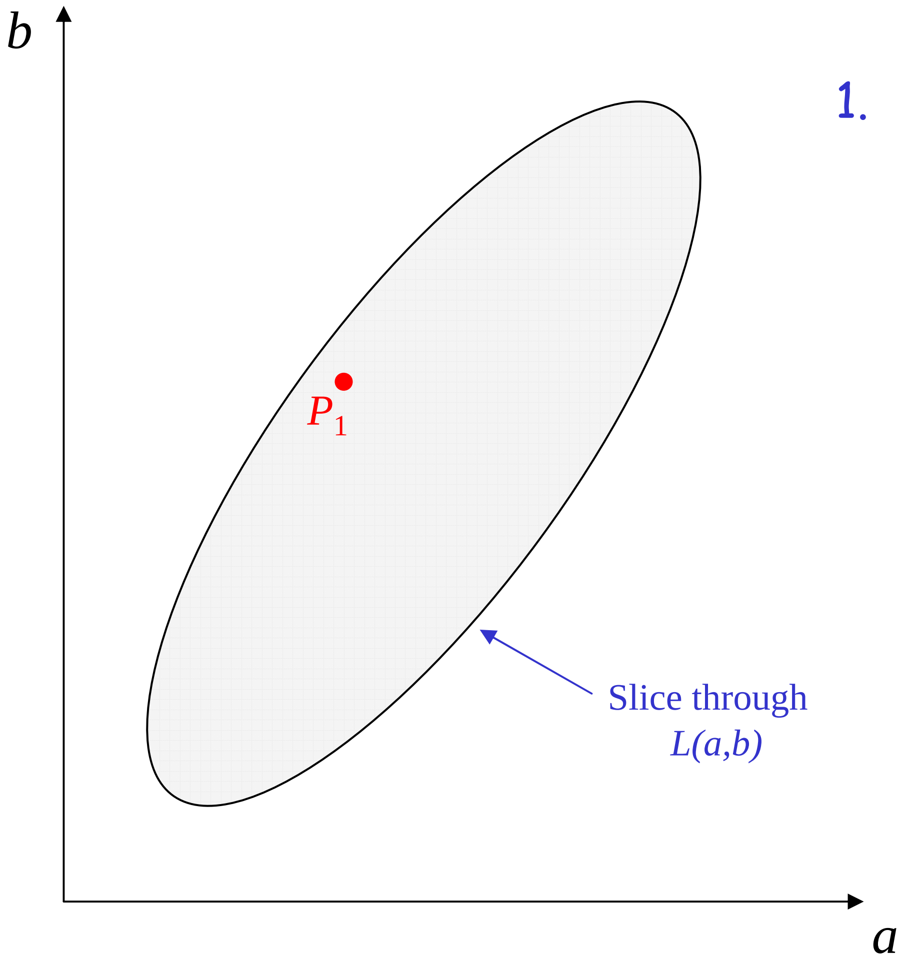

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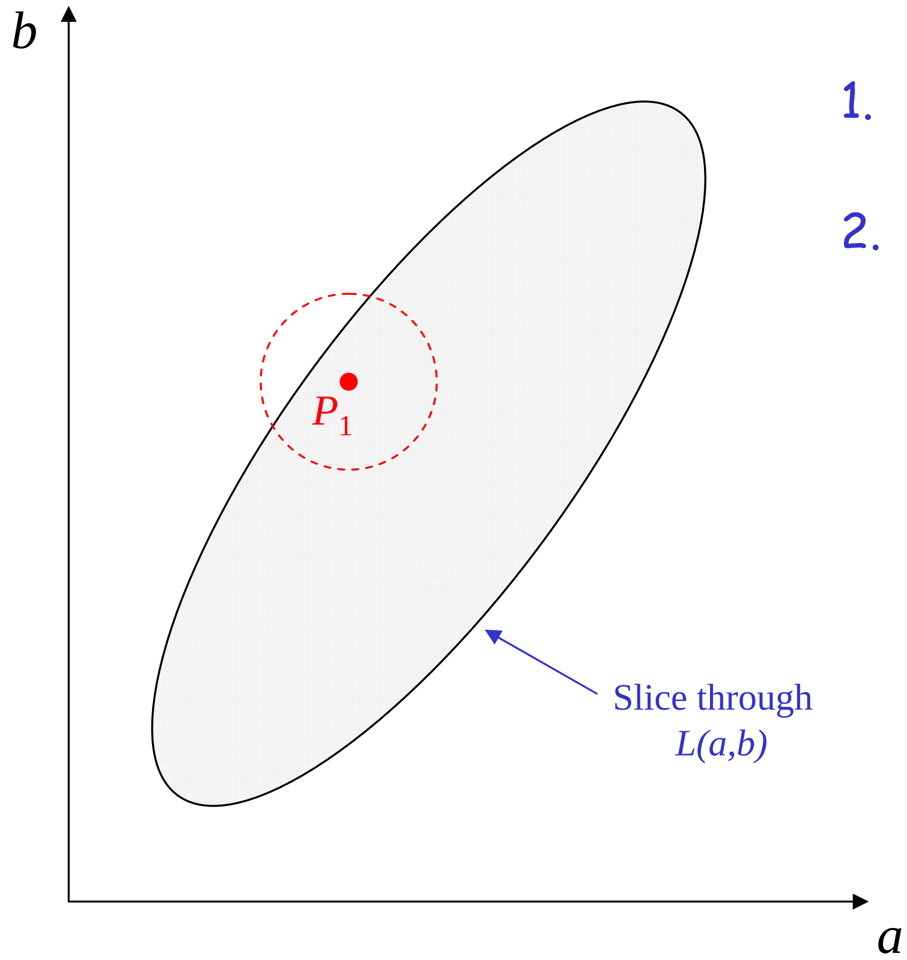

1. Sample random initial point $P_1 = (a_1, b_1)$
2. Centre a new pdf, Q, called the **proposal density**, on P_1

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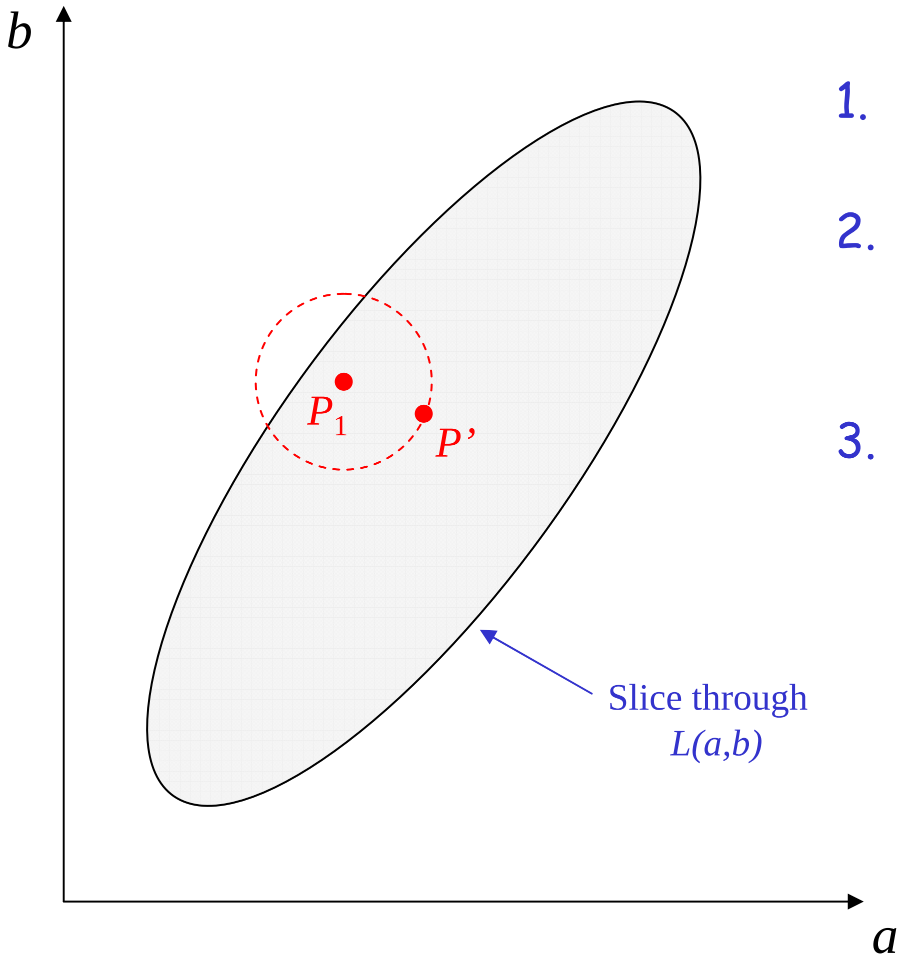

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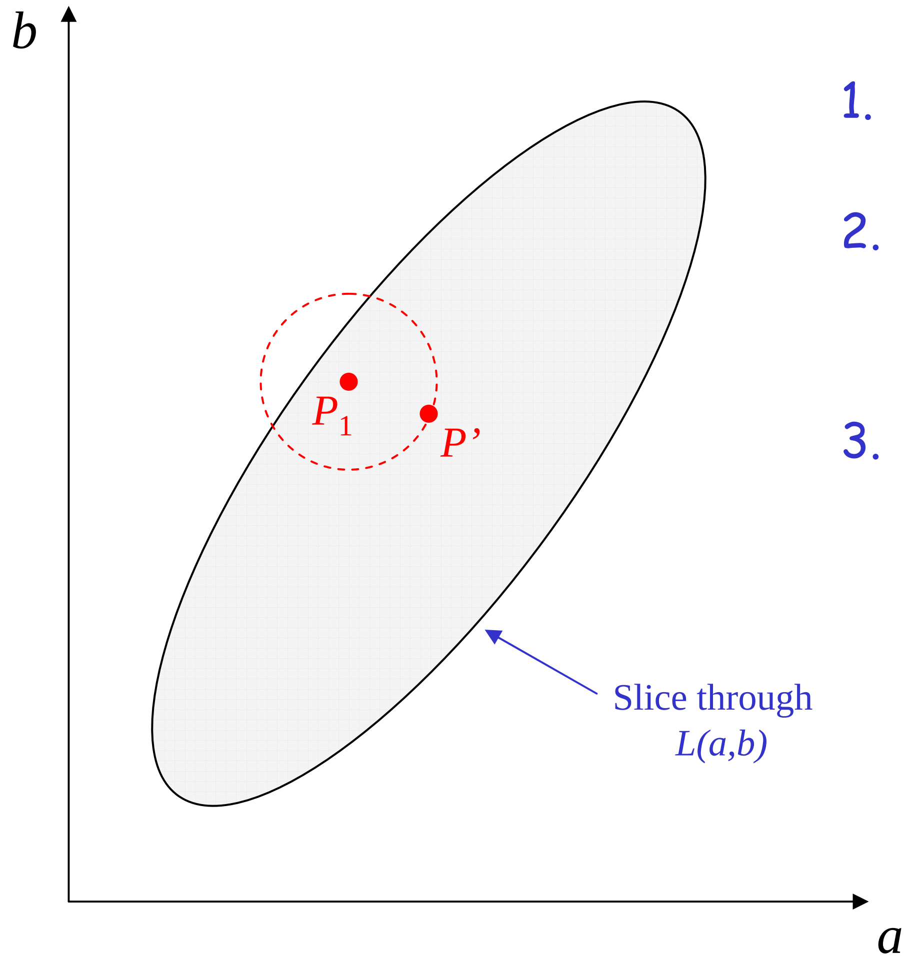

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2. Centre a new pdf, Q, called the **proposal density**, on P_1

3. Sample tentative new point $P' = (a', b')$ from Q

4. Compute $$R = \frac{L(a',b')}{L(a_1,b_1)}$$

5. If $R > 1$ this means P' is **uphill** from P_1 .

We **accept** P' as the next point in our chain, i.e. $P_2 = P'$

6. If $R < 1$ this means P' is **downhill** from P_1 .

In this case we ***may*** reject P' as our next point.

In fact, we accept P' with probability R .

How do we do this?...

(a) Generate a random number $x \sim \mathrm{U}[0,1]$

(b) If $x < R$ then accept P' and set $P_2 = P'$

(c) If $x > R$ then reject P' and set $P_2 = P_1$

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*Acceptance probability depends only on the previous point - **Markov Chain***

So the Metropolis Algorithm generally (but not always) moves uphill, towards the peak of the Likelihood Function.

Remarkable facts

- Sequence of points $\{ P_1, P_2, P_3, P_4, P_5, \ldots \}$
  represents a sample from the LF $L(a,b)$

- Sequence for each coordinate, e.g. $\{ a_1, a_2, a_3, a_4, a_5, \ldots \}$
  samples the marginalised likelihood of a

- We can make a histogram of $\{ a_1, a_2, a_3, a_4, a_5, \ldots, a_n \}$
  and use it to compute the mean and variance of a (i.e.
  to attach an error bar to a)

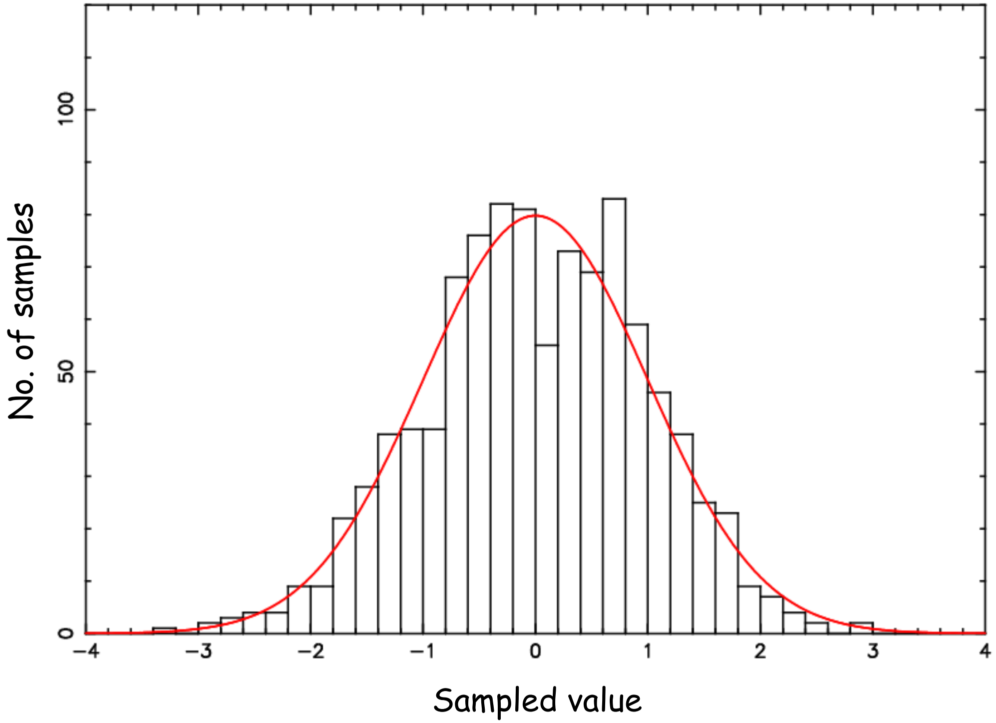
Why is this so useful?…

Suppose our LF was a 1-D Gaussian. We could estimate the mean and variance quite well from a histogram of e.g. 1000 samples.

But what if our problem is, e.g. 7 dimensional?

'Exhaustive' sampling could require $(1000)^7$ samples!

MCMC provides a short-cut.



To compute a new point in our Markov Chain we need to compute the LF. But the computational cost does **not** grow so dramatically as we increase the number of dimensions of our problem.

This lets us tackle problems that would be impossible by 'normal' sampling.

Outline of talk

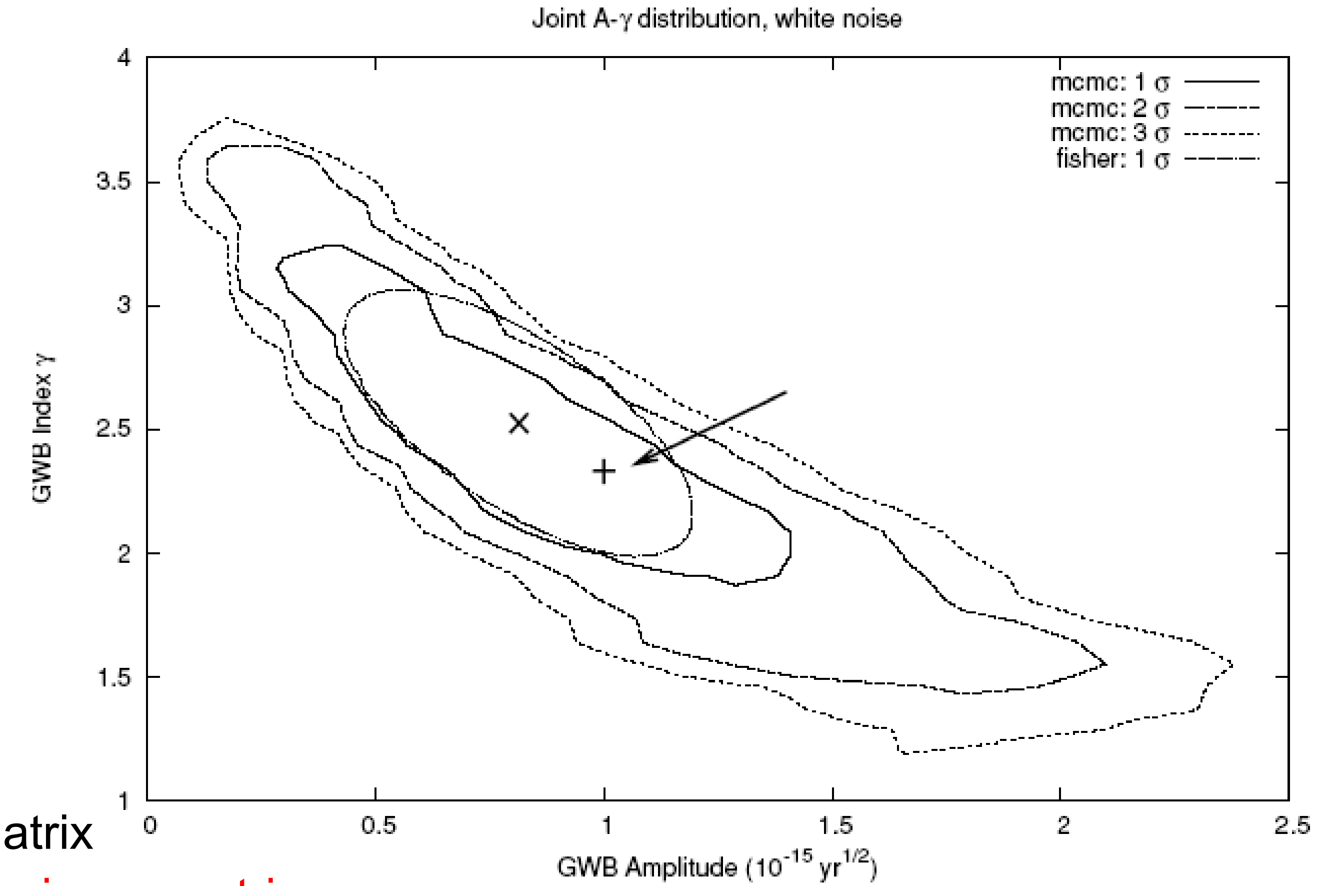
- A Stochastic Background of GWs – basic concepts
- Astrophysical and primordial sources
- Probing the SB with pulsar timing arrays: methods
- **Current and future GW limits with PTAs**
- Probing GWs with the CMBR: methods
- Current and future GW limits with the CMBR

Some simulation results from van Haasteren et al (2009)

20 mock pulsars;
100 data points per pulsar over 5 years.

'White' timing noise of 100 ns.

