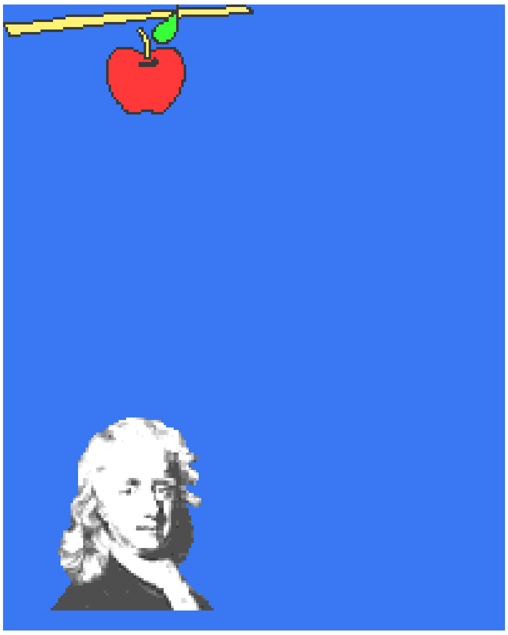
Section 2: Newton's Law of Gravitation

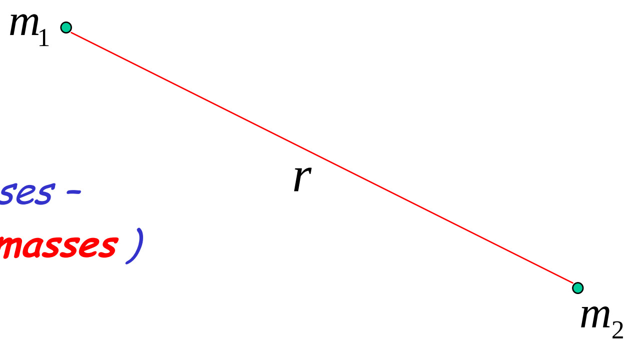


In 1686 Isaac Newton published his **Universal Law of Gravitation**.

This explained gravity as a force of attraction between all matter in the Universe, causing e.g. apples to fall from trees and the Moon to orbit the Earth.

(See also A1X Dynamical Astronomy)

Consider two masses m_1 and m_2, separated by distance r



(we ignore for the moment the physical extent of the two masses – i.e. we say that they are ***point masses*** *)*

Gravitational force on m_1 due to m_2 is

$$\boxed{\vec{F}_{12} = \frac{G\,m_1 m_2}{r^2}\,\hat{r}_{12}} \qquad (2.1)$$

Notes

1. The gravitational force is a **vector** – i.e. it has both magnitude and direction.

2. $\hat{r}_{12}$ is a unit vector from m_1 to m_2. In other words, $\vec{F}_{12}$ acts along the straight line joining the two masses.

3. The **gravitational constant** G is a fundamental constant of nature, believed to be the same everywhere in the Universe.

$$\boxed{G = 6.673\times10^{-11}\ \mathrm{Nm^2\,kg^{-2}}} \qquad (2.2)$$

4. The gravitational force on m_2 due to m_1 is of equal magnitude, but in the opposite direction, i.e.

$$\boxed{\vec{F}_{21} = -\vec{F}_{12}} \qquad (2.3)$$

5. Gravity is described as an **Inverse-Square Law**. i.e. the gravitational force between two bodies is inversely proportional to the **square** of their separation.

6. The gravitational force per unit mass is known as the **gravitational field**, or **gravitational acceleration**.

It is usually denoted by $\vec{g}$

Aside

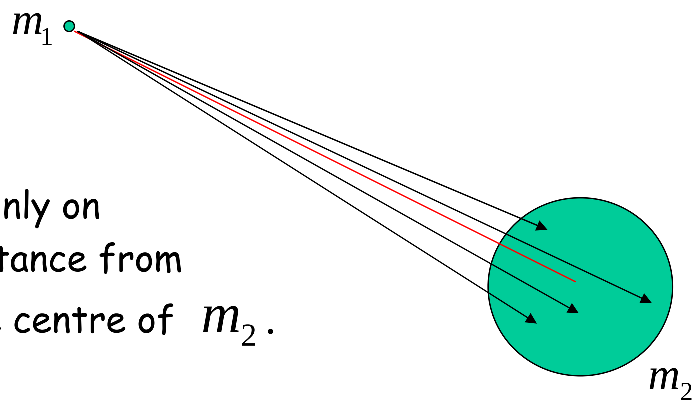
We shouldn't be too surprised that $\vec{g}$ is an acceleration: Newton's 2nd law states that "Force = mass x acceleration".

