Lectures for the 27th IAU ISYA
Ifrane, 2nd - 23rd July 2004

Statistical Astronomy

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Parameter estimation: the Gaussian approximation

Posterior Likelihood Prior

$$\boxed{p(\text{model} \mid \text{data}, I) \propto p(\text{data} \mid \text{model}, I) \times p(\text{model} \mid I)}$$

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Parameter estimation: the Gaussian approximation

$$\boxed{p(\theta \mid \text{data}, I) \propto p(\text{data} \mid \theta, I) \times p(\theta \mid I)}$$

'Best' estimator: $\left.\frac{\partial p(\theta \mid \text{data}, I)}{\partial \theta}\right|_{\theta=\theta_0} = 0$ Maximise posterior likelihood

Equivalently, we can define $\ell = \log p(\theta \mid \text{data}, I)$ and compute $\left.\frac{\partial \ell}{\partial \theta}\right|_{\theta=\theta_0} = 0$

Taylor expand $\ell(\theta)$ around $\theta=\theta_0$:

$$\boxed{\ell(\theta) = \ell(\theta_0) + \cancel{\left.\frac{\partial \ell}{\partial \theta}\right|_{\theta=\theta_0}(\theta-\theta_0)} + \frac{1}{2}\left.\frac{\partial^2 \ell}{\partial \theta^2}\right|_{\theta=\theta_0}(\theta-\theta_0)^2 + \ldots}$$

Parameter estimation: the Gaussian approximation

$$p(\theta \mid \text{data}, \ I) = \exp\left[\ell(\theta)\right]$$

Neglecting higher order terms in $\ell(\theta)$ ← Gaussian approximation

$$p(\theta \mid \text{data}, \ I) \propto \exp\left(-\frac{A}{2}(\theta - \theta_0)^2\right)$$

where $A = -\left.\dfrac{\partial^2 \ell}{\partial \theta^2}\right|_{\theta=\theta_0}$

This is equivalent to a normal distribution, with $\sigma^{-2} = A = -\left.\dfrac{\partial^2 \ell}{\partial \theta^2}\right|_{\theta=\theta_0}$

Can summarise inference from posterior by

$$\boxed{\theta = \theta_0 \pm \sigma}$$

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Parameter estimation: 2-D case

Recall our definition of *variance*

$$\mathrm{var}[x] \;=\; \int_{-\infty}^{\infty} \left(x - \langle x \rangle\right)^2 p(x \mid I)\,dx$$

Extend to 2 variables - *covariance*

$$\mathrm{cov}[x, y] \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(x - \langle x \rangle\right)\left(y - \langle y \rangle\right) p(x, y \mid I)\,dx\,dy$$

If x and y are independent, $\mathrm{cov}[x, y] \;=\; 0$

This is because $p(x, y \mid I) = p(x \mid I)\, p(y \mid I)$

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Parameter estimation: 2-D case

$$\boxed{p(\theta_1, \theta_2 \mid \text{data}, \; I) \quad \propto \quad p(\text{data} \;\mid \theta_1, \theta_2, I) \times p(\theta_1, \theta_2 \mid I)}$$

'Best' estimator: $$\left.\frac{\partial p(\theta_1, \theta_2 \mid \text{data}, I)}{\partial \theta_j}\right|_{\theta_j = \theta_{0\,j}} = 0$$

Compute $$\left.\frac{\partial \ell}{\partial \theta_j}\right|_{\theta_j = \theta_{0\,j}} = 0$$ where $\ell = \log p(\theta_1, \theta_2 \mid \text{data}, I)$

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Parameter estimation: 2-D case

Taylor expand $\ell(\theta_1, \theta_2)$ around θ_{0j} :

$$\boxed{\begin{aligned}\ell(\theta_1,\theta_2) \;=\; & \ell(\theta_{01},\theta_{02}) + \cancel{\left.\frac{\partial \ell}{\partial \theta_1}\right|_{\theta_j=\theta_{0j}} (\theta_1 - \theta_{01})} + \cancel{\left.\frac{\partial \ell}{\partial \theta_2}\right|_{\theta_j=\theta_{0j}} (\theta_2 - \theta_{02})} + \\ & \frac{1}{2}\left[\left.\frac{\partial^2 \ell}{\partial \theta_1^{\,2}}\right|_{\theta_j=\theta_{0j}} (\theta_1-\theta_{01})^2 + \left.\frac{\partial^2 \ell}{\partial \theta_2^{\,2}}\right|_{\theta_j=\theta_{0j}} (\theta_2-\theta_{02})^2 + 2\left.\frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_2}\right|_{\theta_j=\theta_{0j}} (\theta_1-\theta_{01})(\theta_2-\theta_{02}) \right] + \ldots\end{aligned}}$$

$$p(\theta_1, \theta_2 \mid \text{data}, \; I) \;\propto\; \exp\left[\ell(\theta_1,\theta_2)\right]$$

$$\propto\; \exp\left[-\tfrac{1}{2} Q\right]$$

← Gaussian approximation

χ^2

Maximising likelihood
$\equiv$ Minimising χ^2

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Parameter estimation: 2-D case