'Fisher' contour assumes posterior pdf is Gaussian – confidence region computed from the Fisher information matrix = Inverse of the covariance matrix.



See http://www.astro.gla.ac.uk/users/martin/supa-da.html

Advanced Data Analysis for the Physical Sciences

University of Glasgow

January 6th - 7th 2009

Overview

This 2-day course, organised by the SUPA Graduate School, and hosted by the Department of Physics and Astronomy at the University of Glasgow, will provide a comprehensive introduction to the principles and practice of advanced data analysis, with particular focus on their application within the physical sciences.

During the course, a range of topics will be explored, via a series of lectures and discussions.

SUPA ADA Intended Learning Outcomes: download

Day One: Core Topics

- Theoretical foundations, and the nature of probability
- The essentials of line and curve fitting
- An introduction to Bayesian inference
- Bayesian vs frequentist hypothesis testing, and goodness of fit statistics

Day Two: Advanced Topics

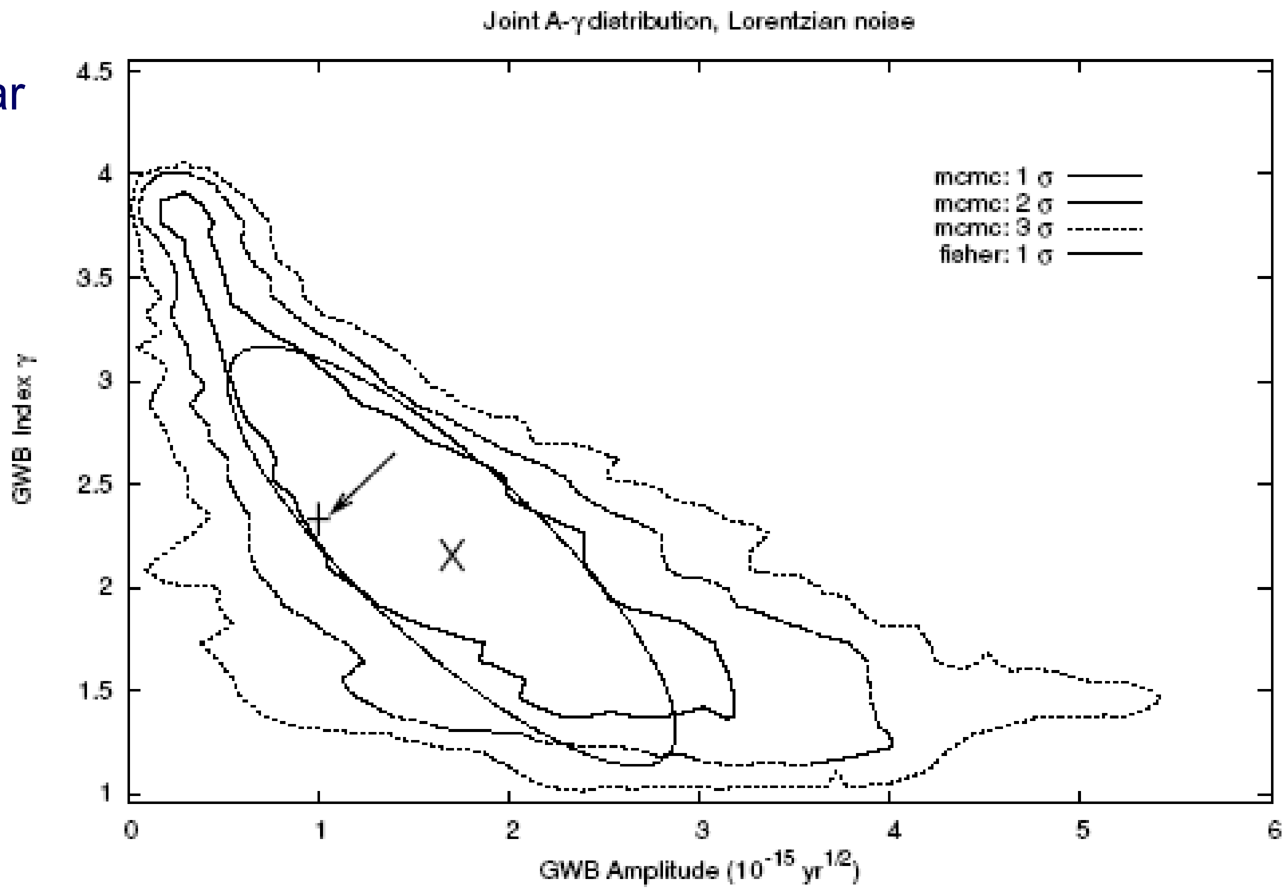
- Covariance and the Fisher matrix
- Bayesian evidence and model selection
- Efficient techniques for generating random numbers
- Bayesian inference with very large parameter spaces: Markov Chain Monte

Some simulation results from van Haasteren et al (2009)

Strong dependence of results on form of pulsar timing noise.

For Lorentzian TN, greater degeneracy between fitted amplitude and power law index.

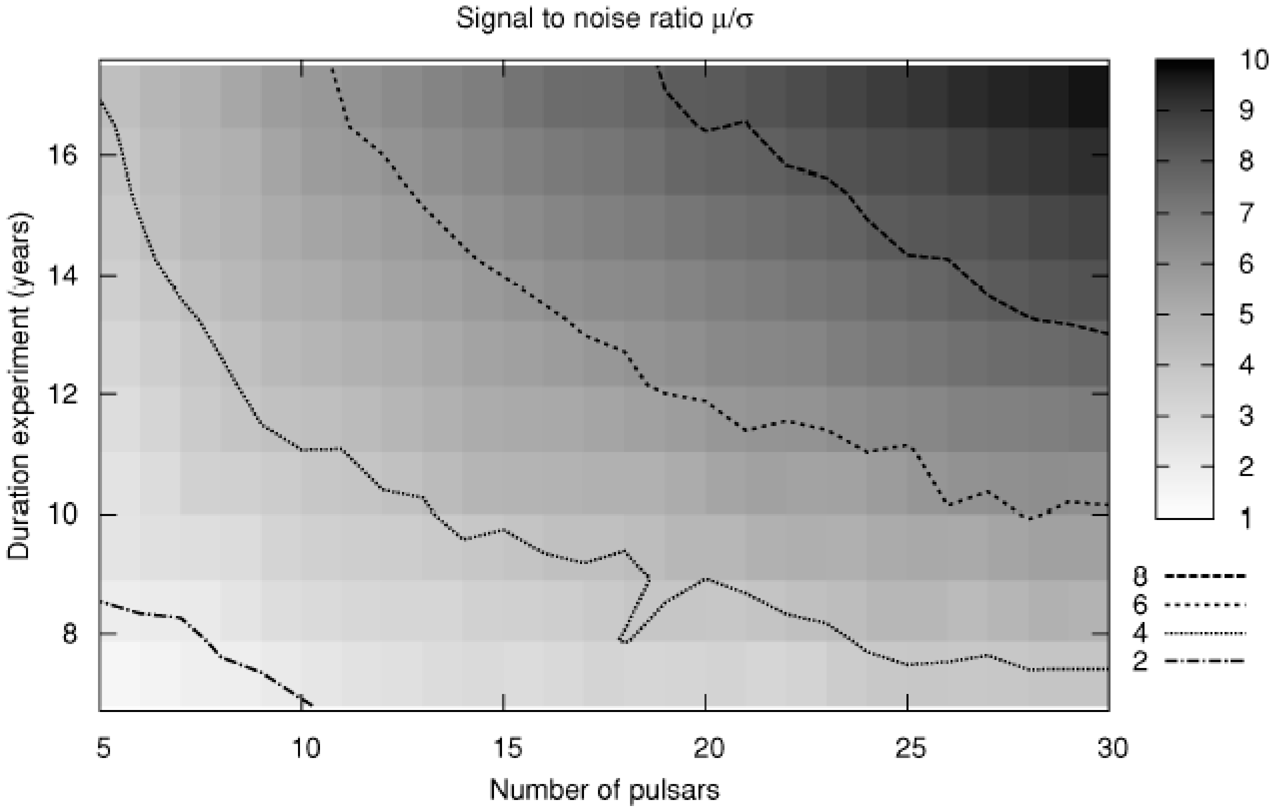
This would impact significantly on our ability to detect the GW background.



Some simulation results from van Haasteren et al (2009)

Investigation of various issues:

1) Duration of experiment



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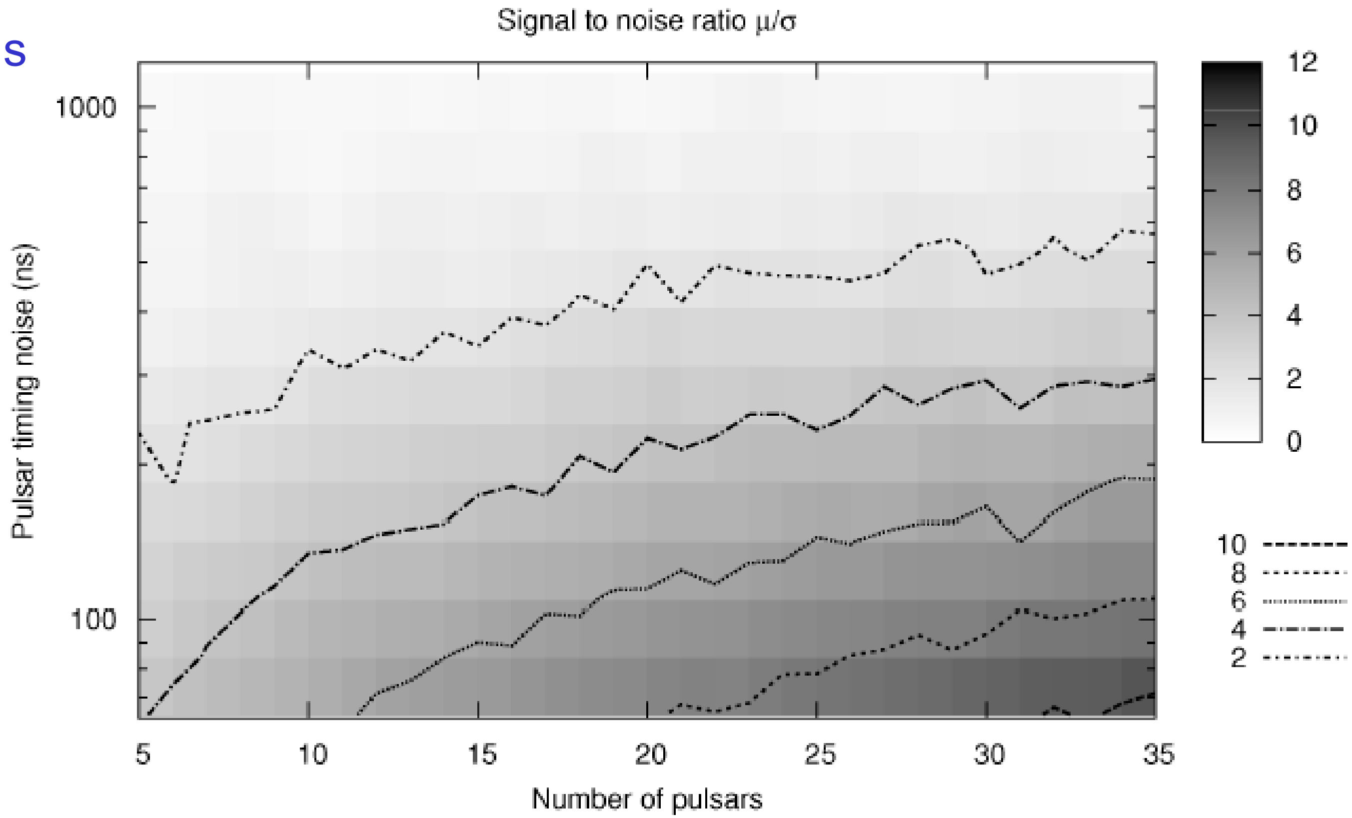
2) Magnitude of pulsar timing noise



Some simulation results from van Haasteren et al (2009)

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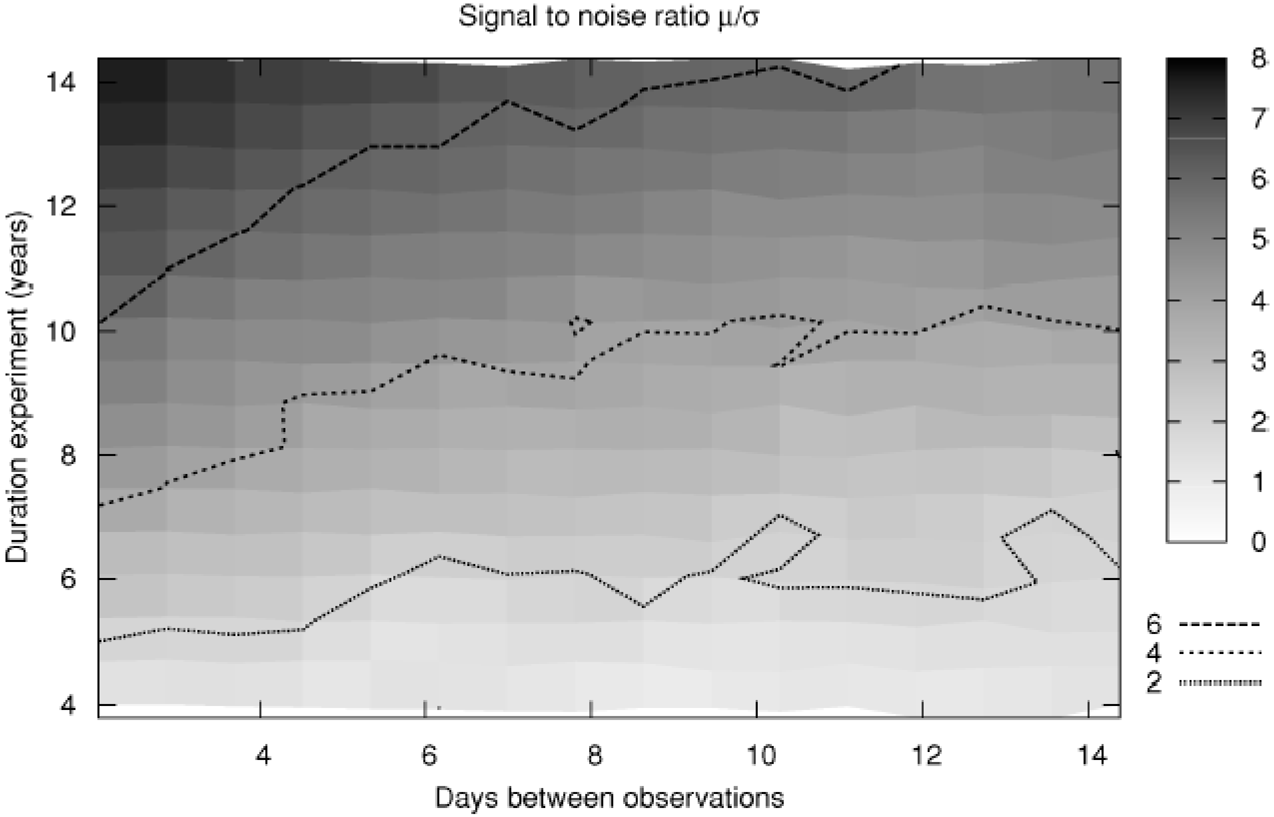
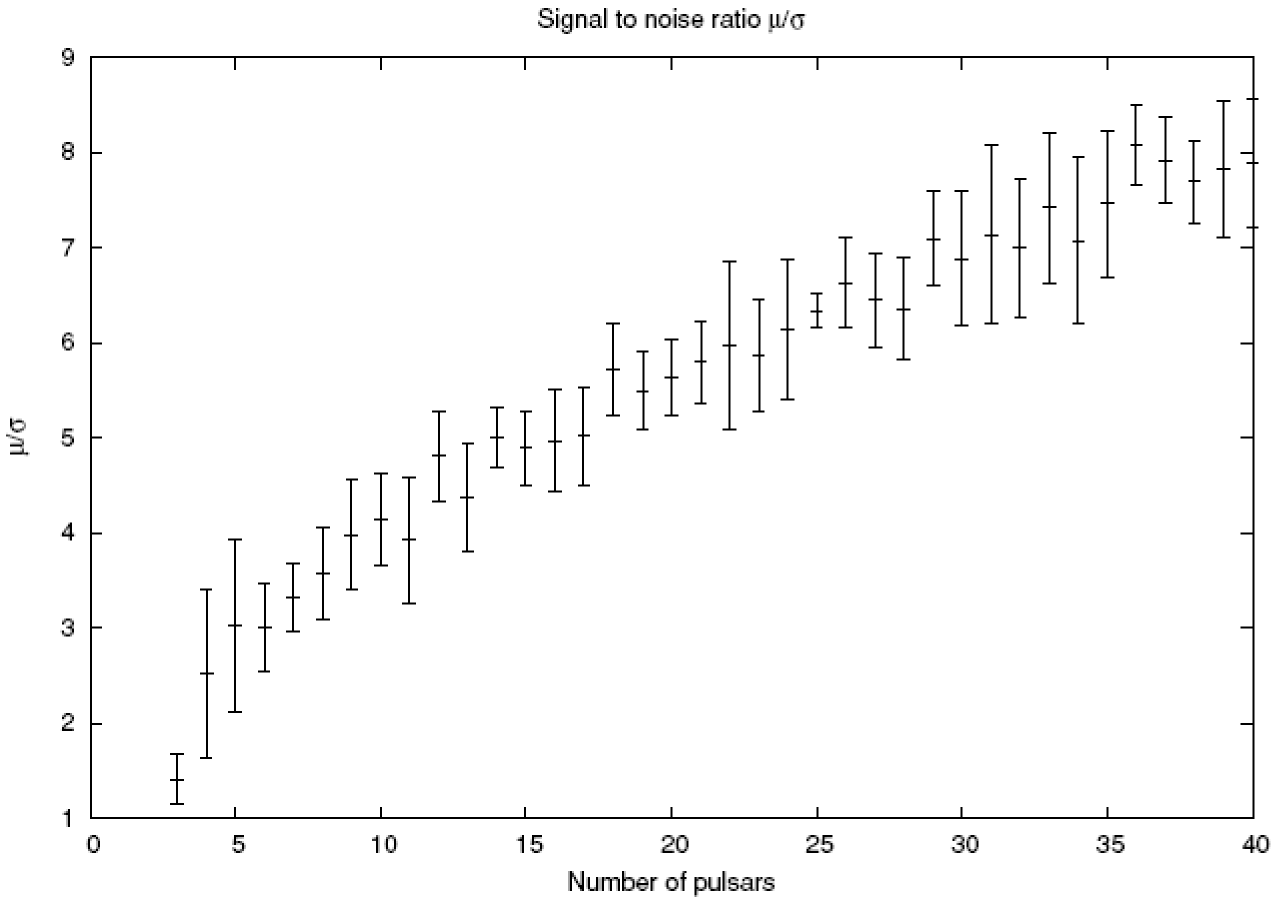


Some simulation results from van Haasteren et al (2009)

Investigation of various issues:

1) Duration of experiment
2) Magnitude of pulsar timing noise
3) Gaps between observations
4) Number of pulsars



Square Kilometre Array

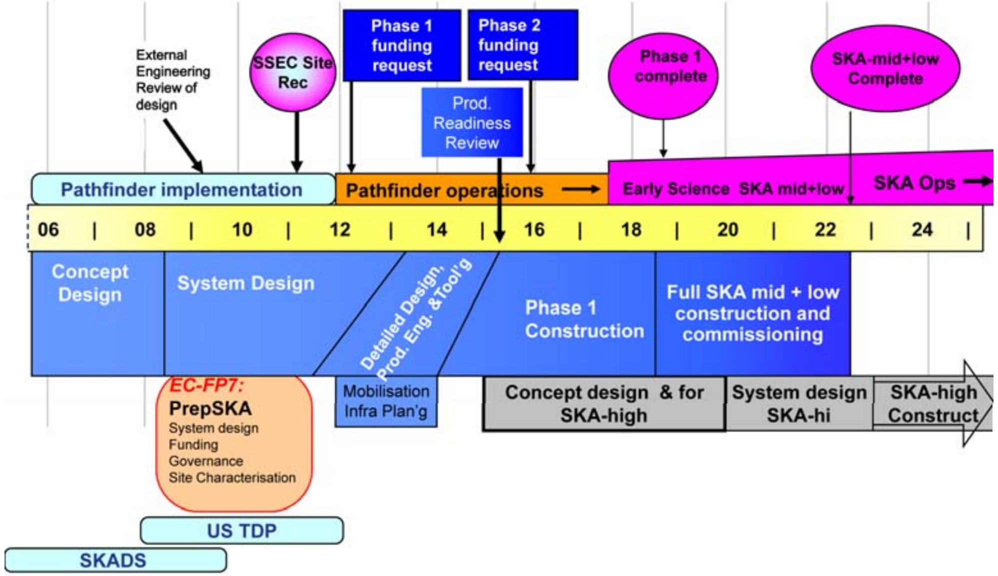
International consortium of more than 15 countries.