*However, Newton's 2nd law concerns **inertial mass** while Newton's law of gravitation concerns **gravitational mass**. That these two quantities are measured to be identical to each other is a very profound fact, for which Newton had no explanation, but which much later led Einstein to his theory of relativity.*

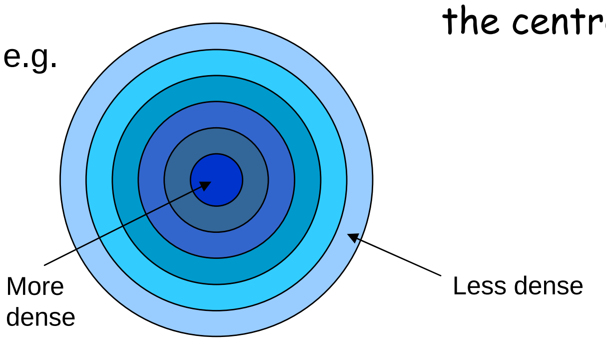
See P1 dynamics & relativity, and A2 special relativity

Planets and stars are ***not*** point mass objects. To determine the net force on m_1 due to m_2 we must add together the forces from all parts of m_2.

Suppose that m_2 is **spherical**, and its **density** (amount of mass per unit volume) depends only on distance from the centre of m_2.

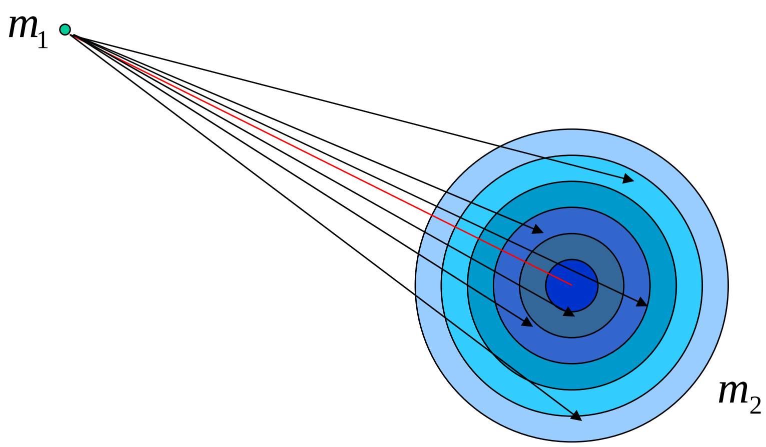


e.g.



*We say that the density is **spherically symmetric***

In this special case, the net gravitational force on m_1 due to m_2 is exactly the same as if all of the matter in m_2 were concentrated in a point at the centre of m_2 .



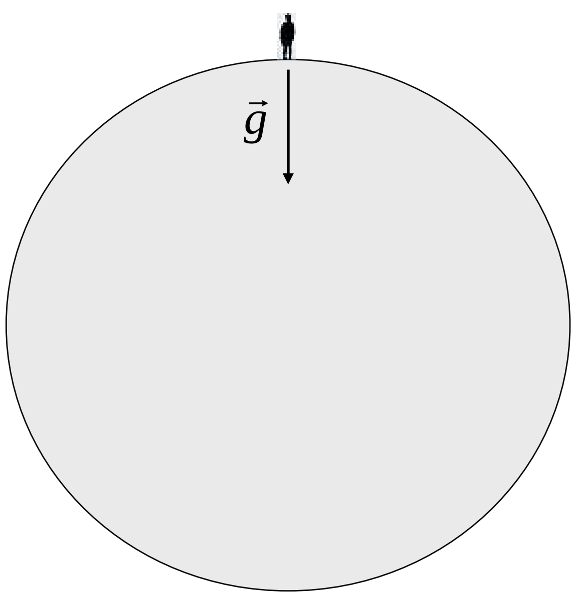
Section 3: Surface Gravity and Escape Speed

Consider, therefore, a spherical planet of radius R and total mass M which has a spherically symmetric density distribution.