Taylor expand $\ell(\theta_1, \theta_2)$ around θ_{0j} :

$$\boxed{Q = \begin{pmatrix} \theta_1 - \theta_{10} & \theta_2 - \theta_{20} \end{pmatrix} \begin{bmatrix} A & C \\ C & B \end{bmatrix} \begin{pmatrix} \theta_1 - \theta_{10} \\ \theta_2 - \theta_{20} \end{pmatrix}}$$

Quadratic form

where $A = \left.\dfrac{\partial^2 \ell}{\partial \theta_1{}^2}\right|_{\theta_j = \theta_{0j}}$ $\quad B = \left.\dfrac{\partial^2 \ell}{\partial \theta_2{}^2}\right|_{\theta_j = \theta_{0j}}$ $\quad C = \left.\dfrac{\partial^2 \ell}{\partial \theta_1 \partial \theta_2}\right|_{\theta_j = \theta_{0j}}$

This is a bivariate normal distribution with covariance matrix

$$\sigma_{ij}^2 = \mathrm{cov}_{ij} = \left\langle (\theta_i - \theta_{i0})(\theta_j - \theta_{j0}) \right\rangle = \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \right]^{-1}$$

Fisher information matrix

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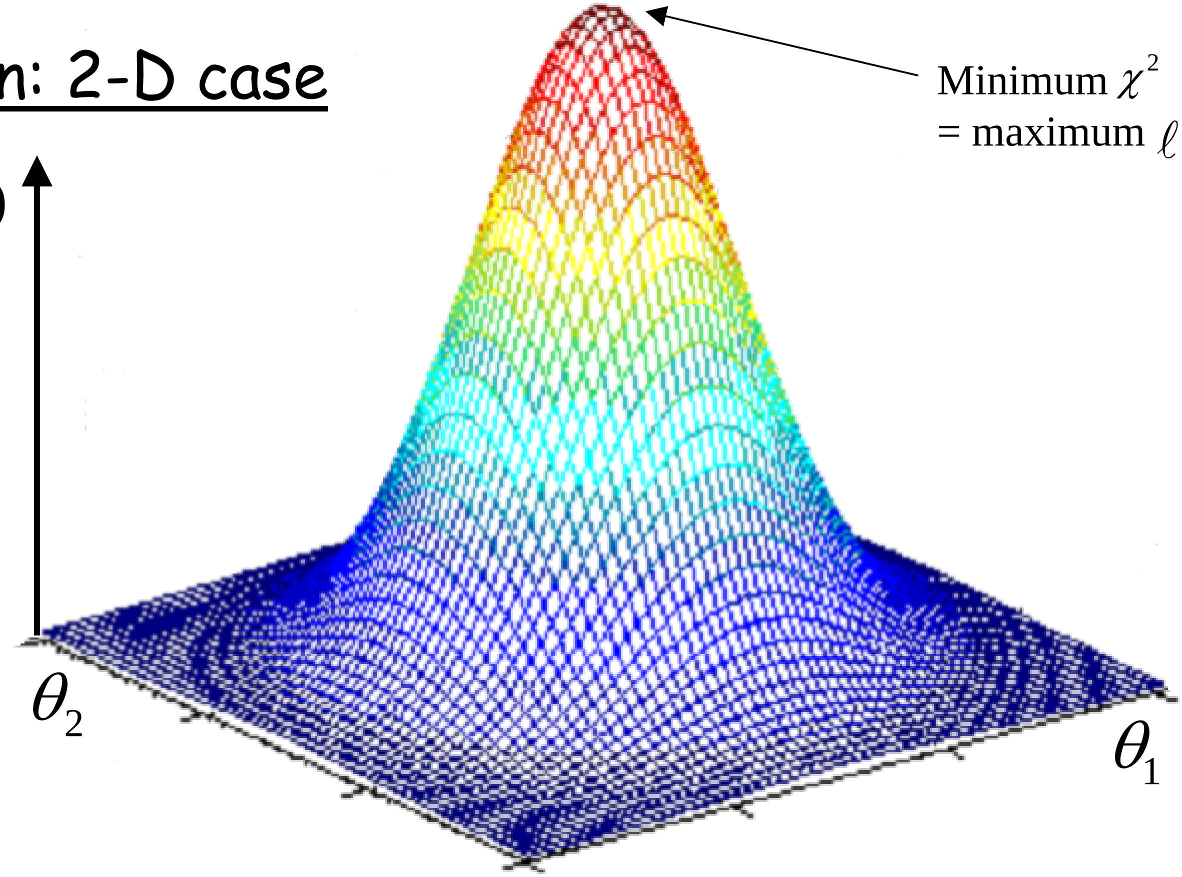
Parameter estimation: 2-D case