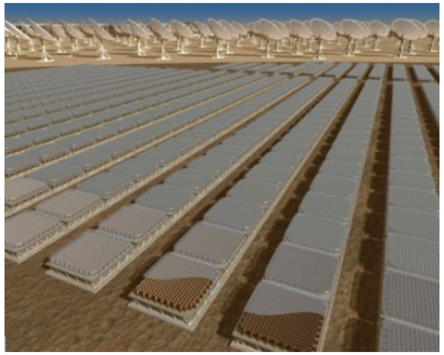
Site to be chosen ~2011

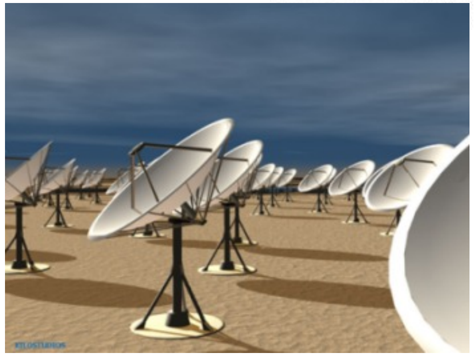
Precision pulsar timing one of 5 key science projects.

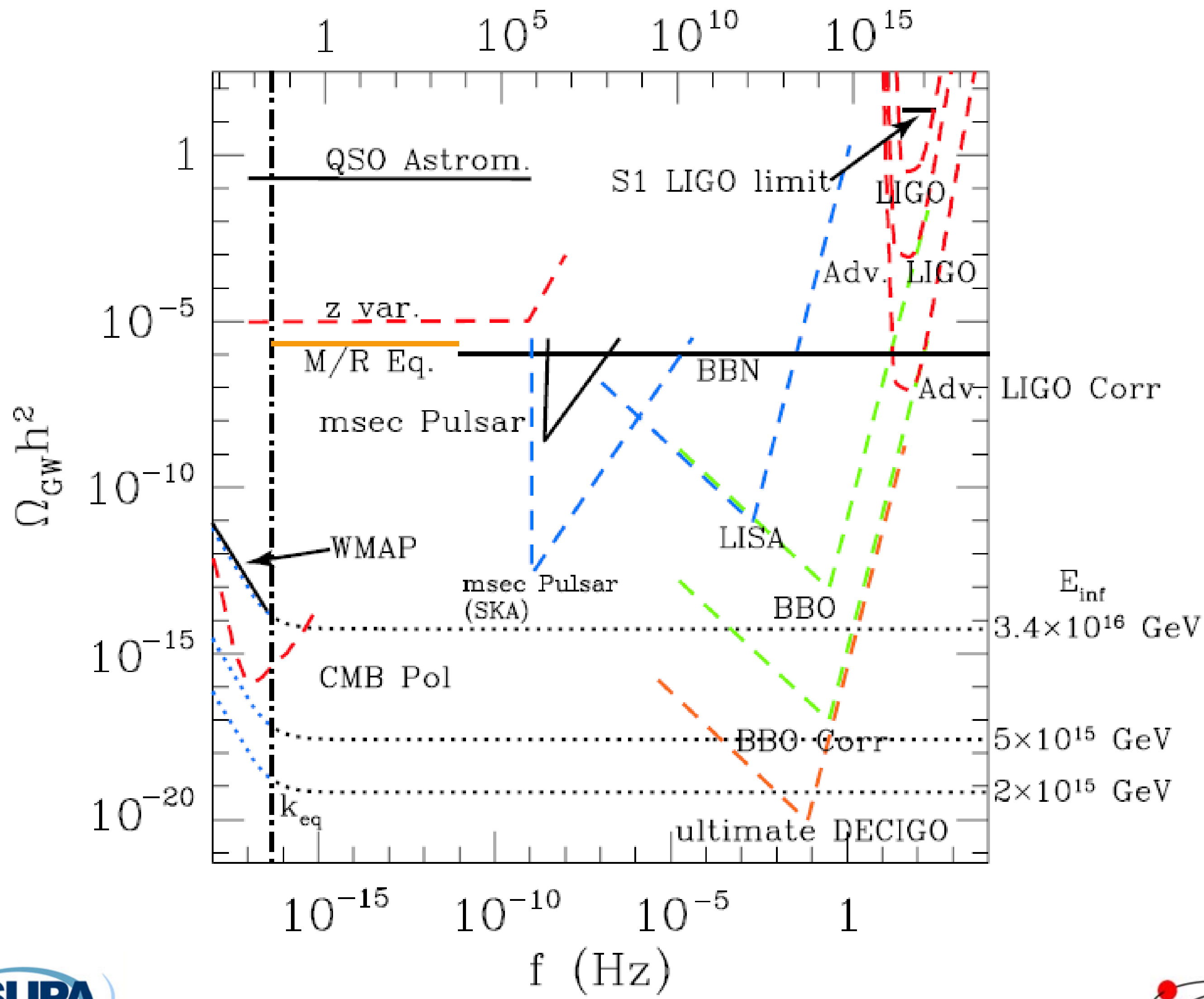
SKA should observe >1000 millisecond pulsars, with a timing accuracy of < 100ns.

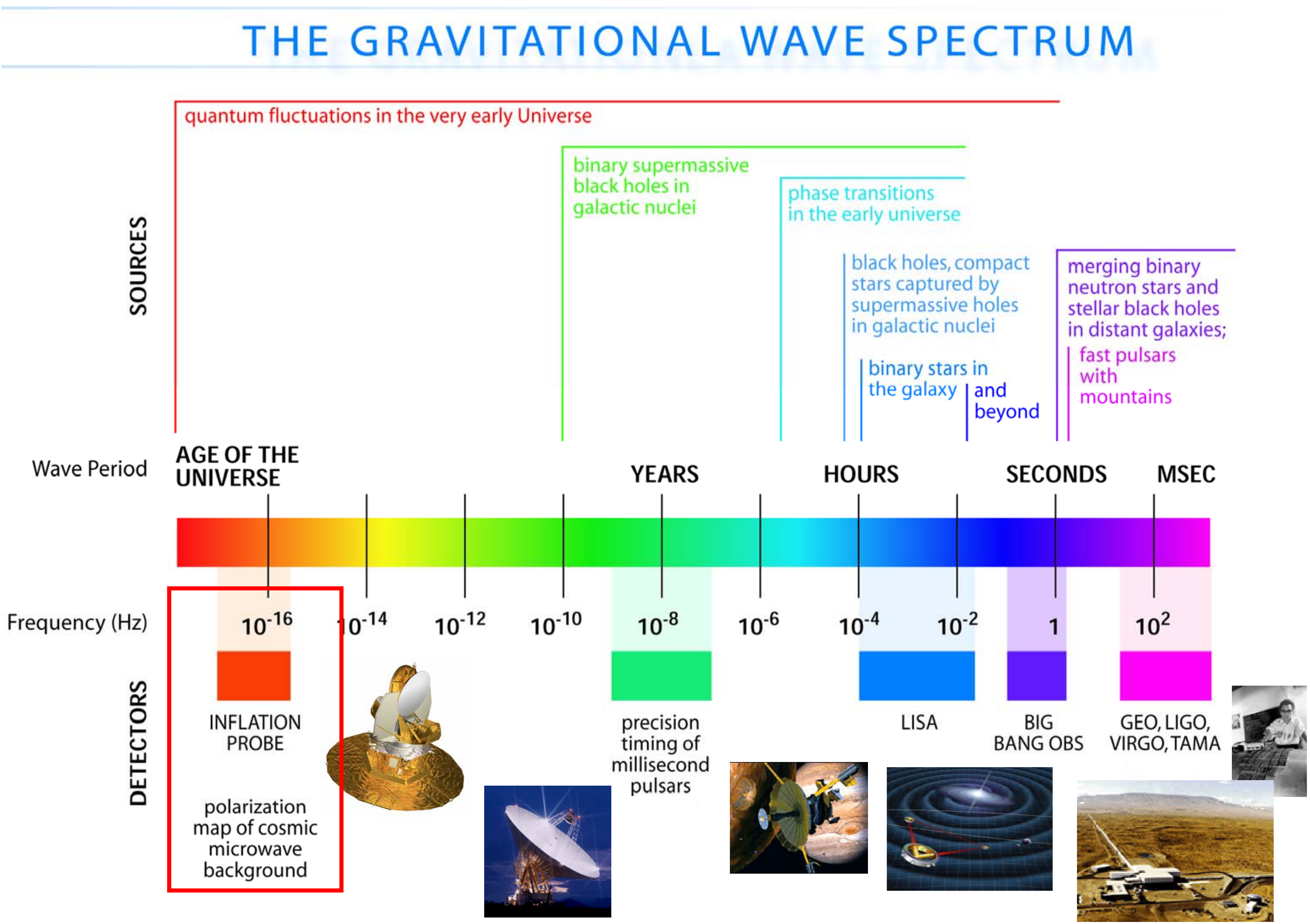
www.skatelescope.org





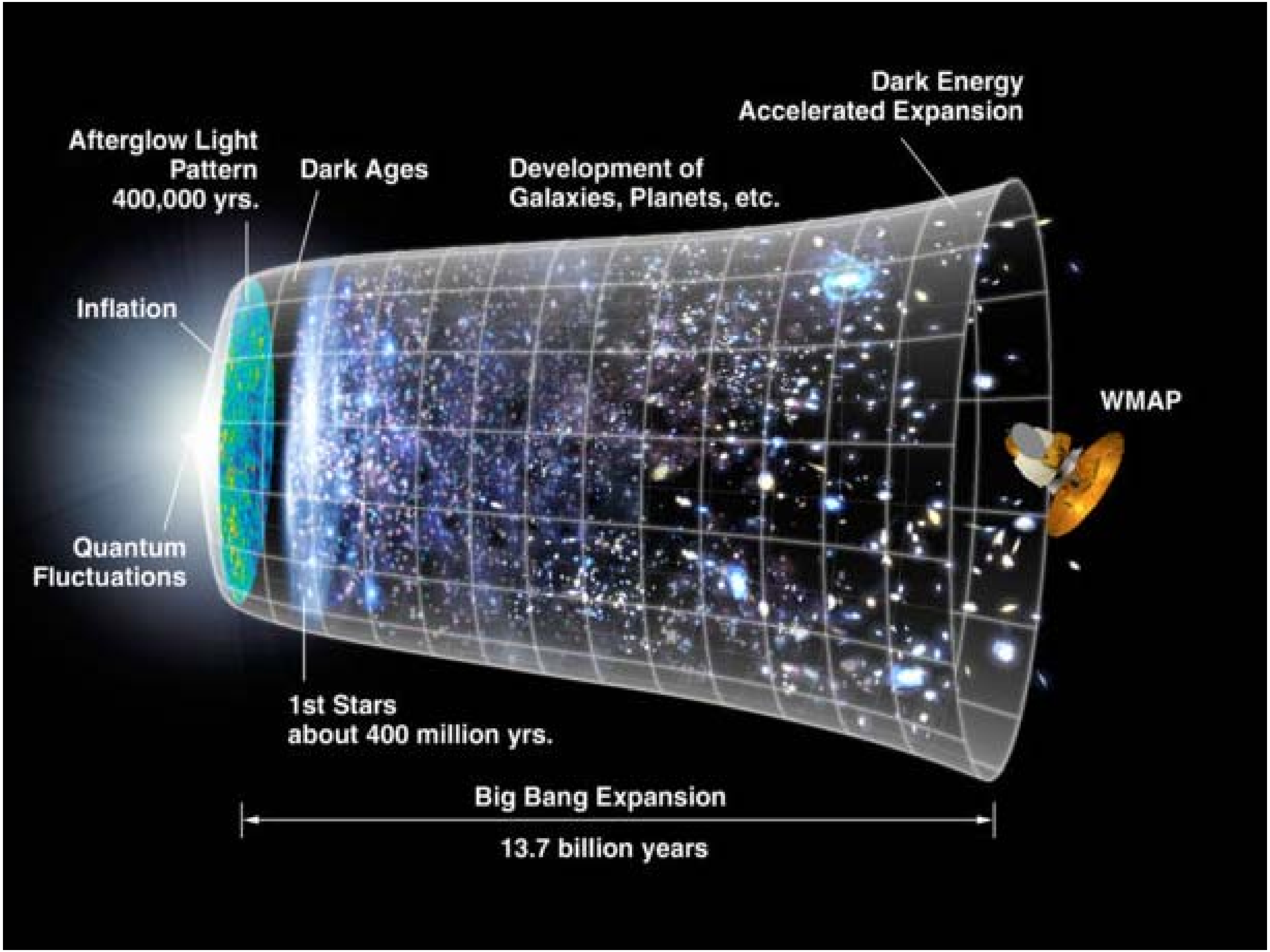



1
10^5
10^10
10^15
1
10^−5
10^−10
10^−15
10^−20
Ω_GW h^2
QSO Astrom.
S1 LIGO limit
LIGO
Adv. LIGO
z var.
M/R Eq.
BBN
Adv. LIGO Corr
msec Pulsar
WMAP
LISA
msec Pulsar
(SKA)
BBO
E_inf
3.4×10^16 GeV
CMB Pol
BBO Corr
5×10^15 GeV
2×10^15 GeV
k_eq
ultimate DECIGO
10^−15
10^−10
10^−5
1
f (Hz)


THE GRAVITATIONAL WAVE SPECTRUM
SOURCES
quantum fluctuations in the very early Universe
binary supermassive black holes in galactic nuclei
phase transitions in the early universe
black holes, compact stars captured by supermassive holes in galactic nuclei
binary stars in the galaxy
and beyond
merging binary neutron stars and stellar black holes in distant galaxies;
fast pulsars with mountains
Wave Period
AGE OF THE UNIVERSE
YEARS
HOURS
SECONDS
MSEC
Frequency (Hz)
10-16
10-14
10-12
10-10
10-8
10-6
10-4
10-2
1
102
DETECTORS
INFLATION PROBE
polarization map of cosmic microwave background
precision timing of millisecond pulsars
LISA
BIG BANG OBS
GEO, LIGO, VIRGO, TAMA

Outline of talk

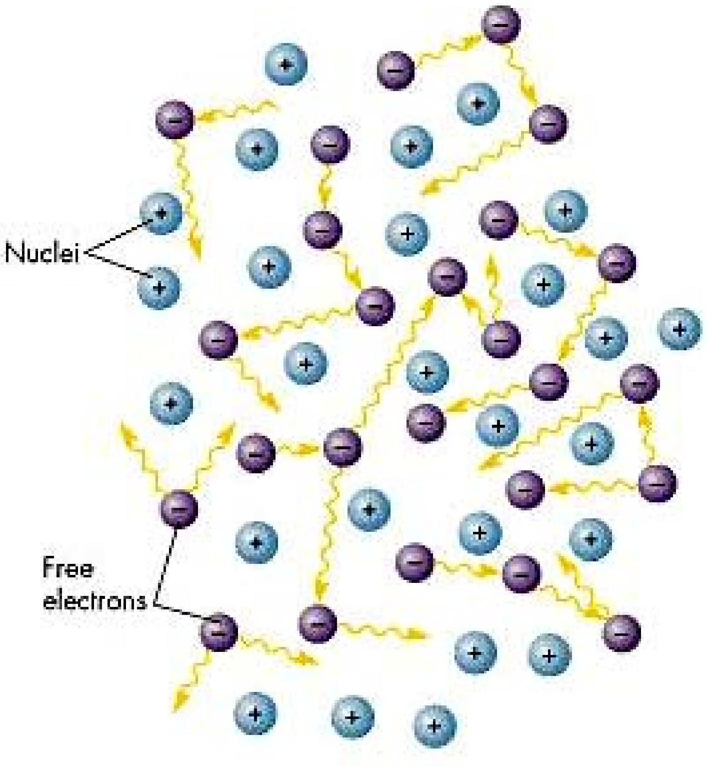
- A Stochastic Background of GWs – basic concepts
- Astrophysical and primordial sources
- Probing the SB with pulsar timing arrays: methods
- Current and future GW limits with PTAs
- **Probing GWs with the CMBR: methods**
- Current and future GW limits with the CMBR

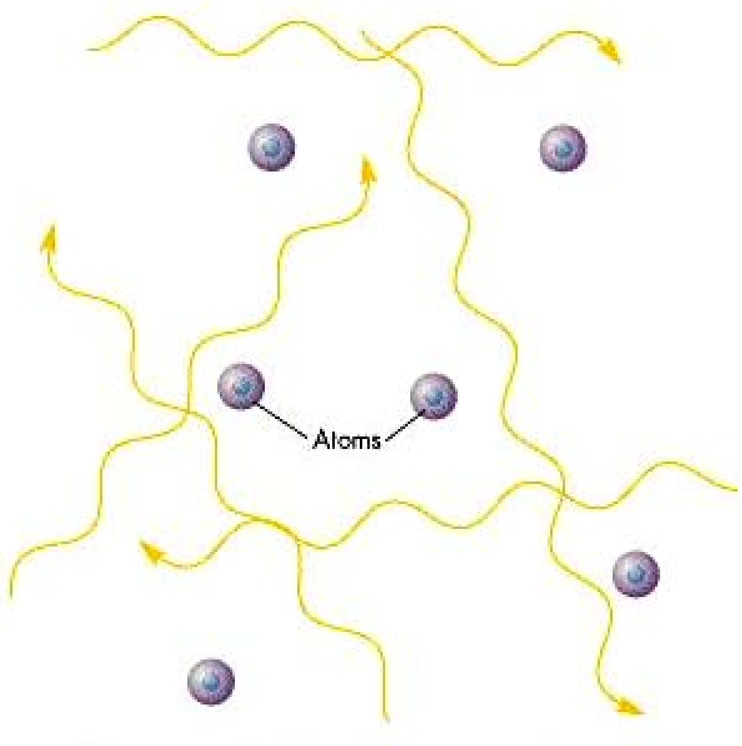

Dark Energy
Accelerated Expansion
Afterglow Light
Pattern
400,000 yrs.
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
WMAP
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.7 billion years

