Section 5: The Ideal Gas Law

The atmospheres of planets (and the Sun too) can be modelled as an **Ideal Gas** – i.e. consisting of point-like particles (atoms or molecules) moving in random directions and interacting through perfectly elastic collisions.

We assume that the atmosphere has an **equation of state**, which links its pressure, density and temperature:

Volume of the gas
m^3

Gas temperature, a measure of the K.E. of the particles
degrees Kelvin K

$$\boxed{PV = NkT} \qquad (5.1)$$

Gas pressure, a measure of the force of the collisions
$\mathrm{Pa} \equiv \mathrm{Nm}^{-2}$

Number of particles in the gas

Boltzmann constant:
$1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}}$

We can also write eq. (5.1) in the form

$$\boxed{P = nkT} \qquad (5.2)$$

Here $n = N/V$ is the **number density** of gas particles.

Also we can write

Mass density of the gas

$$\boxed{n = \frac{\rho}{\overline{m}}} \qquad (5.3)$$

Average mass of a gas particle

So if we introduce

$$\boxed{\mu = \frac{\overline{m}}{m_H}} \qquad (5.4)$$

$1.674 \times 10^{-27}\ \mathrm{kg}$

Average mass of a gas particle
in units of mass of hydrogen atom

Then, from eqs. (5.2) – (5.4)

$$\boxed{P = \frac{\rho k T}{\mu m_H}} \qquad (5.5)$$

The temperature of the gas is a measure of the **average kinetic energy** of the particles.

Suppose all particles have mass m. Then we define

$$\boxed{\tfrac{1}{2} m \overline{\upsilon^2} = \tfrac{3}{2} kT} \qquad (5.6)$$

Note

- $\overline{\upsilon^2}$ is the "mean square speed" of the gas particles
- Factor of 3 on the RHS comes from the 3 dimensions ("degrees of freedom") in which the particles can move.

  The gas has energy of $\tfrac{1}{2} kT$ per degree of freedom

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From eq. (5.5)

$$\boxed{\tfrac{1}{3} m \overline{\upsilon^2} = kT} \qquad (5.7)$$

Substituting from eqs. (5.2) and (5.3)

$$\boxed{P = \tfrac{1}{3} \rho \overline{\upsilon^2}} \qquad (5.8)$$

At a temperature of **absolute zero**, i.e. $0\,\mathrm{K}$, all gas motions cease. Gas pressure drops to zero.

Section 6: Hydrostatic Equilibrium

The pressure (and hence the density and temperature) is not constant throughout a planetary atmosphere. A balance is maintained between the outward **pressure force** and the inward **gravitational force**.

We call this balance **hydrostatic equilibrium**.

Let's assume (as we did in section 2 for the interior of a planet) that the density of gas in the atmosphere is **spherically symmetric**.

We can then derive an expression for how the pressure changes as a function of height in the planet's atmosphere.

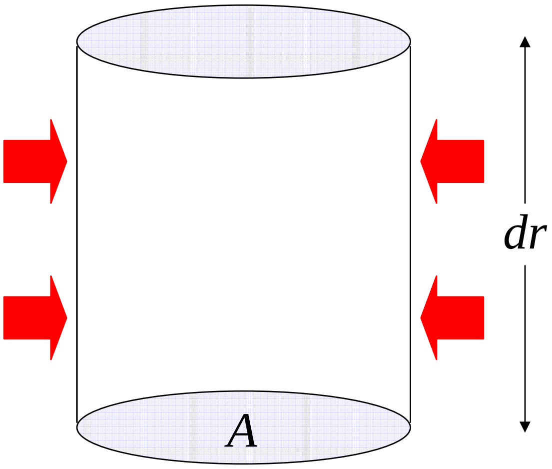
(we do this using calculus, forming a ***differential equation****)*

Consider a small cylinder of gas in the planet's atmosphere, the bottom of which is a distance r from the centre of the planet.

Let the area of the cylinder be A and its height be dr.

Suppose the cylinder contains a mass m of gas.