The gravitational field $\vec{g}$ at the planet's surface is directed radially towards the planet's centre, and

$$|\vec{g}| = \frac{GM}{R^2} \qquad (3.1)$$

Magnitude of $\vec{g}$



The magnitude of $\vec{g}$ is known as the **surface gravity**, often just denoted by g (i.e. it is not a vector).

For example
$$M_{\text{Earth}} = 5.974 \times 10^{24}\ \text{kg}$$
$$R_{\text{Earth}} = 6.378 \times 10^{6}\ \text{m}$$

This means (assuming the Earth is spherical)

$$\boxed{g_{\text{Earth}} = 9.80\,\text{ms}^{-2}} \qquad (3.2)$$

g measures the rate of acceleration of falling objects (neglecting air resistance).

For any other body P (e.g. another planet or moon) it is useful to write

$$\boxed{\frac{g_P}{g_{\text{Earth}}} = \left(\frac{R_{\text{Earth}}}{R_P}\right)^2 \left(\frac{M_P}{M_{\text{Earth}}}\right)} \qquad (3.3)$$

e.g. for **Mars** :
$$R_{\text{Mars}} = 0.533\, R_{\text{Earth}}$$
$$M_{\text{Mars}} = 0.107\, M_{\text{Earth}}$$

So
$$\boxed{g_{\text{Mars}} = 0.377\, g_{\text{Earth}} = 3.69\,\text{ms}^{-2}} \qquad (3.4)$$

Exercise: Use the table of planetary data from the textbook and Section 1 to compute g for all the planets.

We can also express g in terms of average density $\overline{\rho}$

$$\overline{\rho} = \frac{\text{mass}}{\text{volume}} = \frac{M}{\frac{4}{3}\pi R^3}$$

i.e.

$$\boxed{M = \tfrac{4}{3}\pi R^3 \overline{\rho}} \qquad (3.5)$$

So from (3.1)

$$\boxed{g = \frac{GM}{R^2} = \tfrac{4}{3}\pi G R \overline{\rho}} \qquad (3.6)$$

From eq. (3.6)

1. If two planets have the same average density, the **larger** planet will have the higher surface gravity.

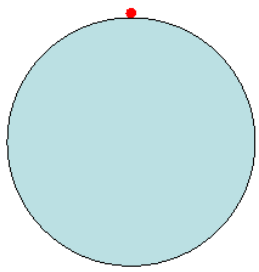
2. If two planets have the same radius, the **denser** planet will have the higher surface gravity.

Escape Speed

Consider a projectile launched vertically upwards at speed υ from the surface of a planet, with surface gravity g.

As the projectile climbs, the planet's gravity slows it down – its **kinetic energy** converted to **potential energy**.

In the animation we see the projectile slow to a stop, then accelerate back to the surface.



If the initial speed is high enough, the projectile will *never* return to the surface.

We say that the projectile **escapes** the planet's gravity.

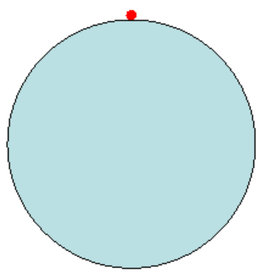
The minimum speed required to achieve this escape is known as the **escape speed**, and

$$\boxed{\upsilon_{\text{escape}} = \sqrt{\frac{2GM}{R}}} \qquad (3.7)$$

M = mass of planet

R = radius of planet

See A1X Dynamical Astronomy



For the Earth $\quad \upsilon_{\text{escape}} = 11.2\,\text{kms}^{-1}$

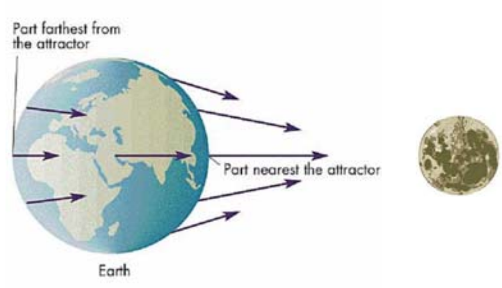
For Jupiter $\quad \upsilon_{\text{escape}} = 59.6\,\text{kms}^{-1}$

Note that the escape speed does not depend on the mass of the projectile.