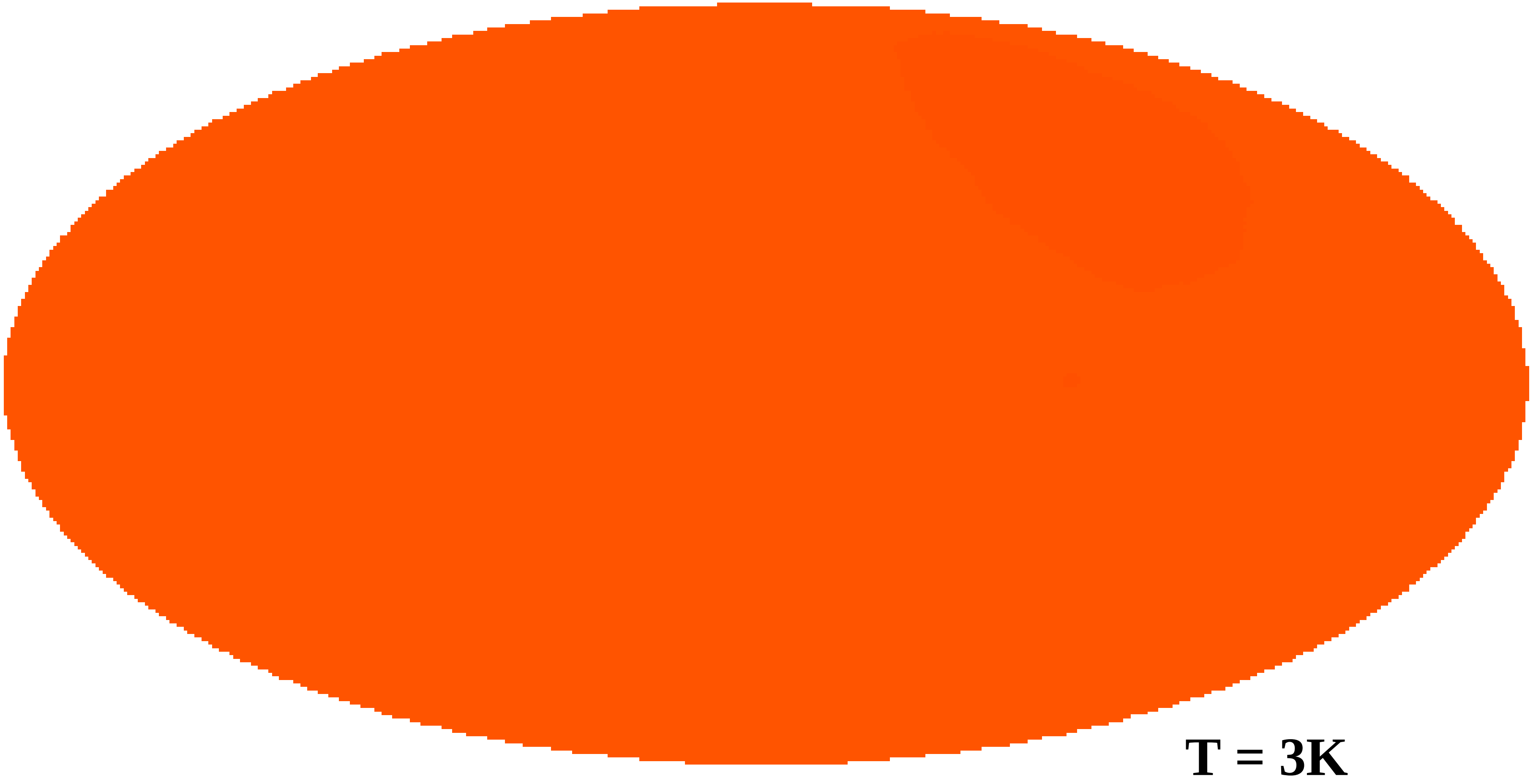
Early Universe too hot for neutral atoms

Free electrons scattered light (as in a fog)

After ~380,000 years, cool enough for atoms; fog clears!







Strong support for the Cosmological Principle:

"The Universe is homogeneous and isotropic on large scales"

Modelling the Universe:-

Background cosmological model described by the Robertson-Walker metric

$$ds^2 = -dt^2 + R(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]$$

$$R(t) = \text{cosmic scale factor}$$

$$\frac{R(t)}{R_0} = \frac{1}{1+z}$$

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} = \text{redshift}$$

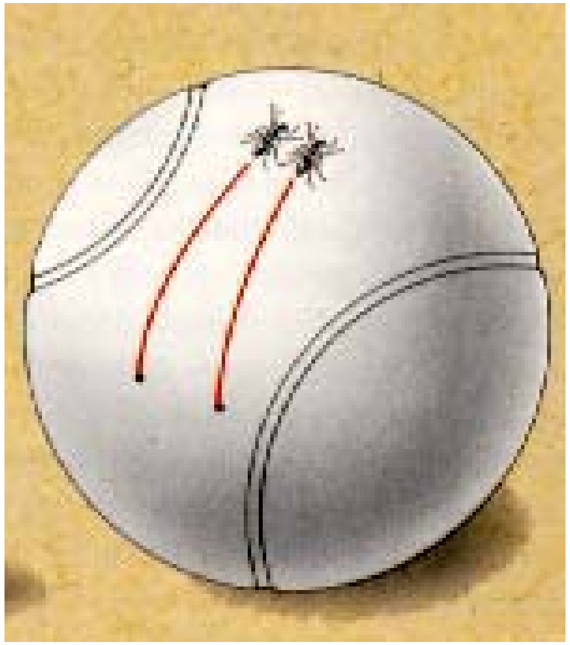
Modelling the Universe:-

Background cosmological model described by the Robertson-Walker metric

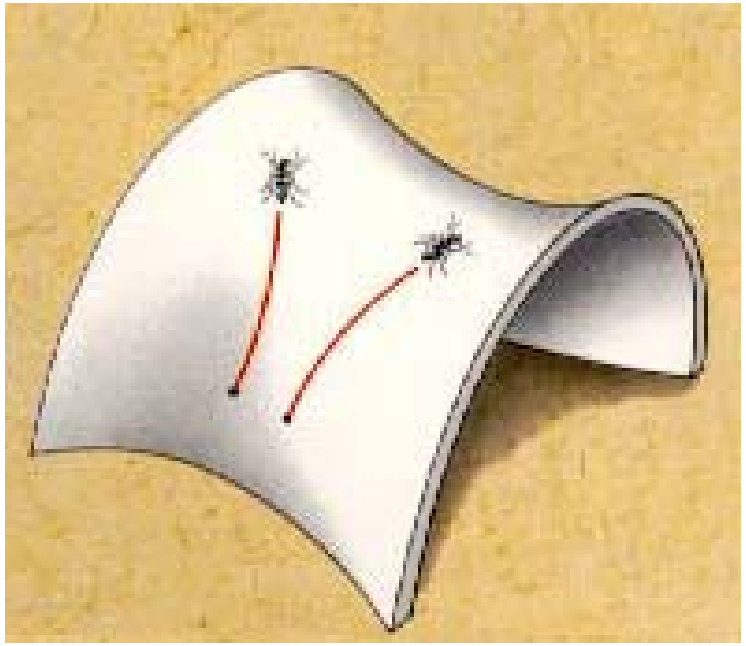
$$ds^2 = -dt^2 + R(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

Metric describes the geometry of the Universe

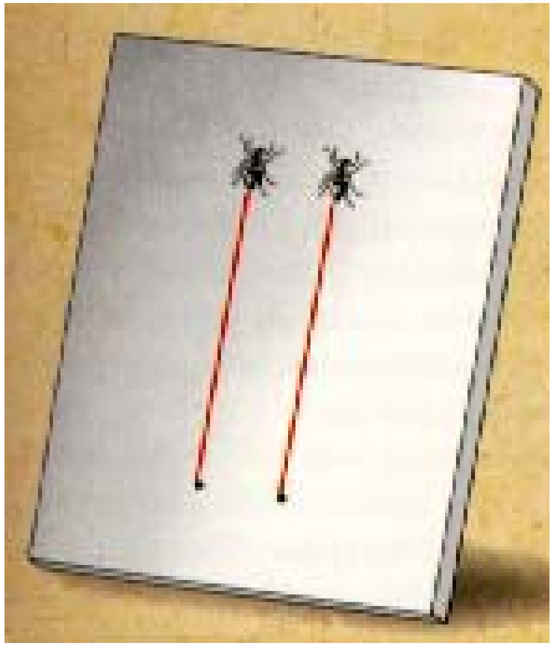
$$k = \text{curvature constant} = \begin{cases} -1, & \text{open} \\ 0, & \text{flat} \\ +1, & \text{closed} \end{cases}$$



Closed



Open



Flat

$$k = \text{curvature constant} = \begin{cases} -1, & \text{open} \\ 0, & \text{flat} \\ +1, & \text{closed} \end{cases}$$

General Relativity:-

Geometry ⟷ matter / energy

> *"Spacetime tells matter how to move and matter tells spacetime how to curve"*

Einstein's Field Equations

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}$$

Einstein tensor

Ricci tensor

Metric tensor

Curvature scalar

Energy-momentum tensor of gravitating mass-energy

General Relativity:-

Geometry ⟷ matter / energy

> *"Spacetime tells matter how to move and matter tells spacetime how to curve"*

Einstein's Field Equations

Given $g^{\mu\nu}$ can compute $R^{\mu\nu}$ and R ;

These are *generated* by $T^{\mu\nu}$

Treat Universe as a *perfect fluid*

$$T^{\mu\nu} = (\rho + P)u^{\mu}u^{\nu} + Pg^{\mu\nu}$$

Density — Pressure — Four-velocity

N.B. $c = 1$

Solve to give *Friedmann's Equations*

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3P)$$

Einstein originally sought *static solution* i.e. :-

$$\dot{R} = 0 \quad \text{for all } t$$

But if $\rho, P \geq 0$ can't have $\ddot{R} = 0$

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But if $\rho, P \geq 0$ can't have $\ddot{R} = 0$

However, GR actually gives

Covariant derivative

$$G^{\mu\nu}{}_{;\nu} = T^{\mu\nu}{}_{;\nu} = 0$$

Can add a constant times $g^{\mu\nu}$ to $G^{\mu\nu}$

Einstein's cosmological constant

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + g^{\mu\nu} \Lambda$$

Friedmann's Equations now give:-

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - \frac{k}{R^2}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}$$

Can tune Λ to give $\dot{R} = 0$ for all t but *unstable*

(and *Hubble expansion* made idea redundant)

Einstein’s greatest blunder?

Friedmann's Equations now give:-

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - \frac{k}{R^2}$$

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Can tune Λ to give $\dot{R} = 0$ for all t but *unstable*

(and *Hubble expansion* made idea redundant)

But Lambda term could still be non-zero anyway !

Can instead think of Lambda term as added to energy-momentum tensor:-

$$T^{\mu\nu} \rightarrow T^{\mu\nu} + g^{\mu\nu}\Lambda$$

But what is Λ ?…

Particle physics motivates Λ as *energy density of the vacuum* but scaling arguments suggest:-

$$\left|\frac{\rho_\Lambda(\text{obs})}{\rho_\Lambda(\text{theory})}\right| \geq 10^{-120}$$

So historically it was easier to believe $\Lambda = 0$

Re-expressing Friedmann's Equations:-

For $\Lambda = 0$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2} \quad \Rightarrow \quad k = 0 \Leftrightarrow \rho = \left[\frac{8\pi G}{3H^2}\right]^{-1} = \rho_{\text{crit}}$$

Define

$$\Omega_m = {\rho}/{\rho_{crit}} = \frac{8\pi G\rho}{3H^2} \qquad \Omega_\Lambda = \frac{\Lambda}{3H^2} \qquad \Omega_k = -\frac{k}{R^2 H^2}$$

It follows that, at *any time*

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1$$

Re-expressing Friedmann's Equations:-

Consider *pressureless* fluid (dust); assume *mass conservation*

$$\rho R^3 = \rho_0 R_0^3 = \text{ constant} \quad \Rightarrow \rho = \rho_0 \, {R_0^3}/{R^3} = \rho_0 (1+z)^3$$

and

$$\Omega_m = {\rho}/{\rho_{crit}} = \frac{8\pi G \rho_0 (1+z)^3}{3H^2} \frac{H_0^2}{H_0^2} = \Omega_{m0} \frac{H_0^2}{H^2} (1+z)^3$$

More generally:-

$$H = H_0 \left(\sum_i \Omega_{i0} (1+z)^{n_i} \right)^{1/2}$$

	n_i
Matter	3
Radiation	4
Curvature	2
Vacuum	0

Expansion rate dominated by different terms at different redshifts

"Concordance model" predicts:-

$$\Omega_{k0} = 0 \qquad \Omega_{m0} \approx 0.3 \qquad \Omega_{\mathrm{rad},0} = 0 \qquad \Omega_{\Lambda 0} \approx 0.7$$

But at redshift, $z = 2$

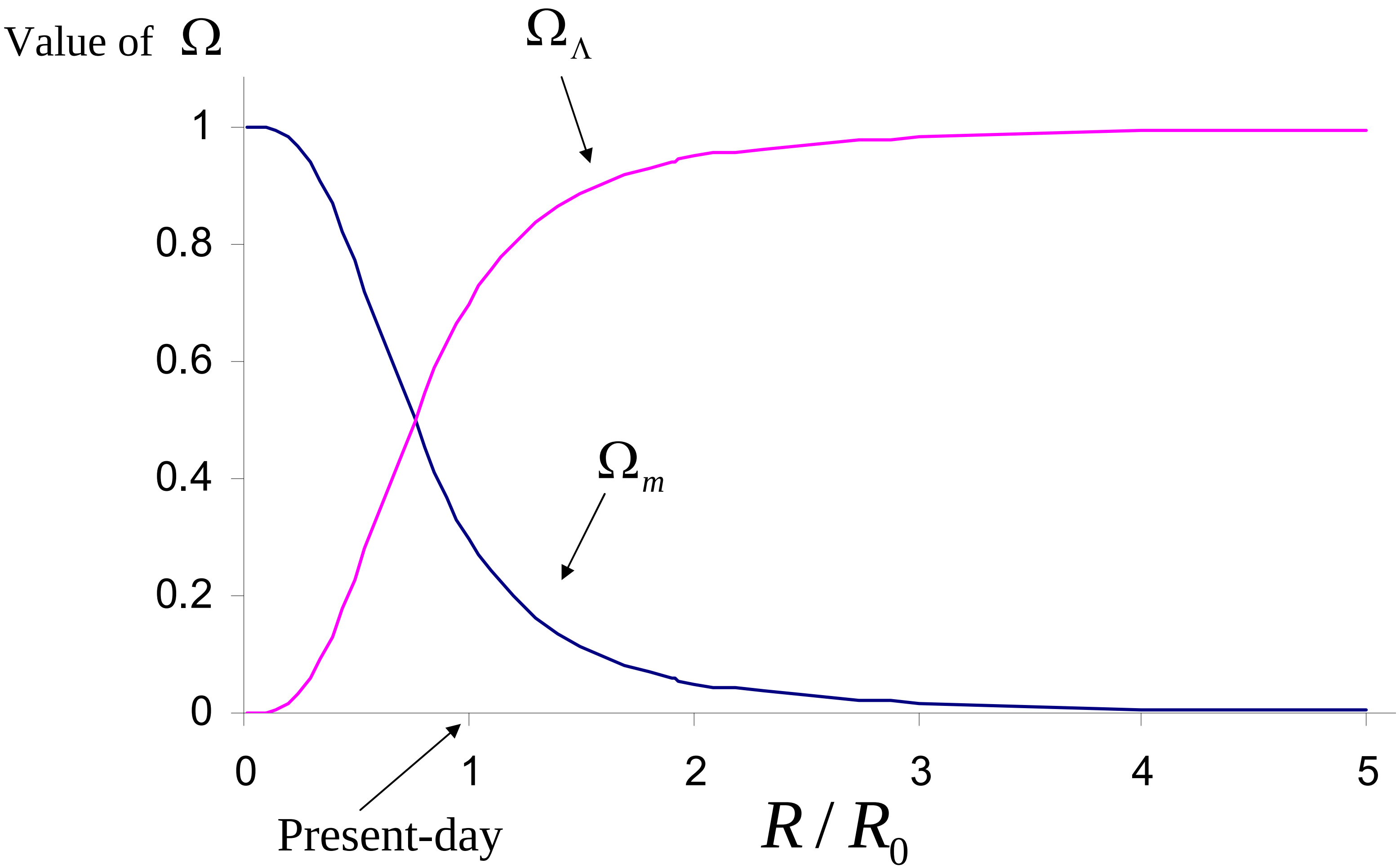
$$\Omega_{k2} = 0 \qquad \Omega_{m2} = 0.92 \qquad \Omega_{\mathrm{rad},2} = 0 \qquad \Omega_{\Lambda 2} = 0.08$$

At redshift, $z = 6$

$$\Omega_{k6} = 0 \qquad \Omega_{m6} = 0.993 \qquad \Omega_{\mathrm{rad},6} = 0 \qquad \Omega_{\Lambda 6} = 0.007$$

And in about another 15 billion years

$$\Omega_{k} = 0 \qquad \Omega_{m} = 0.05 \qquad \Omega_{\mathrm{rad}} = 0 \qquad \Omega_{\Lambda} = 0.95$$



If the Concordance Model is right, we live at a special epoch. Why?...

This has led to more general Dark Energy or Quintessence models:

Evolving scalar field which 'tracks' the matter density

Convenient parametrisation: 'Equation of State'

$$P = w\,\rho$$

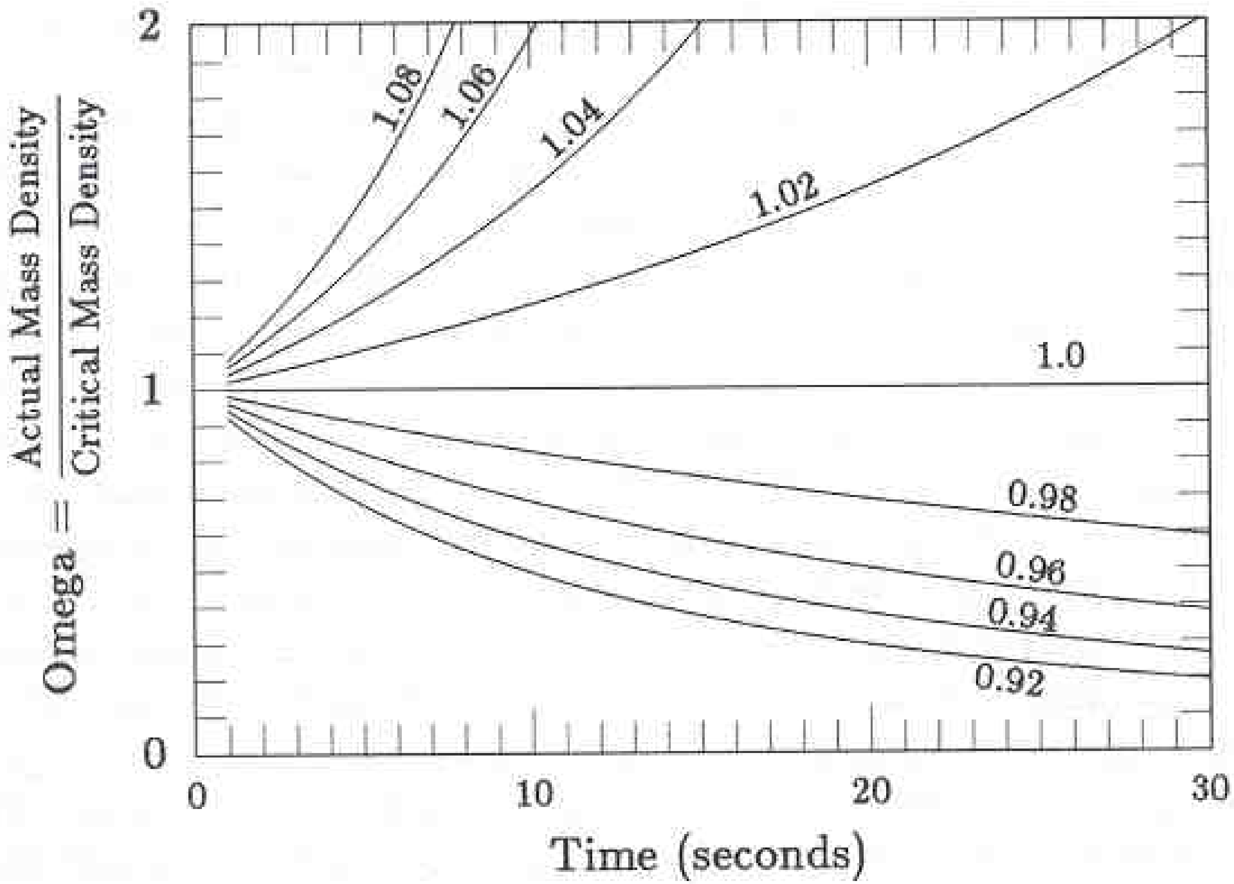
$$H = H_0 \left(\sum_i \Omega_{w_i 0} (1+z)^{3(1+w_i)} \right)^{1/2}$$

	w_i
Matter	0
Radiation	1/3
Curvature	-1/3
'Lambda'	-1
Quintessence	$w(z)$

Can we measure $w(z)$?