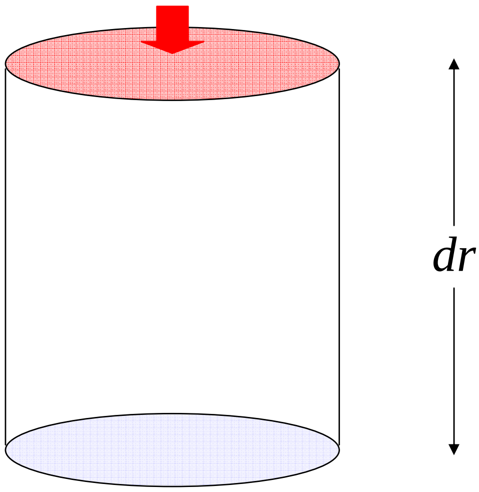
What forces will be exerted on this cylinder by the rest of the atmosphere?...



The horizontal forces on the walls of the cylinder will cancel out

Upper face:

Downward force, due to pressure exerted by gas above the cylinder *



$$\boxed{F_{\text{upper}} = -A\,P(r+dr)} \qquad (6.1)$$

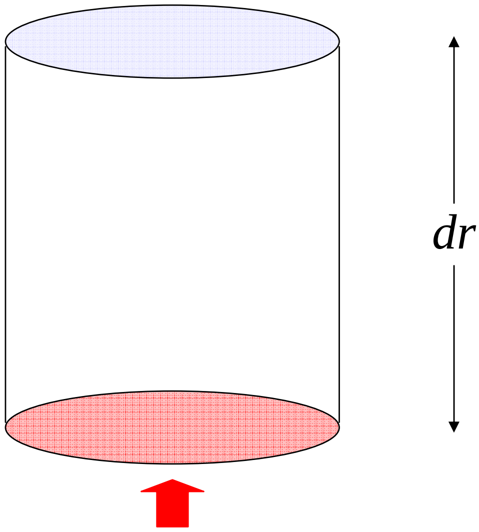
Notes: (1) *we are taking upwards as positive*

(2) *we are using here the relation*
pressure = force per unit area

* There will also be a downward force due to the weight of atmosphere above the cylinder, but we don't consider that here since it will also apply to the lower face.

Lower face:

Upward force, due to pressure of the gas below the cylinder, and *downward* force, due to the weight of the gas in the cylinder.



$$\boxed{F_{\text{lower}} = A\,P(r) - \frac{GM(r)m}{r^2}} \qquad (6.2)$$

Notes: (1) $M(r)$ *is the mass contained within radius* r *from the planet's centre*

(2) *The gravitational force term in eq. 6.2 is an approximation, since the cylinder does not all lie at distance* r *from the centre.*

This is OK provided $dr << r$

We can re-write eq. (6.2) as

$$\boxed{F_{\text{lower}} = A\,P(r) - m\,g} \qquad (6.3)$$

We can also write the mass of gas in the cylinder as **density x volume**:

$$\boxed{m = \rho\,A\,dr} \qquad (6.4)$$

Substituting into eq. (6.3) gives

$$\boxed{F_{\text{lower}} = A\,P(r) - A\rho g\,dr} \qquad (6.5)$$

To keep the cylinder static, we require that there be no **net force** on it, i.e. $F_{\text{lower}} + F_{\text{upper}} = 0$

So

$$\boxed{AP(r+dr) - AP(r) + A\rho g\,dr = 0} \qquad (6.6)$$

Dividing by $A\,dr$ and re-arranging

$$\boxed{\frac{P(r+dr) - P(r)}{dr} = -\rho g} \qquad (6.7)$$

In the limit as $dr \to 0$ the LHS is the **derivative** of $P(r)$ with respect to r i.e. the rate of change of pressure with radius.

Finally,then, we have

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g} \qquad (6.8)$$

Also referred to as the ***pressure gradient***

Since the density and gravitational acceleration are both **positive**, this means that

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}r} < 0} \qquad (6.9)$$

i.e. $P(r)$ *decreases* with increasing radius.

How fast?...