Section 4: Tidal Forces

In Section 2 we pointed out that planets and moons (and indeed stars) are not point mass objects. Consequently, they will be subjected to **tidal forces** since different parts of their interior and surface experience a different gravitational pull from neighbouring bodies.

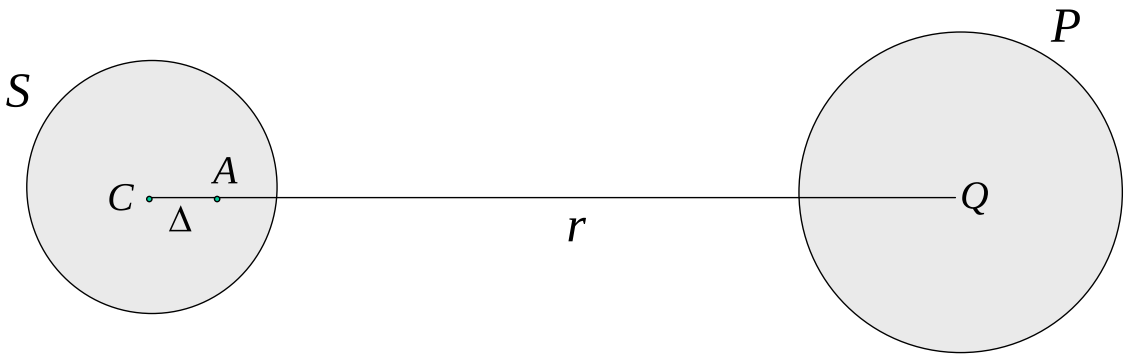
We see this differential effect with e.g. the Earth's tides, due to the Moon (and the Sun).



We now consider briefly the maths of tidal forces, before later exploring some applications to planets and moons in the Solar System.

Suppose that planet P and moon S are separated by distance r (centre to centre)

Consider a small mass m at position A and C



Tidal force between A and C due to the planet is equal to the *difference* in the gravitational force on A and C due to P

Let the distance from A to C equal Δ, and assume $\Delta << r$

Gravitational force on C due to P : $$F_C = \frac{Gm_P m}{r^2}$$

Gravitational force on A due to P : $$F_A = \frac{Gm_P m}{(r-\Delta)^2}$$

(We needn't worry about forces being vectors here, since A, C and Q lie along a straight line)

So $$F_A - F_C = \frac{Gm_P m}{(r-\Delta)^2} - \frac{Gm_P m}{r^2}$$

$$= \frac{Gm_P m}{r^2(1-\Delta/r)^2} - \frac{Gm_P m}{r^2}$$

$$= \frac{Gm_P m}{r^2}\left[\frac{1}{(1-\Delta/r)^2} - 1\right] \qquad (4.1)$$

We can write eq. (4.1) as $F_A - F_C = \frac{G m_P m}{r^2}\left[\left(1-\frac{\Delta}{r}\right)^{-2} - 1\right]$

If we now use that $\Delta << r$ then $\left(1-\frac{\Delta}{r}\right)^{-2} \approx 1 + \frac{2\Delta}{r}$ (4.2)

(Aside: Eq. (4.2) follows from the **Binomial expansion** for $(1+x)^n$ which is approximately $1+nx$ if $x<<1$)

So

$$\boxed{F_A - F_C = \frac{2 G m_P m \Delta}{r^3}} \qquad (4.3)$$

The important point here is that the magnitude of the tidal force is an **inverse-cube law**: i.e. it falls off more rapidly with distance than does the force of gravity.

So, if the planet P is far from the moon S, the tidal force experienced by the moon (and vice versa) will be small.

Conversely, however, if the moon lies very close to the planet, then the tidal forces on its interior may be considerable.

In a later section we will explore the consequences of this for the stability of moons in the Solar System.