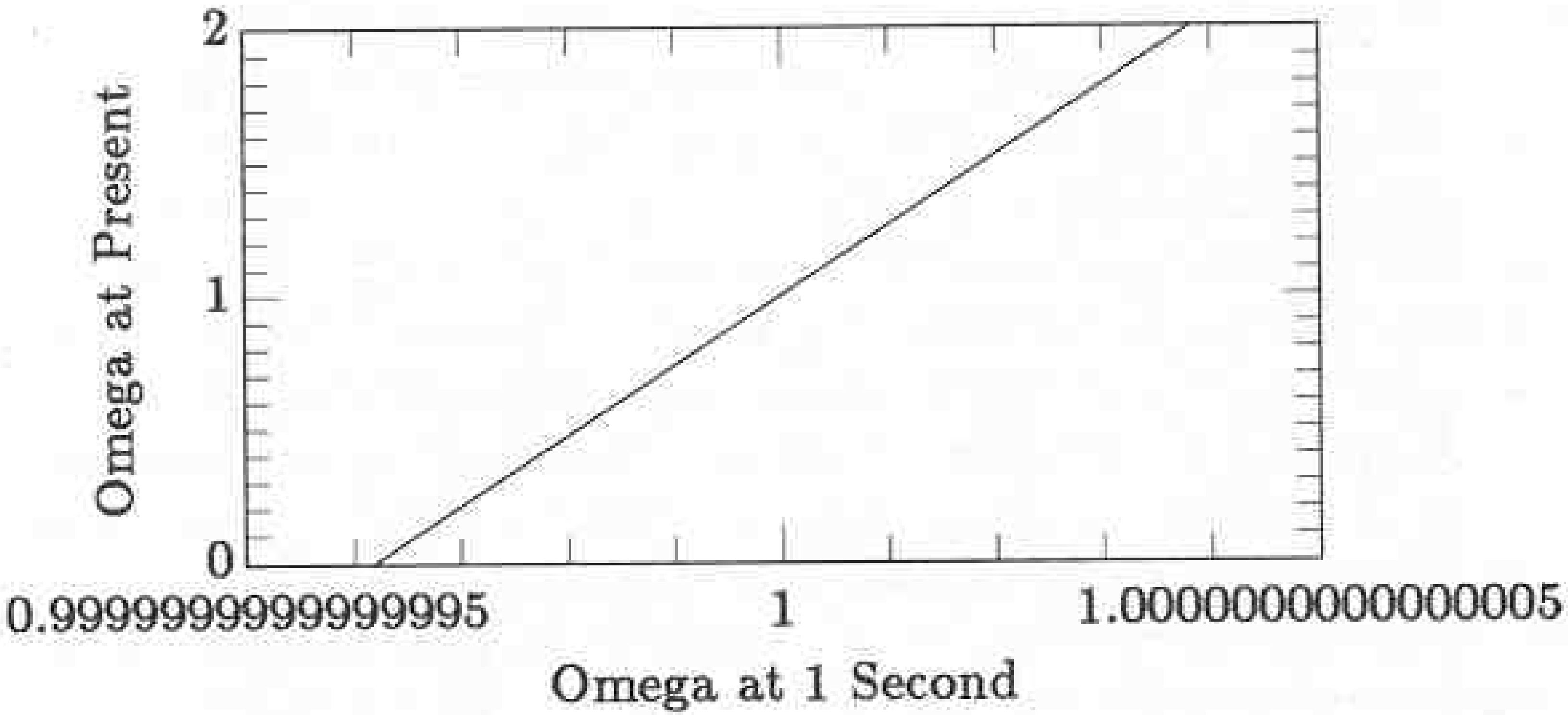
Inflation for astronomers

We have been considering $\Omega_{k0} = 0$ but suppose that in the past $\Omega_k \neq 0$. From the Friedmann equations it would then be very difficult to explain why it is so close to zero today.



Present day 'closeness' of matter density to the critical density appears to require an incredible degree of 'fine tuning' in the very early Universe.

FLATNESS PROBLEM



Inflationary solution to the Flatness Problem

Suppose that in the very early Universe: $\Omega_{k,\text{init}} \neq 0 \quad \Omega_{\text{rad,init}} \neq 0$

Suppose there existed $\Omega_{\text{vac,init}} \neq 0$

Easy to show that:-

$$\left|\frac{\Omega_k}{\Omega_{\text{vac}}}\right| = \left(\frac{R_{\text{init}}}{R}\right)^2 \qquad \left|\frac{\Omega_{\text{rad}}}{\Omega_{\text{vac}}}\right| = \left(\frac{R_{\text{init}}}{R}\right)^4$$

i.e. vacuum energy will dominate as the Universe expands, and drives Ω_k to zero

$$\left(\frac{\dot{R}}{R}\right) \approx \sqrt{\frac{\Lambda}{3}} \Rightarrow R \propto \exp(Ht)$$

De Sitter solution; exponential growth

So if we can invoke a physical mechanism in the early Universe which gives an equation of state $P = -\rho$ it can solve the flatness and horizon (and other) problems.

Lots of hard physics problems!!

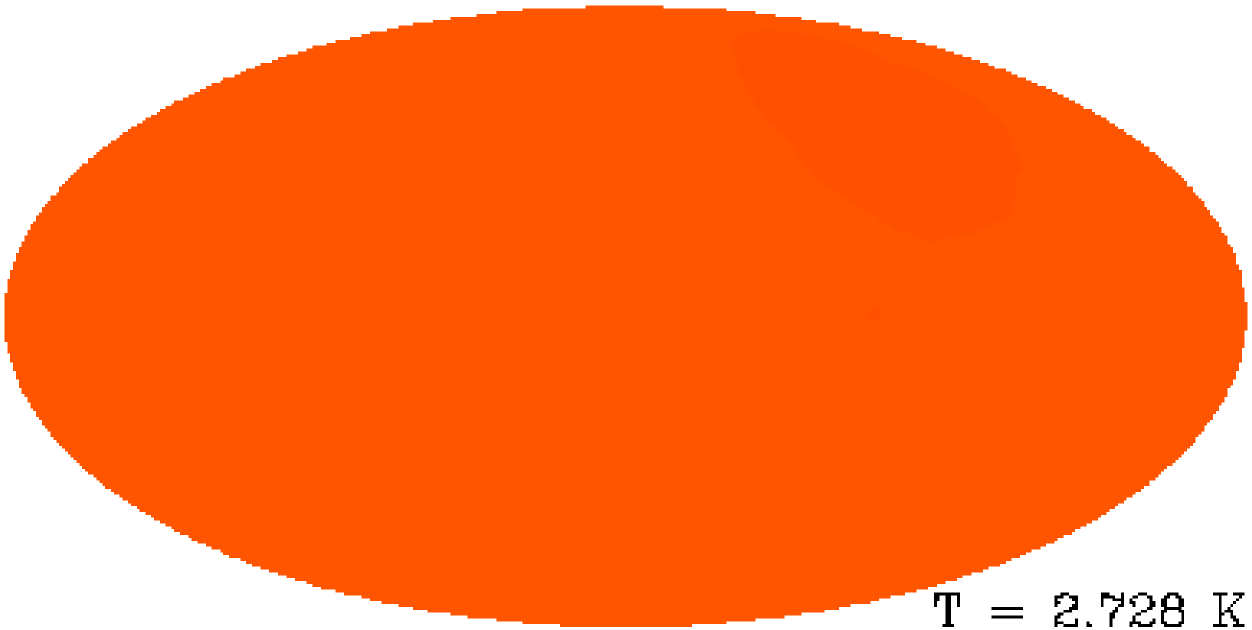
- **What exactly is the mechanism that starts inflation?**
- **When does it happen?**
- **What causes inflation to *stop* ?**
- **What happens when inflation stops?**

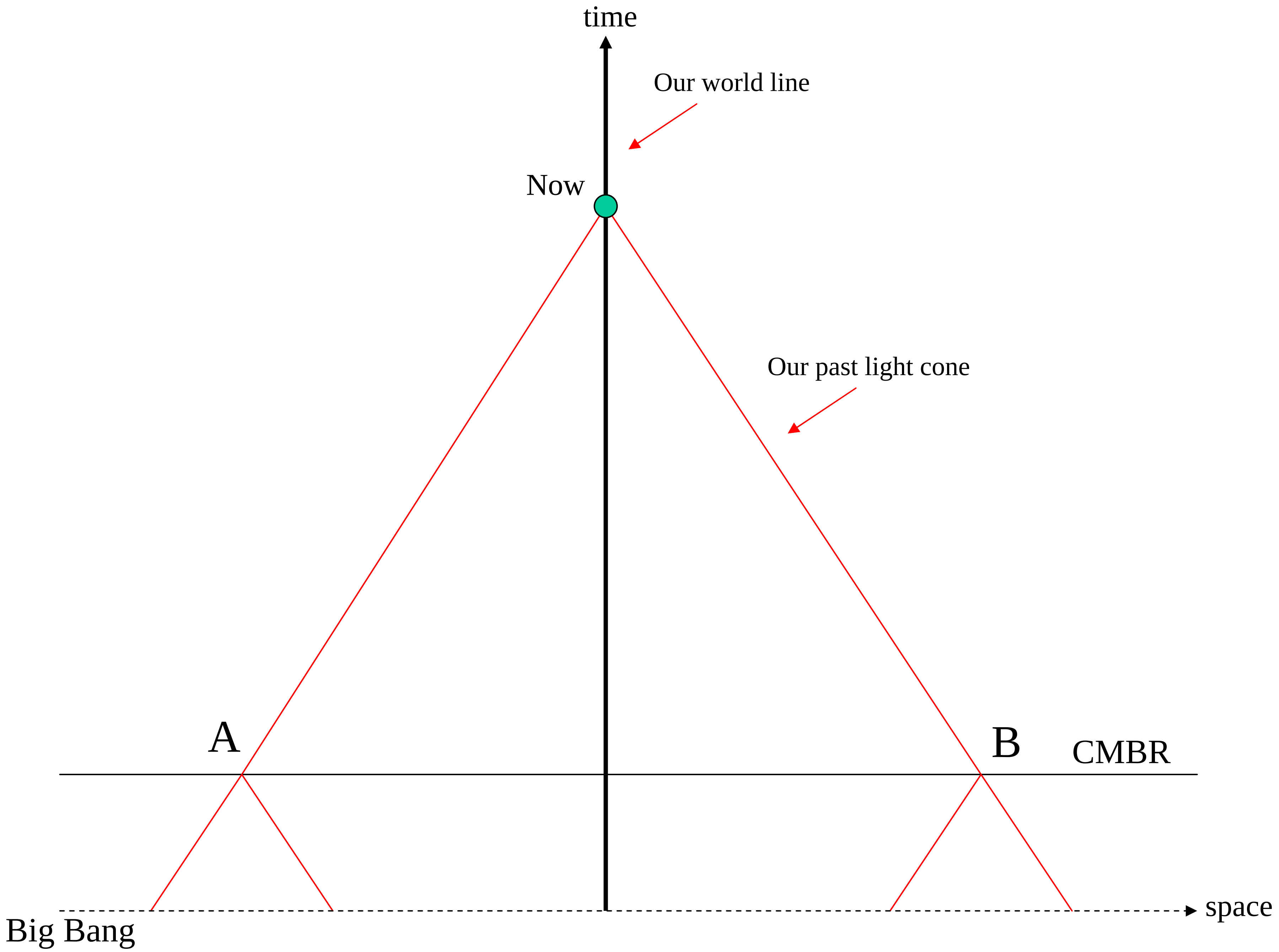
Some specific **predictions** :-

- **The post-inflationary Universe has a flat geometry**
- **The PIU is imprinted with quantum fluctuations in density and temperature, inflated to macroscopic scales and with a particular statistical pattern (seen in the CMBR).**

How do we explain the isotropy of the CMBR, when opposite sides of the sky were ‘causally disconnected’ when the CMBR photons were emitted?

HORIZON PROBLEM



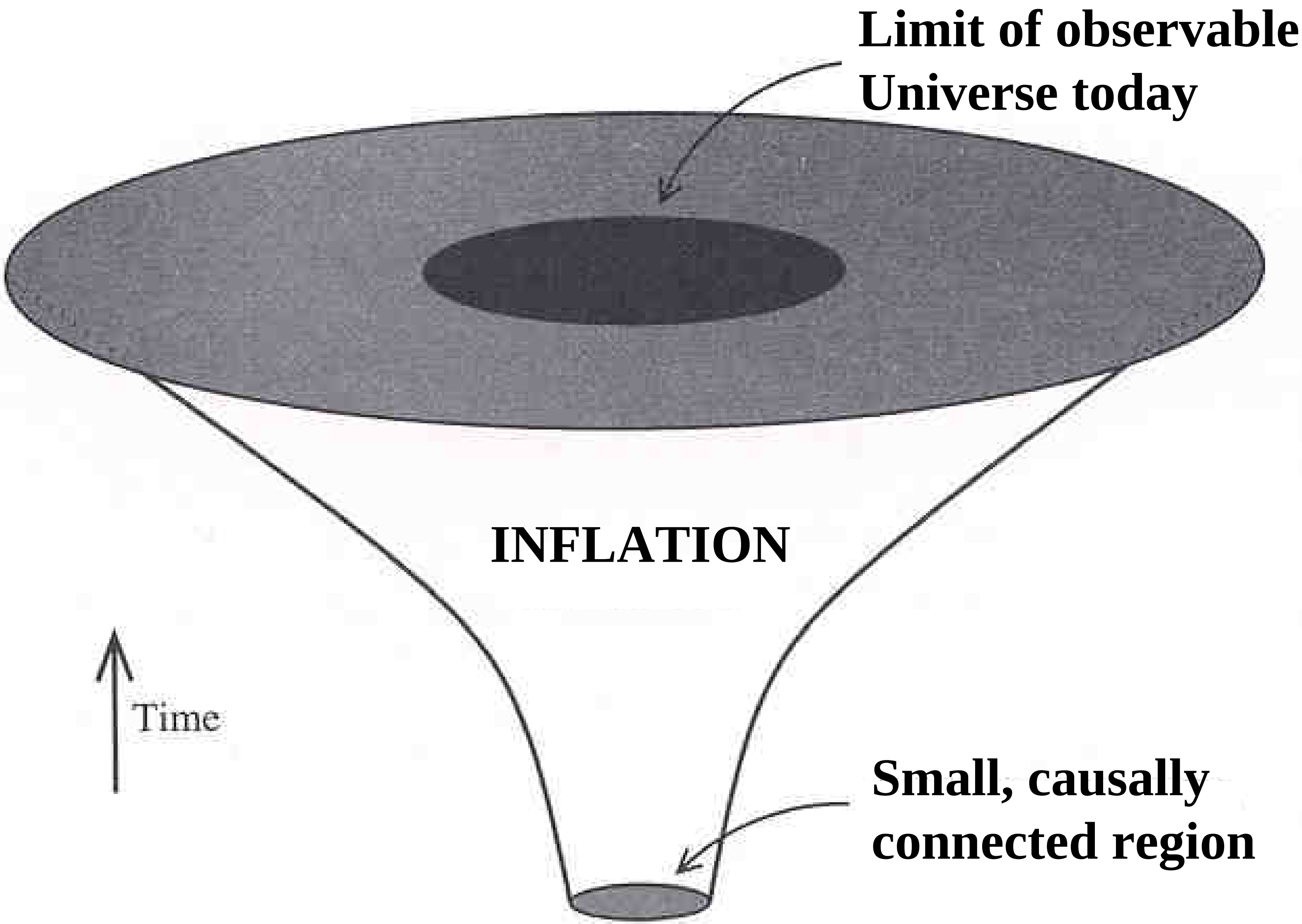

time
Our world line
Now
Our past light cone
A
B
CMBR
space
Big Bang

Solution (first proposed by Guth and Starobinsky in the early 1980s) is…

INFLATION

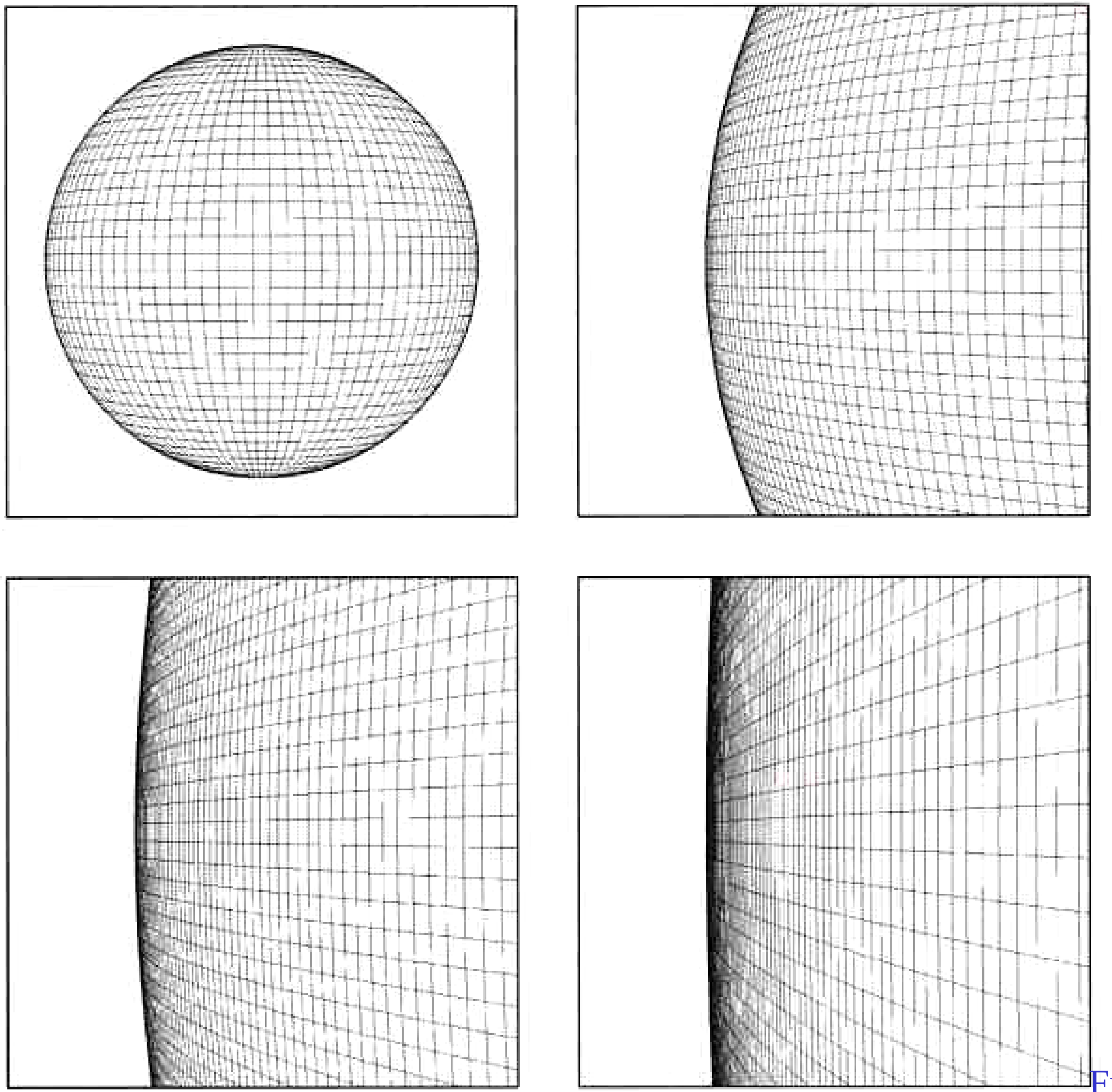
…a period of accelerated expansion in the very early universe.