We define the **pressure scale height** via

$$\boxed{\frac{1}{H_P} = -\frac{1}{P(r)}\frac{\mathrm{d}P}{\mathrm{d}r}} \qquad (6.10)$$

If we make the assumption that H_P is **constant**, then we can find an expression for the pressure as a function of radius.

Re-arranging eq. (6.10)

$$\boxed{\frac{\mathrm{d}P}{P} = -\frac{\mathrm{d}r}{H_P}} \qquad (6.11)$$

This is a **differential equation**. We solve it by integrating both sides.

$$\int \frac{\mathrm{d}P}{P} \quad = \quad -\int \frac{\mathrm{d}r}{H_P} \quad = \quad -\frac{1}{H_P}\int \mathrm{d}r$$

By assuming the scale height is constant, we can take it out of the integral

i.e.

$$\boxed{\log P \quad = \quad -\frac{r}{H_P} \quad + \quad \text{constant}} \qquad (6.12)$$

Natural logarithm

The constant can be fixed by the pressure at $r = 0$, say $P = P_0$

$$\boxed{P(r) \quad = \quad P_0 \exp\left(-\frac{r}{H_P}\right)} \qquad (6.13)$$

Eq. (6.13) makes sense for e.g. the Sun, which is gaseous throughout.

For a planet like the Earth, with a solid interior, we can write

$$r = R + h$$

Radius of surface

Height above surface

We then fix the constant to be the pressure at the surface, $h = 0$

$$\boxed{P(h) = P_S \exp\left(-\frac{h}{H_P}\right)} \qquad (6.14)$$

The Earth's surface pressure is defined as **1 atmosphere**

From eqs. (6.8) and (6.10) $\quad H_P = \dfrac{P}{\rho g}$

For an **ideal gas**, from eq. (5.5) $\quad P = \dfrac{\rho k T}{\mu m_H}$

So

$$\boxed{H_P = \frac{kT}{\mu m_H g}} \qquad (6.15)$$

As T **increases**, so does H_P, i.e. the atmosphere extends **further**.

As μ, g **increase**, H_P **decreases**. i.e. atmosphere **less** extended.

Substituting in eq. (6.14)

$$\boxed{P(h) = P_S \exp\left(-\frac{\mu m_H g h}{kT}\right)} \qquad (6.16)$$

Section 7: Escape of a Planetary Atmosphere

The atoms or molecules in a planet's atmosphere are constantly moving. If they are moving fast enough, they can **escape**.

From eq. (3.7), this requires

$$\upsilon > \upsilon_{\text{escape}} = \sqrt{\frac{2GM_P}{R_P}} \qquad (7.1)$$

But the particles will have a **distribution** of speeds - some will exceed the escape speed, while others will not.

When a sufficient fraction of the particles exceed the escape speed, the planet will effectively 'lose' its atmosphere.

A good 'rule of thumb' is:

> A particular component of a planet's atmosphere will be lost if, for that component, $\upsilon_{\text{rms}} > \frac{1}{6}\upsilon_{\text{escape}}$

υ_{rms} is the '**root mean square**' speed, the square root of the mean square speed we met in Section 5.

$$\upsilon_{\text{rms}} = \sqrt{\overline{\upsilon^2}} \qquad (7.2)$$

We can use the results of Section 5 to relate the escape criterion to **temperature**, using:

$$\tfrac{1}{3} m \upsilon_{\text{rms}}{}^2 = kT \qquad (7.3)$$

So a particle of mass m will escape if

$$\boxed{kT = \tfrac{1}{3} m \upsilon_{\text{rms}}^{\;\;2} > \tfrac{1}{3} m \tfrac{1}{36} \upsilon_{\text{escape}}^2 = \frac{1}{54} \frac{G M_P m}{R_P}} \tag{7.4}$$

We define the **escape temperature**

$$\boxed{T_{\text{escape}} = \frac{1}{54} \frac{G M_P m}{k R_P}} \tag{7.5}$$

The more massive the planet, the hotter it must be before a given atmospheric component is lost.