Inflationary solution to the Horizon Problem



From Guth (1997)

Inflationary solution to the Flatness Problem

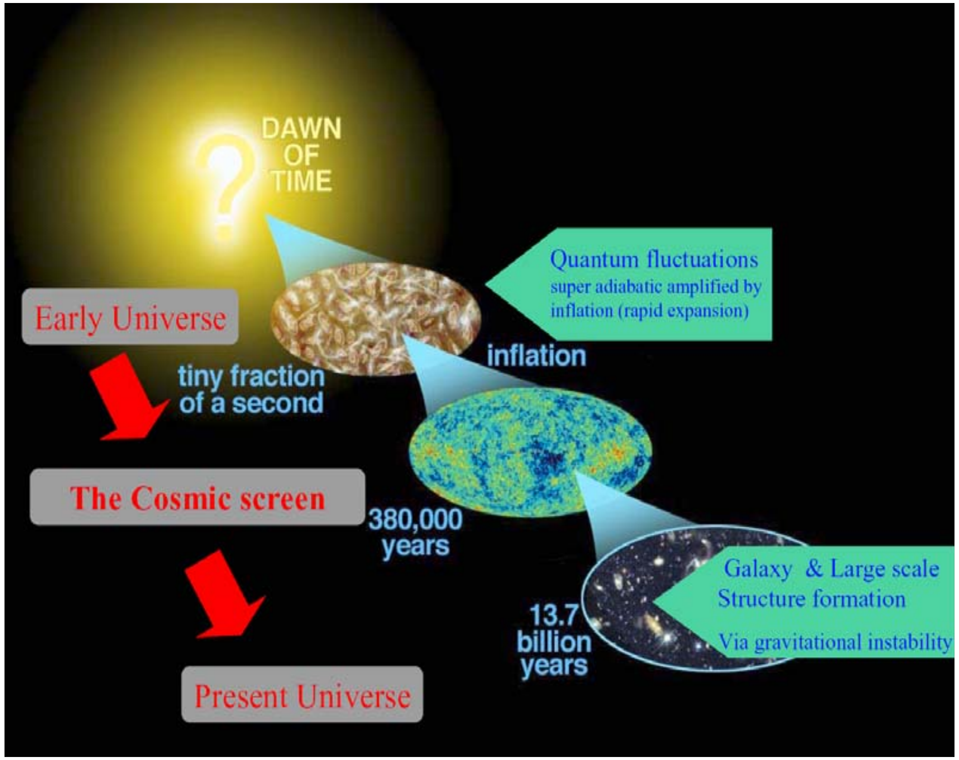


From Guth (1997)

Origin of the Stochastic Background

Cosmological Inflation

- Inflation also provides a mechanism for generating large scale structure in the Universe.
- Primordial quantum fluctuations become the ‘seeds’ of structure that we see in the CMBR.
- These fluctuations are both **scalar** (density perturbations) and **tensor** (gravitational waves).
- We can hope to measure the latter directly, and by the imprint they leave on the temperature distribution of the CMBR (see later).



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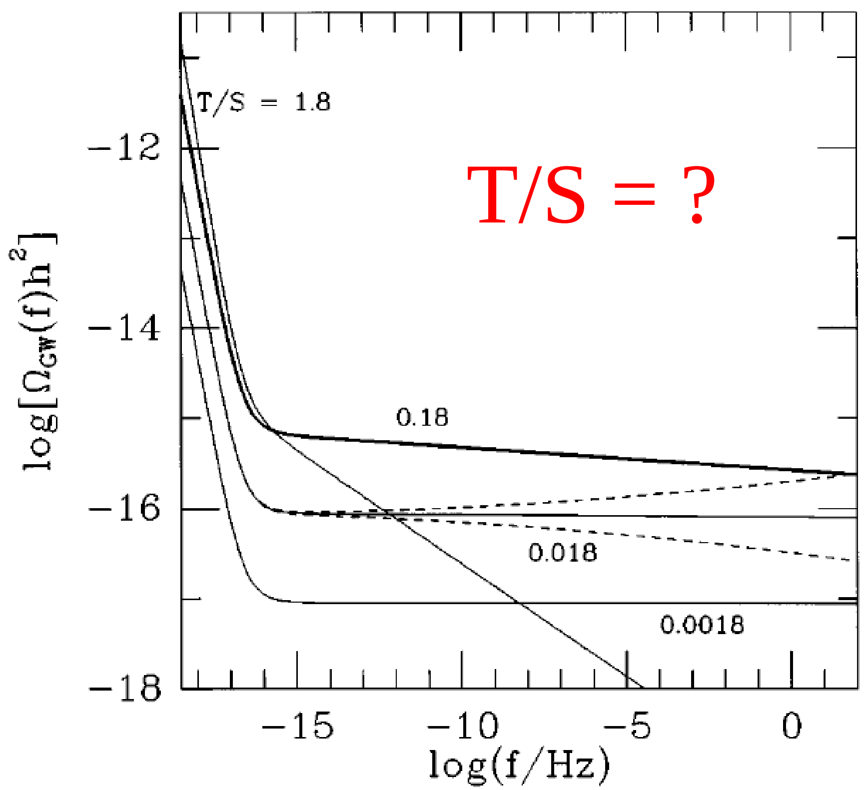
Turner (1997)



Outline of talk

- A Stochastic Background of GWs – basic concepts
- Astrophysical and primordial sources
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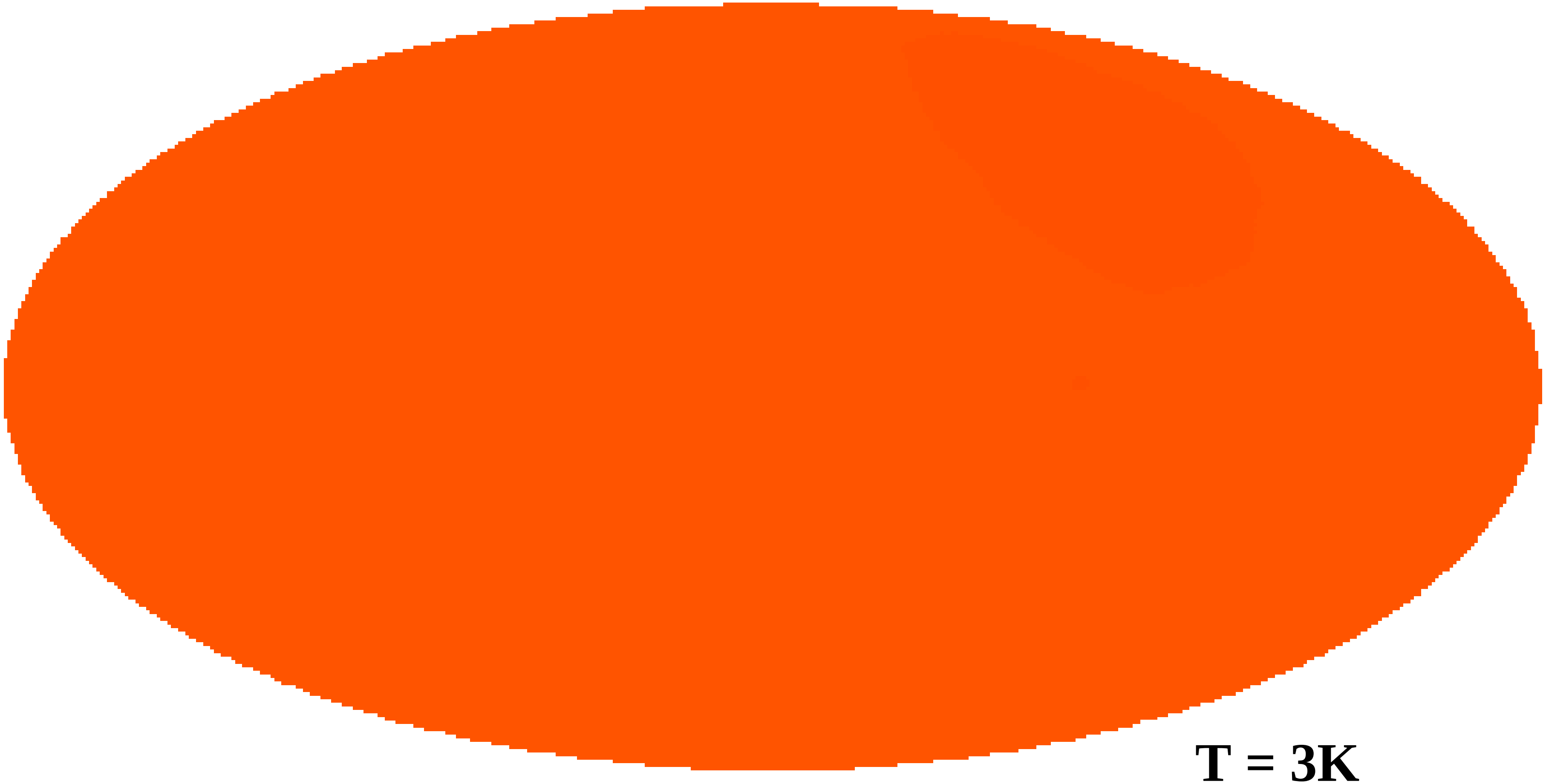
- Current and future GW limits with PTAs
- Probing GWs with the CMBR: methods
- **Current and future GW limits with the CMBR**



Strong support for the Cosmological Principle:

"The Universe is homogeneous and isotropic on large scales"

What can we constrain with CMBR data?

It is usually a good approximation to take the power spectra as being power-laws with scale. So

Density perturbations
$$A_S^2(k) = A_S^2(k_0)\left[\frac{k}{k_0}\right]^{n_S - 1}$$

Gravitational waves
$$A_T^2(k) = A_T^2(k_0)\left[\frac{k}{k_0}\right]^{n_T}$$

Following Melchiorri (2008)

What can we constrain with CMBR data?

The 4 parameters are related to the inflaton potential and to its first two derivatives:

$$n_S - 1 \approx -\frac{m_{Pl}^2}{8\pi}\left(\frac{V_*'}{V_*}\right)^2 + \frac{m_{Pl}^2}{4\pi}\left(\frac{V_*'}{V_*}\right)'$$

$$n_T \approx -\frac{m_{Pl}^2}{8\pi}\left(\frac{V_*'}{V_*}\right)^2$$

$$A_T \approx 0.61\frac{V_*}{m_{Pl}^4}$$

$$A_S \approx -\frac{1}{7}\frac{A_T}{n_T}$$

$$r = \frac{A_T}{A_S}$$

Following Melchiorri (2008)

CMBR fluctuations

- The spectrum of density perturbations produces a pattern of temperature fluctuations on the sky.

Decompose temperature fluctuations in spherical harmonics

$$\frac{\Delta T}{\bar{T}}(\vec{\varphi}) = \sum_{\ell,m} a_{\ell m} Y_{\ell m}(\vec{\varphi})$$

Spherical harmonics

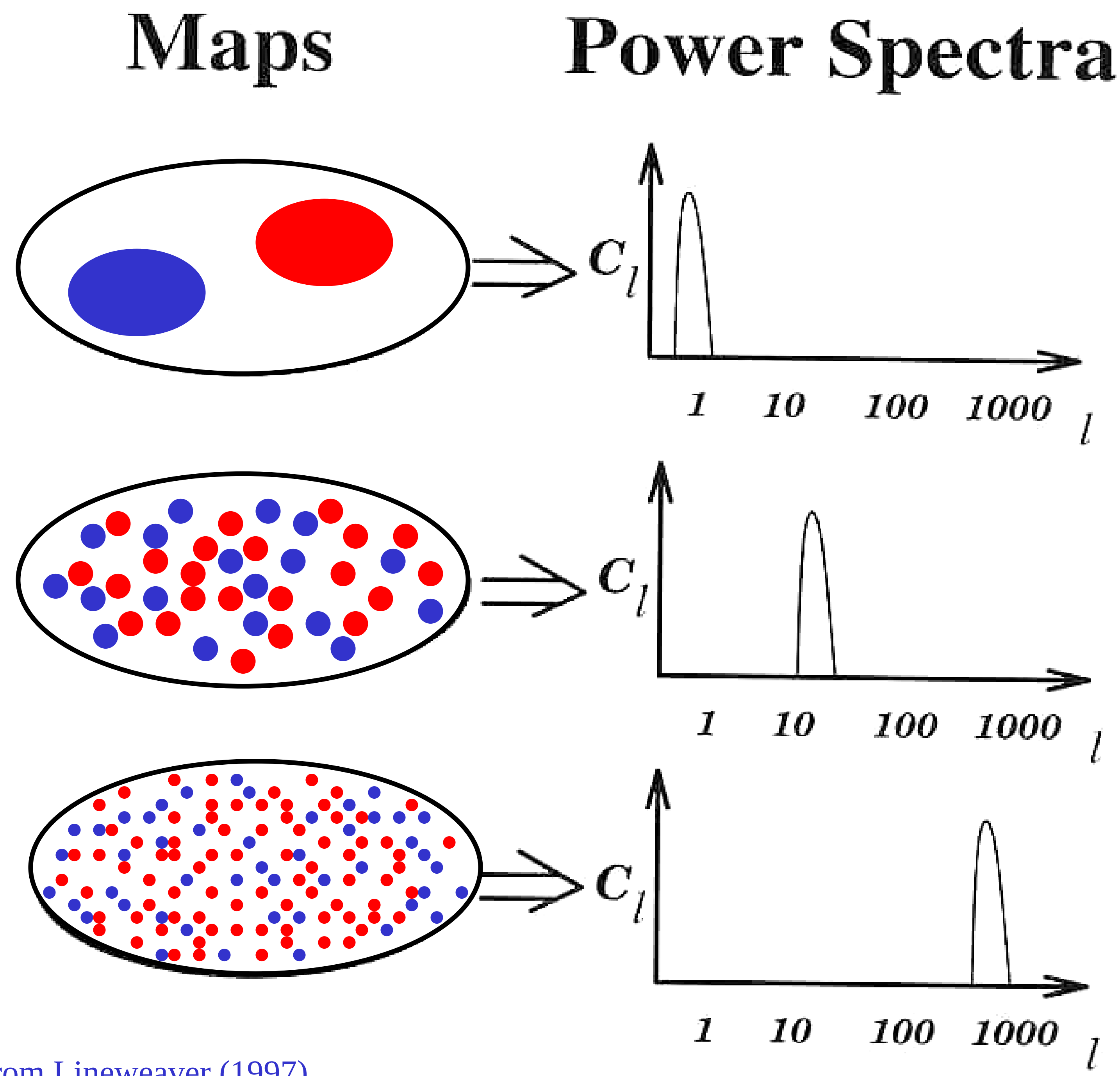
define angular 2-point correlation function:-

$$C(\theta) \equiv \left\langle \frac{\Delta T}{\bar{T}}(\vec{\phi}_1) \frac{\Delta T}{\bar{T}}(\vec{\phi}_2) \right\rangle_{\vec{\phi}_1 \cdot \vec{\phi}_2 = \cos\theta} = \frac{1}{4\pi} \sum_{\ell} (2\ell+1) C_\ell P_\ell(\cos\theta)$$

Legendre polynomials

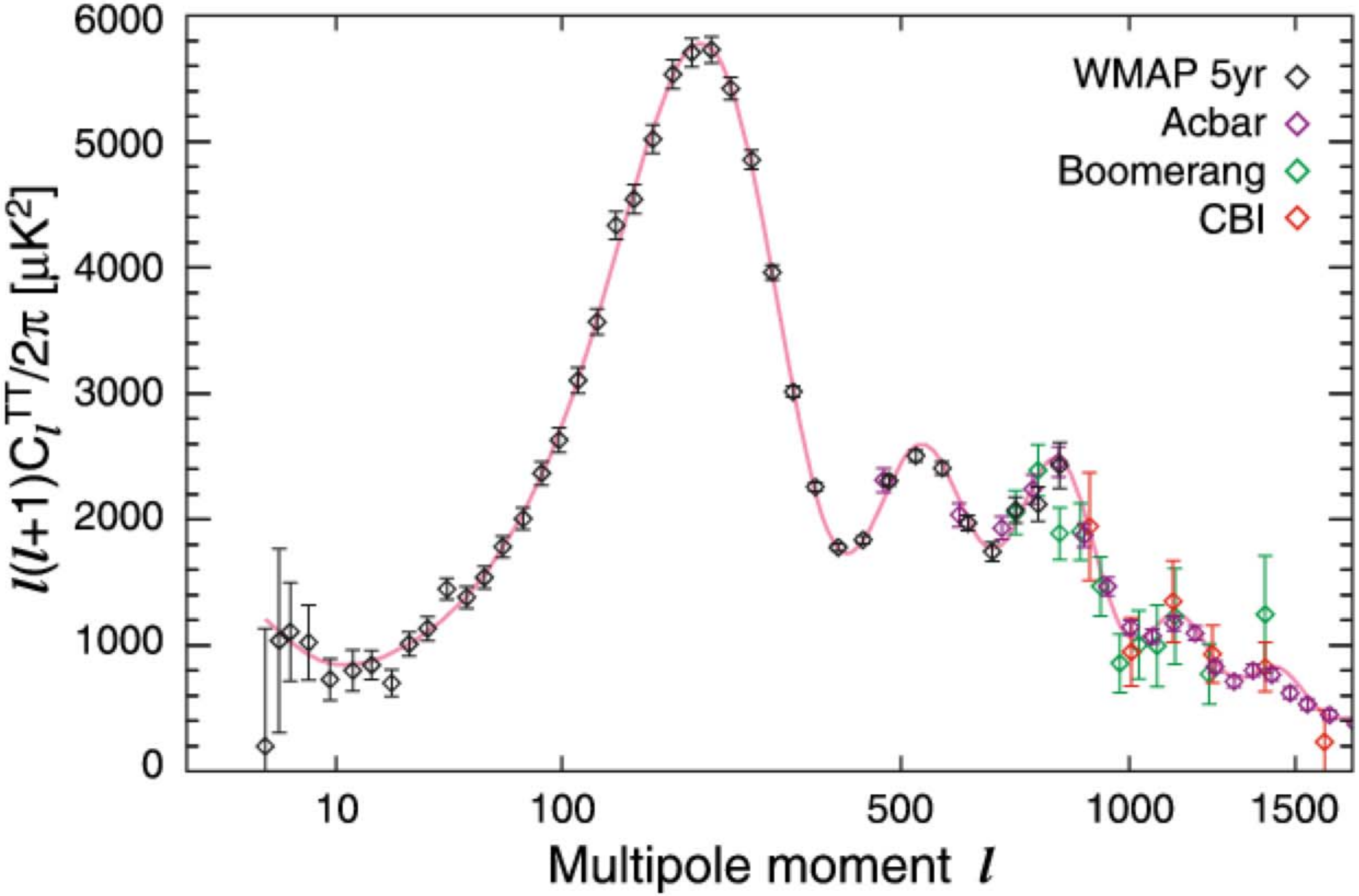
$$C_\ell = \frac{1}{2\ell+1} \sum_m \left\langle |a_{\ell m}|^2 \right\rangle$$

= angular power spectrum



Adapted from Lineweaver (1997)

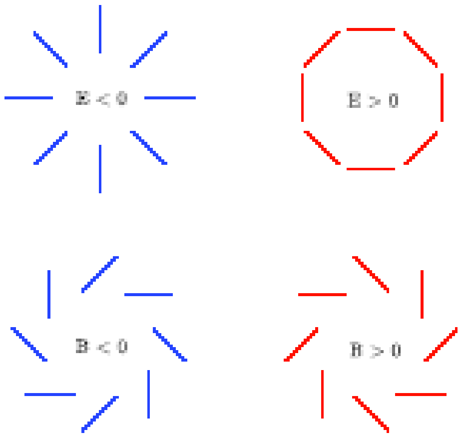
What can we constrain with CMBR data?



Because Thomson scattering is anisotropic, the CMBR is **polarised**.

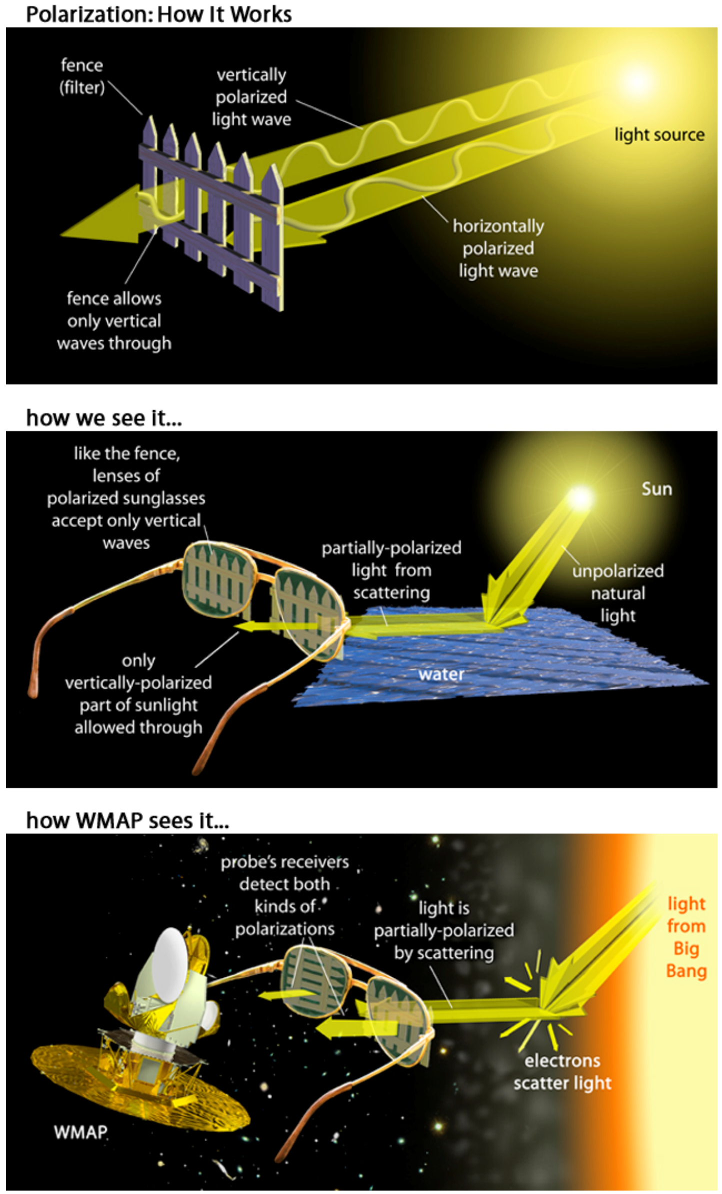
We can decompose the polarisation field into E and B modes.

$$P(\hat{n}) = \nabla E + \nabla \times B$$

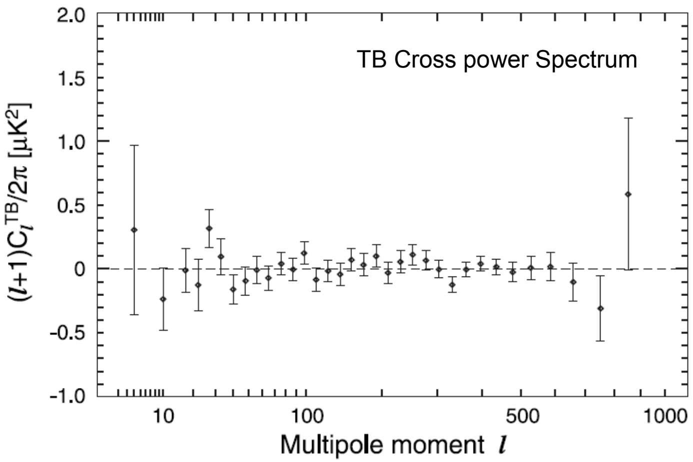


Grad

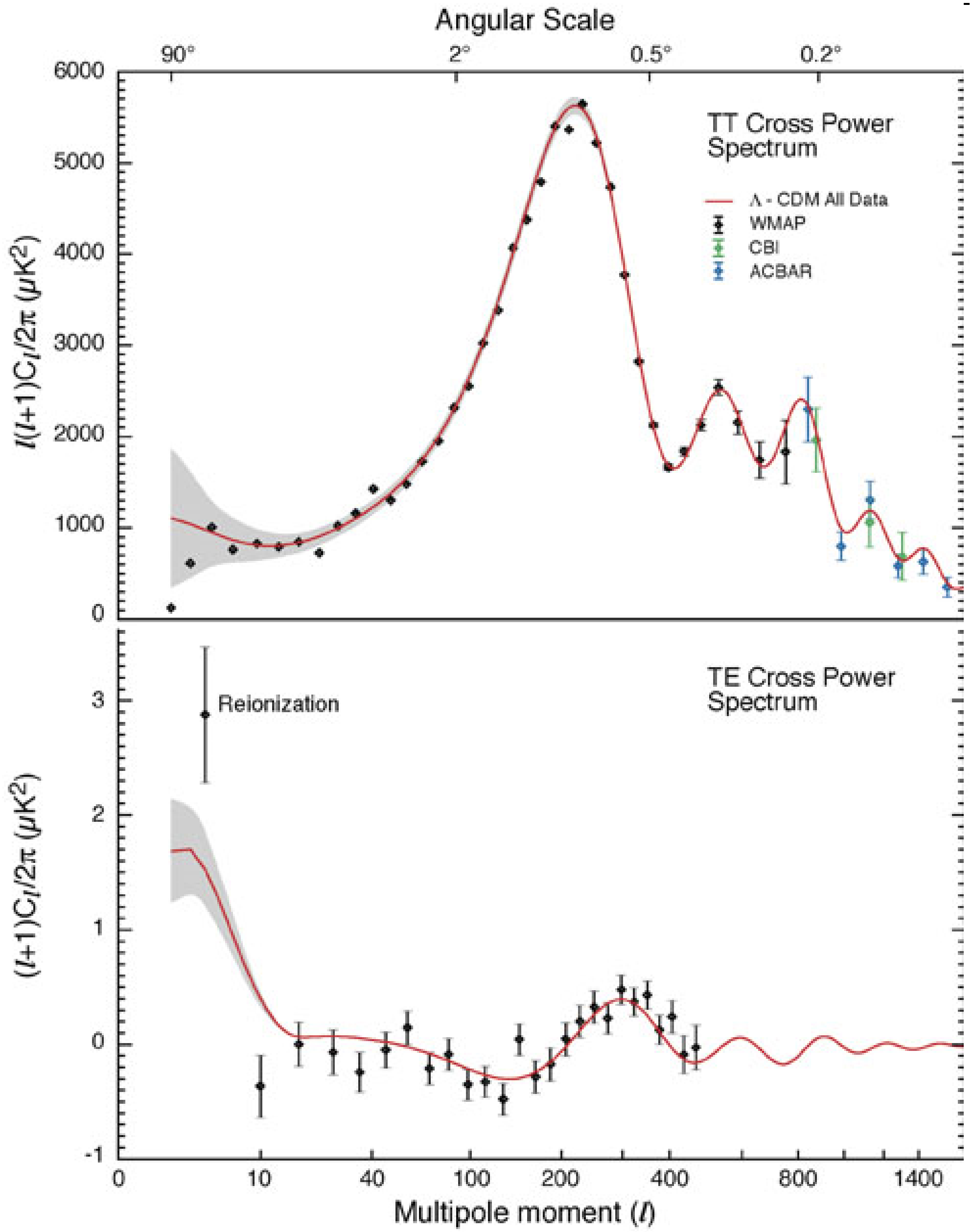
Curl



WMAP is not sensitive enough to detect a B mode signal, but *has* measured an E mode signal.



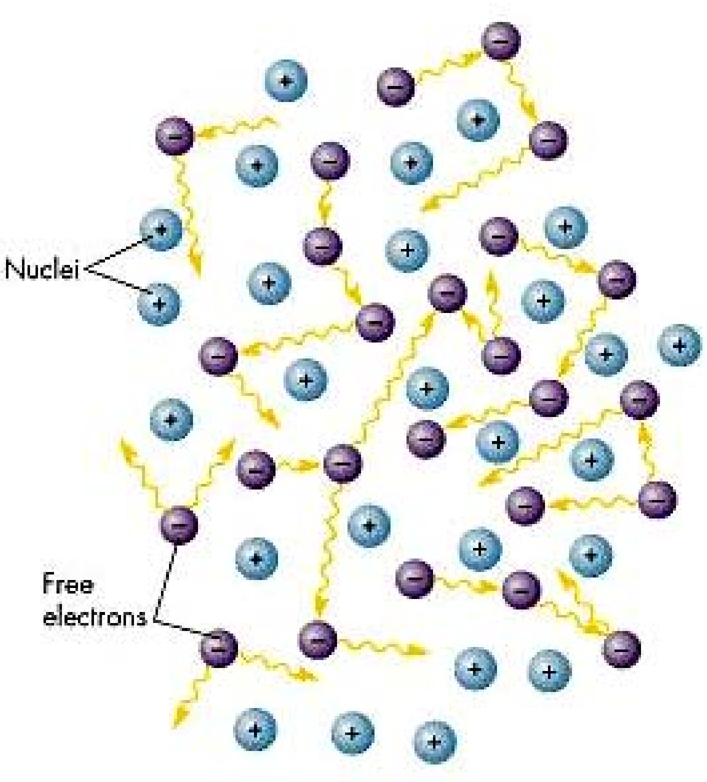
The strong peak in the TE Spectrum due to re-ionization means that the T/S ratio is rather degenerate with the optical depth of re-ionization.

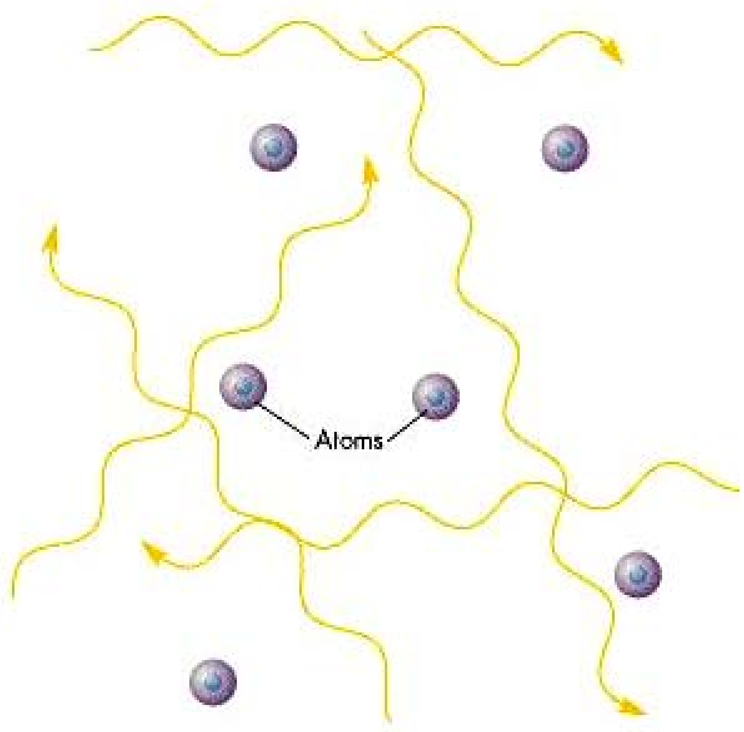


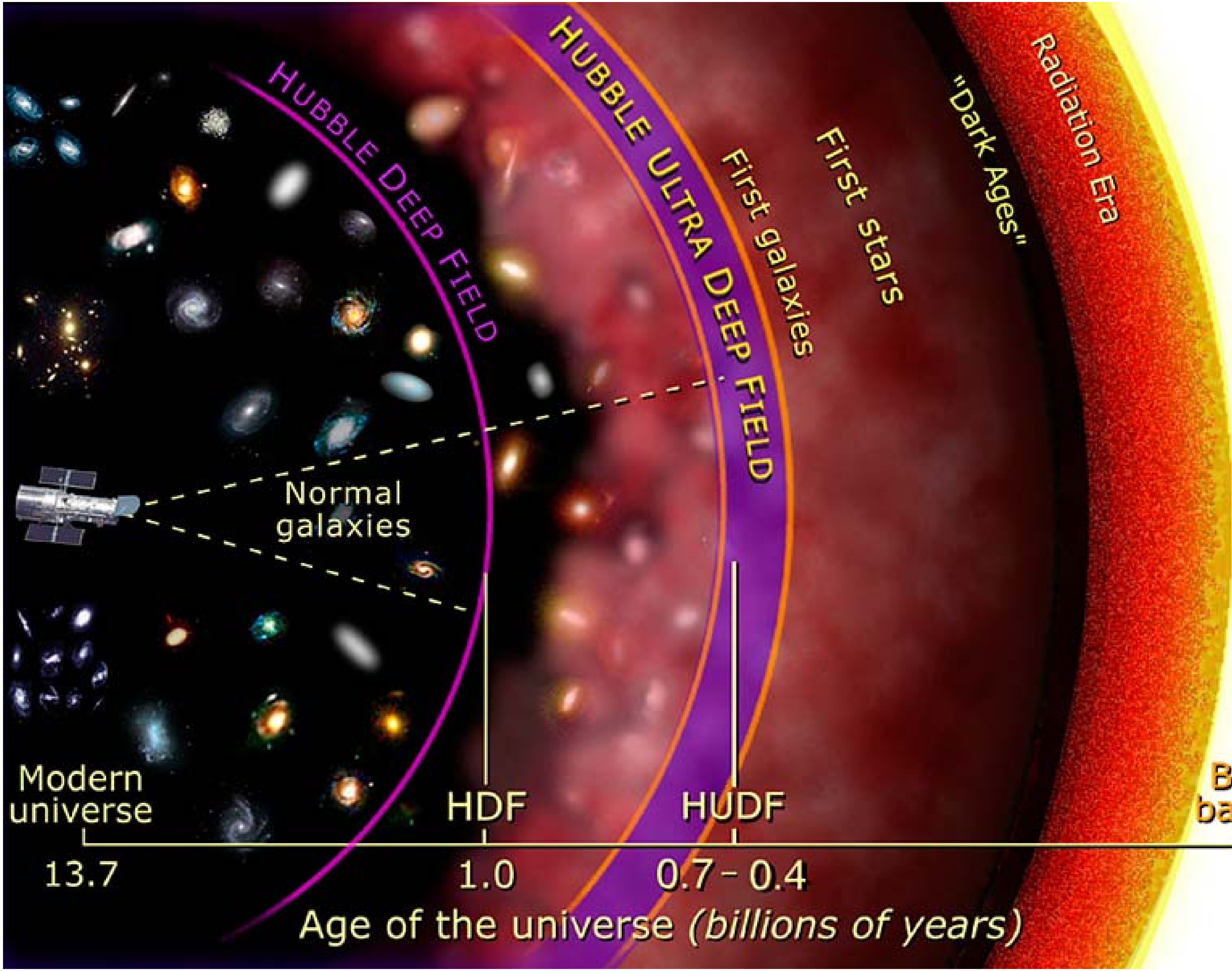
Early Universe too hot for neutral atoms

Free electrons scattered light (as in a fog)

After ~380,000 years, cool enough for atoms; fog clears!

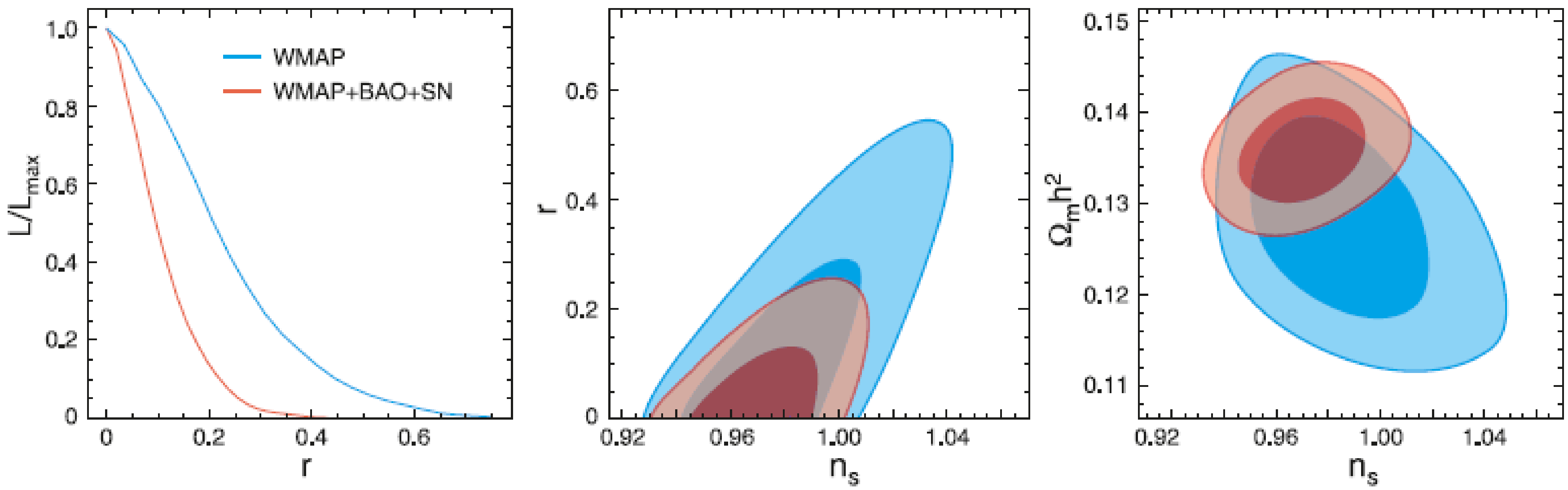






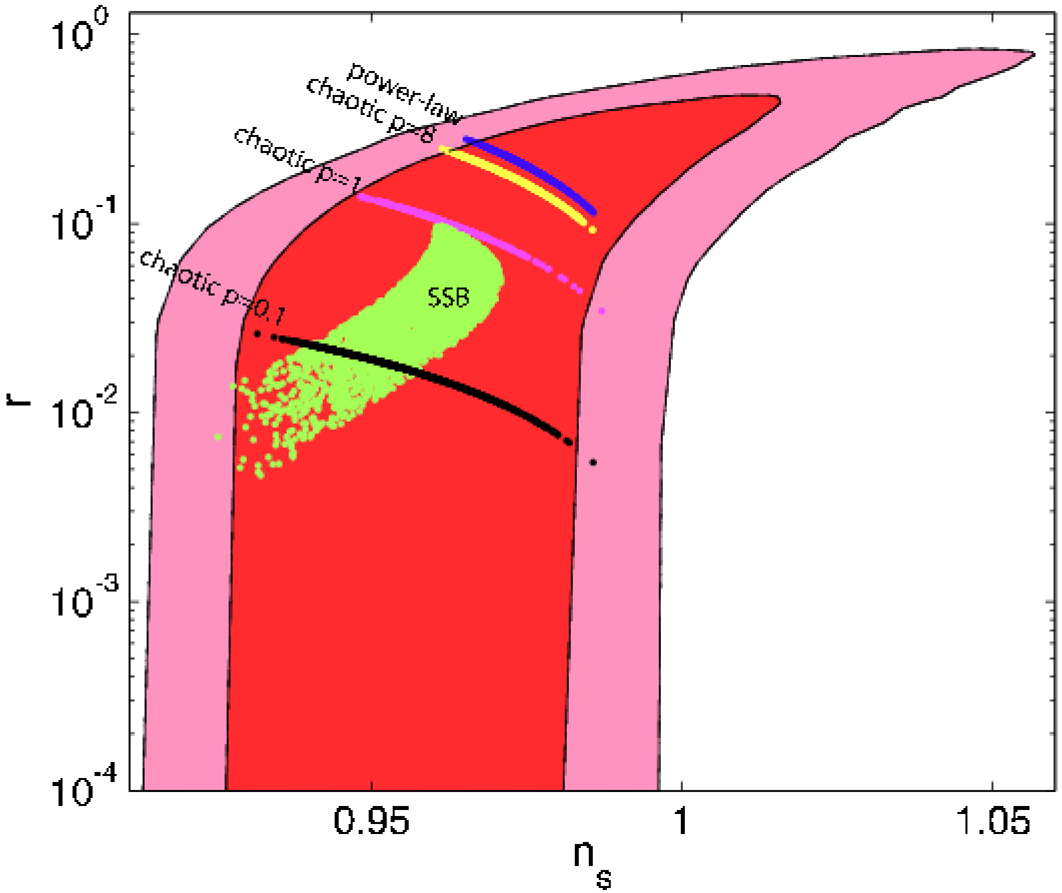
Hubble Deep Field
Hubble Ultra Deep Field
First galaxies
First stars
"Dark Ages"
Radiation Era
Normal galaxies
Modern universe
HDF
HUDF
Big bang
13.7
1.0
0.7 – 0.4
Age of the universe (billions of years)

But we can break this degeneracy somewhat by adding other cosmological information…



Komatsu et al (2008)

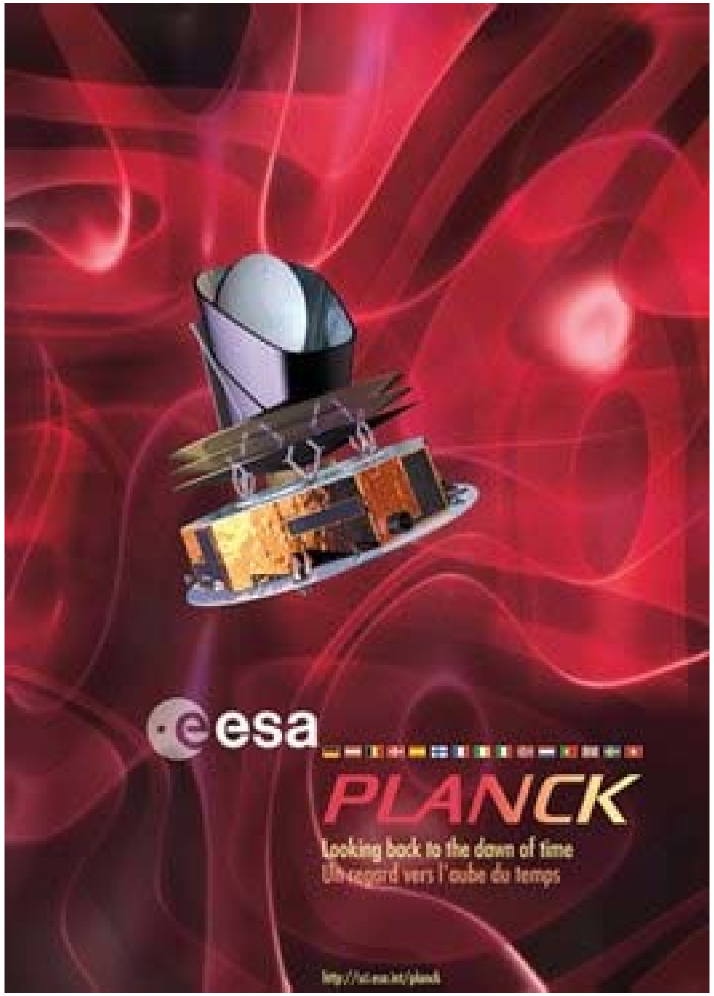
The WMAP5 results already start to place some interesting limits on e.g. inflationary models.

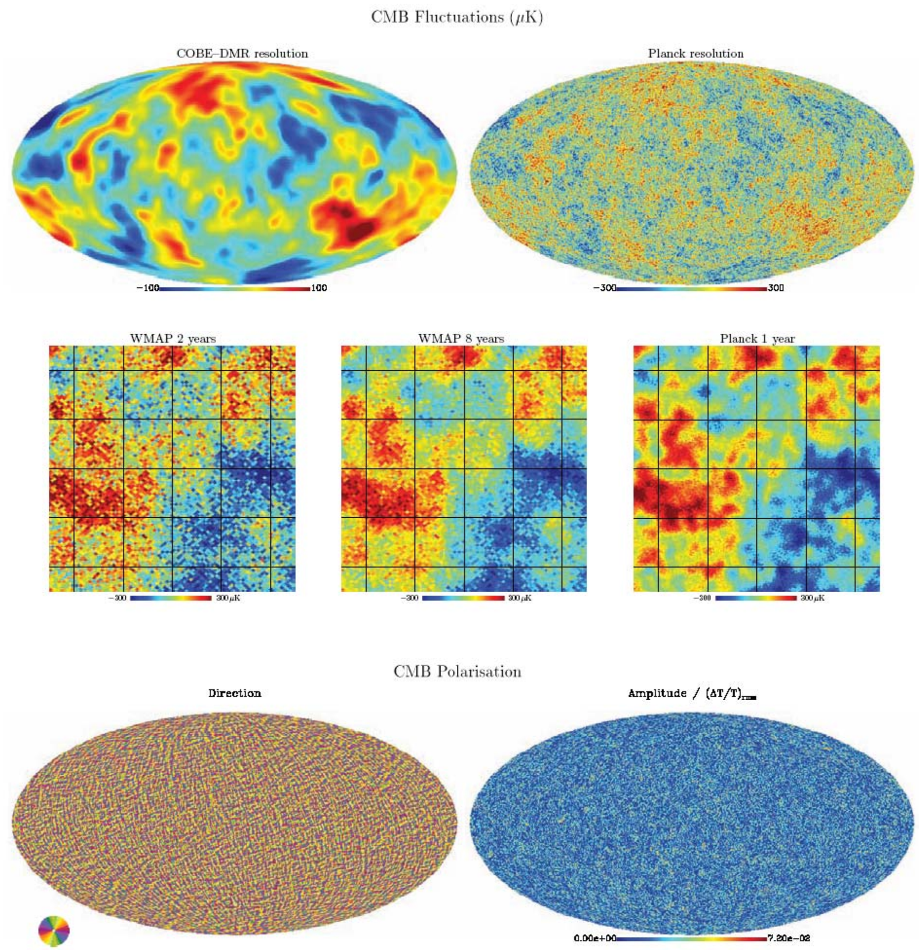


From Melchiorri (2008)

Several inflationary models predict a sizable GW background (r>0.01) if n<1.
If we assume those models, we already have an indication ror r>0 thanks to the WMAP n<1 result.
However there is no reason to believe that inflation is described by one of those models.

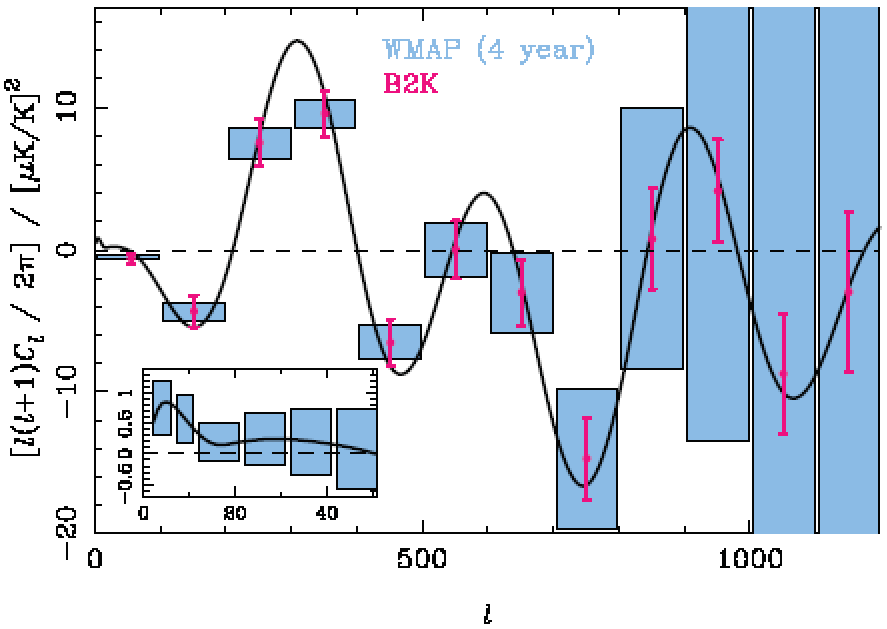
See Pagano, Cooray, Melchiorri And Kamionkowsky, JCAP 08.

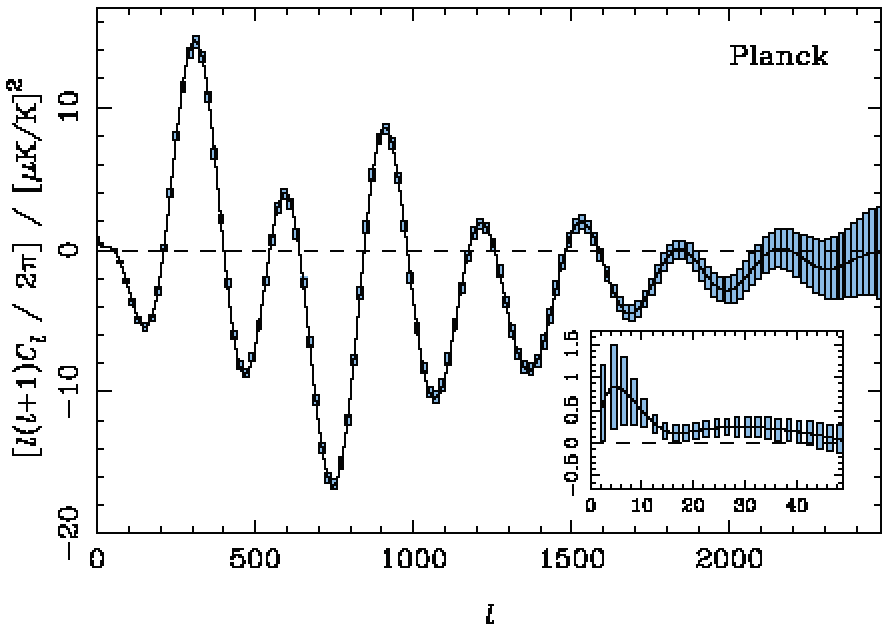


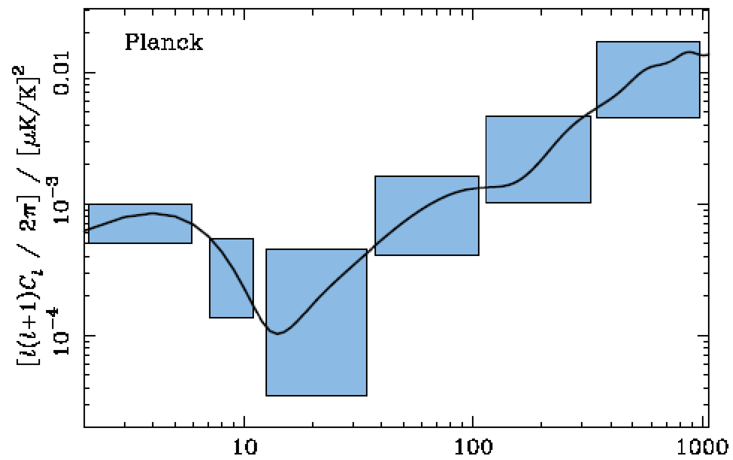


Launched May 14th 2009

TE cross power spectrum: WMAP versus Planck



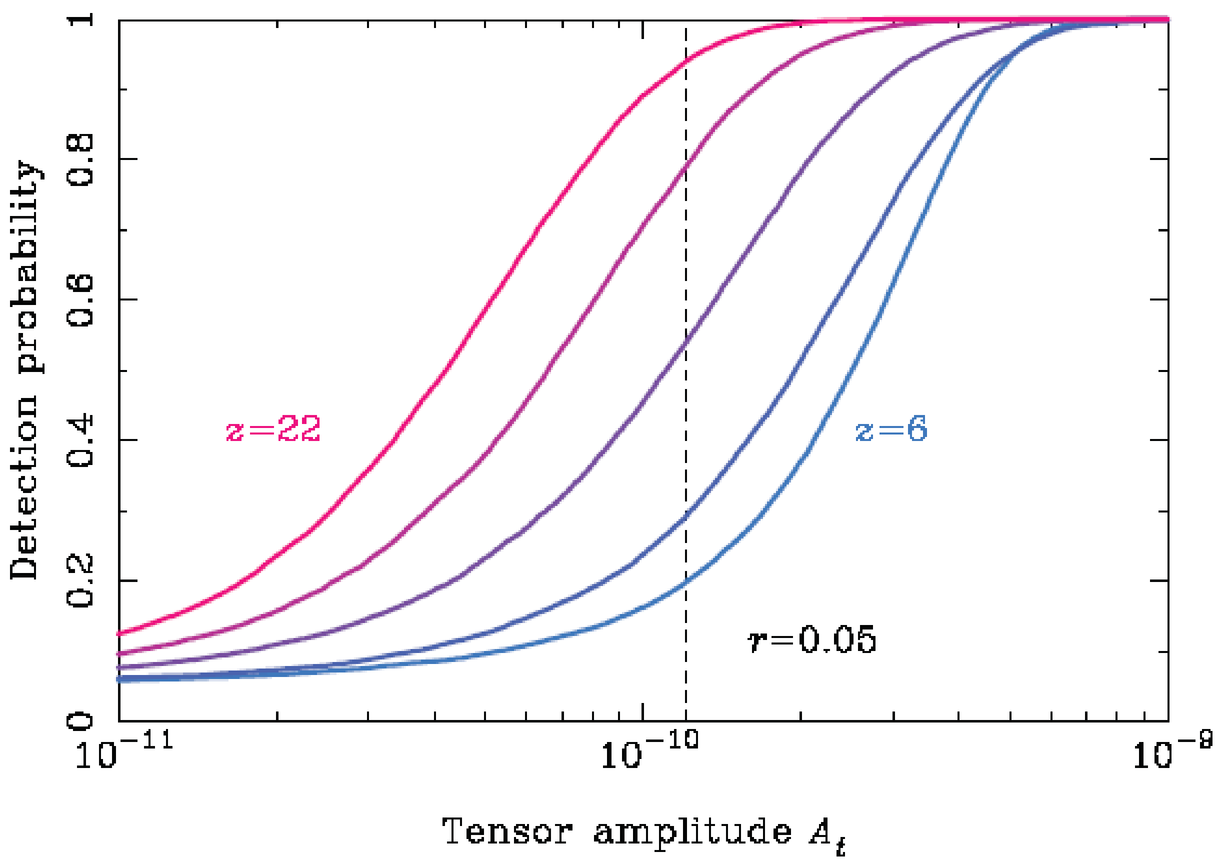




BB power spectrum: Planck

So, *will* Planck detect non-zero B-mode polarisation?

Depends on the actual value of T/S, and on the impact of foreground contamination from gravitational lensing.



Planning already underway for Next generation:

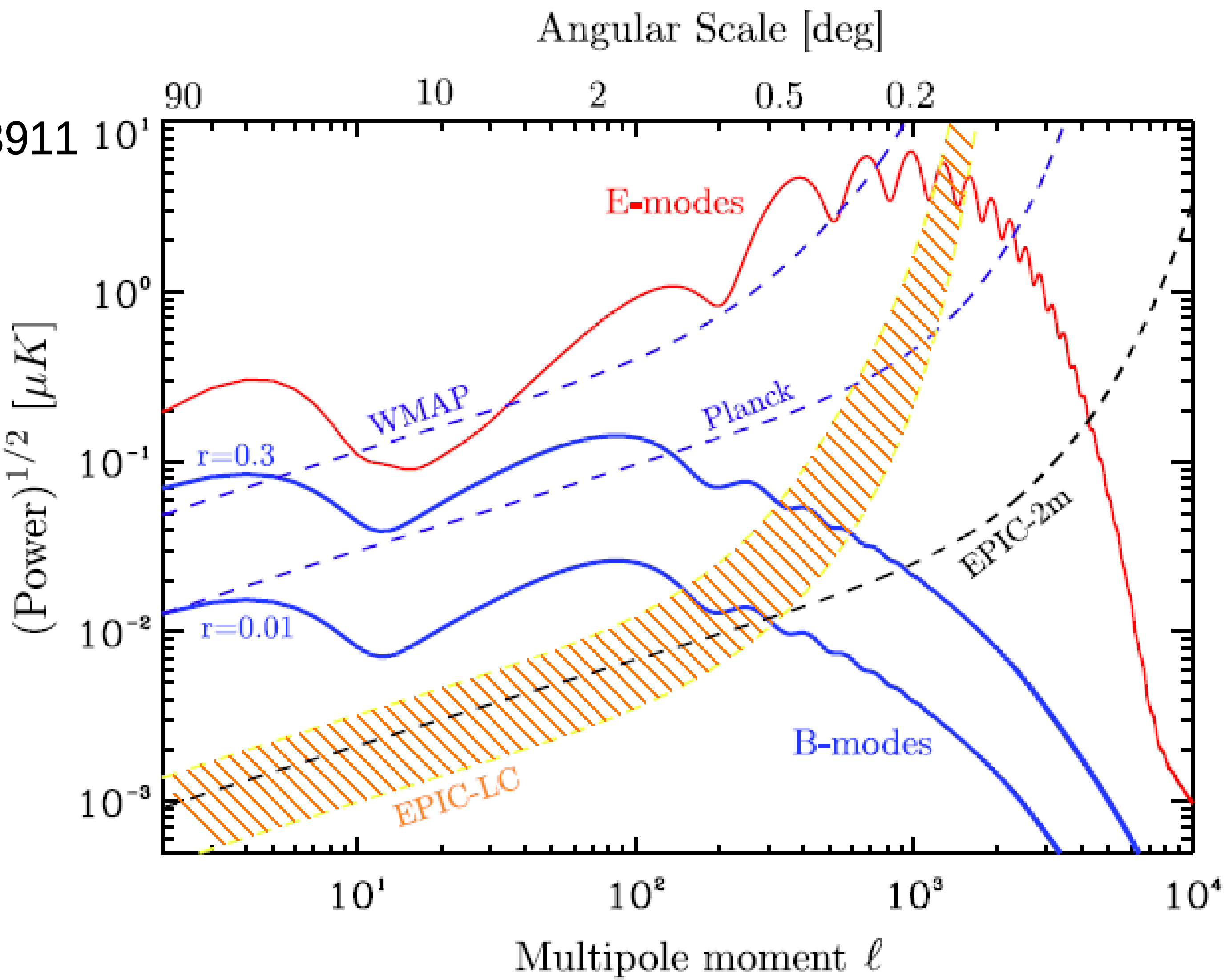
CMBPol

astro-ph/0811.3911

Could push to

T/S ~ 0.001
on largest
scales.

Timescale:
2020?



THE GRAVITATIONAL WAVE SPECTRUM