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Fisher information matrix

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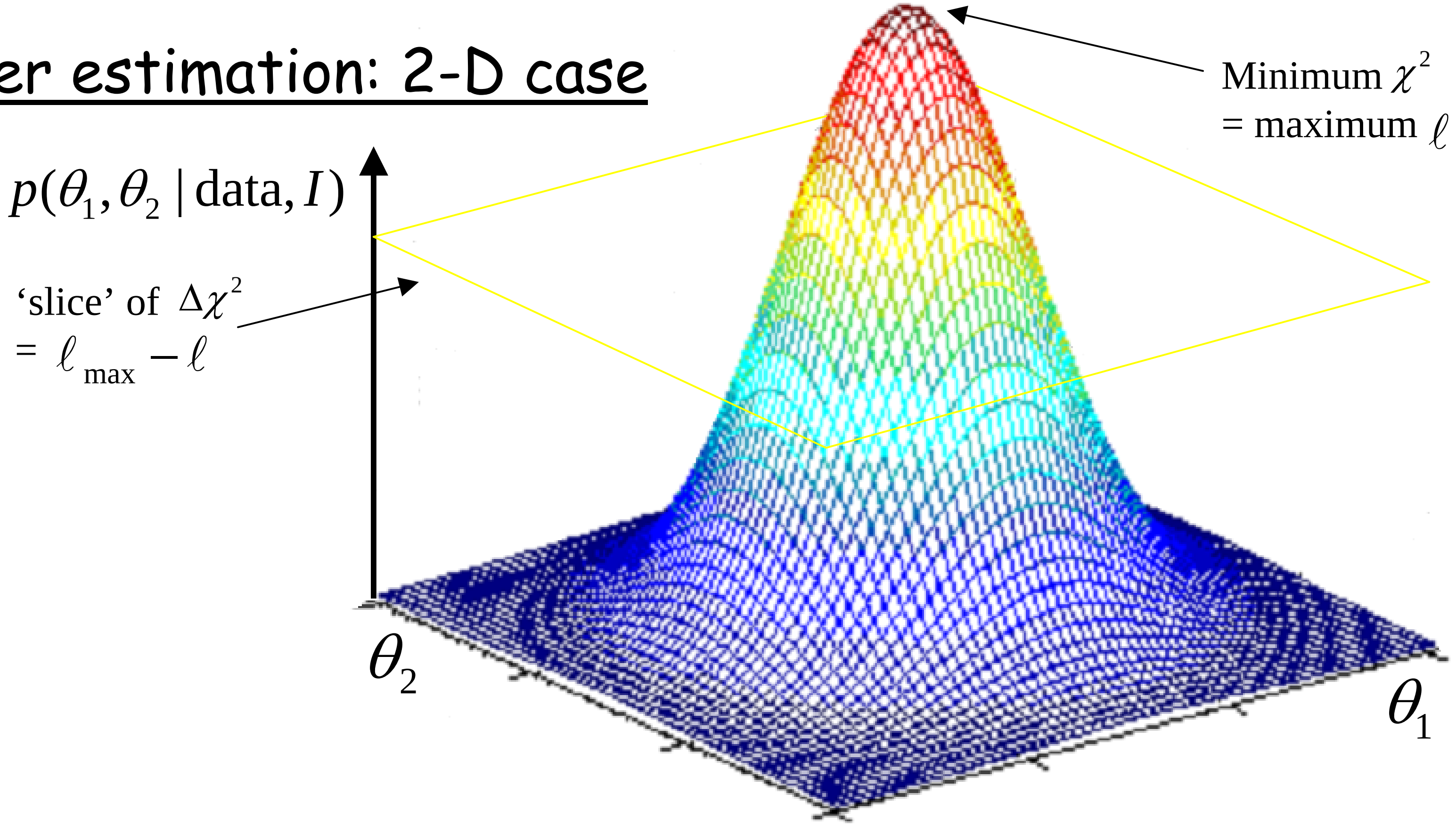

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Fisher information matrix

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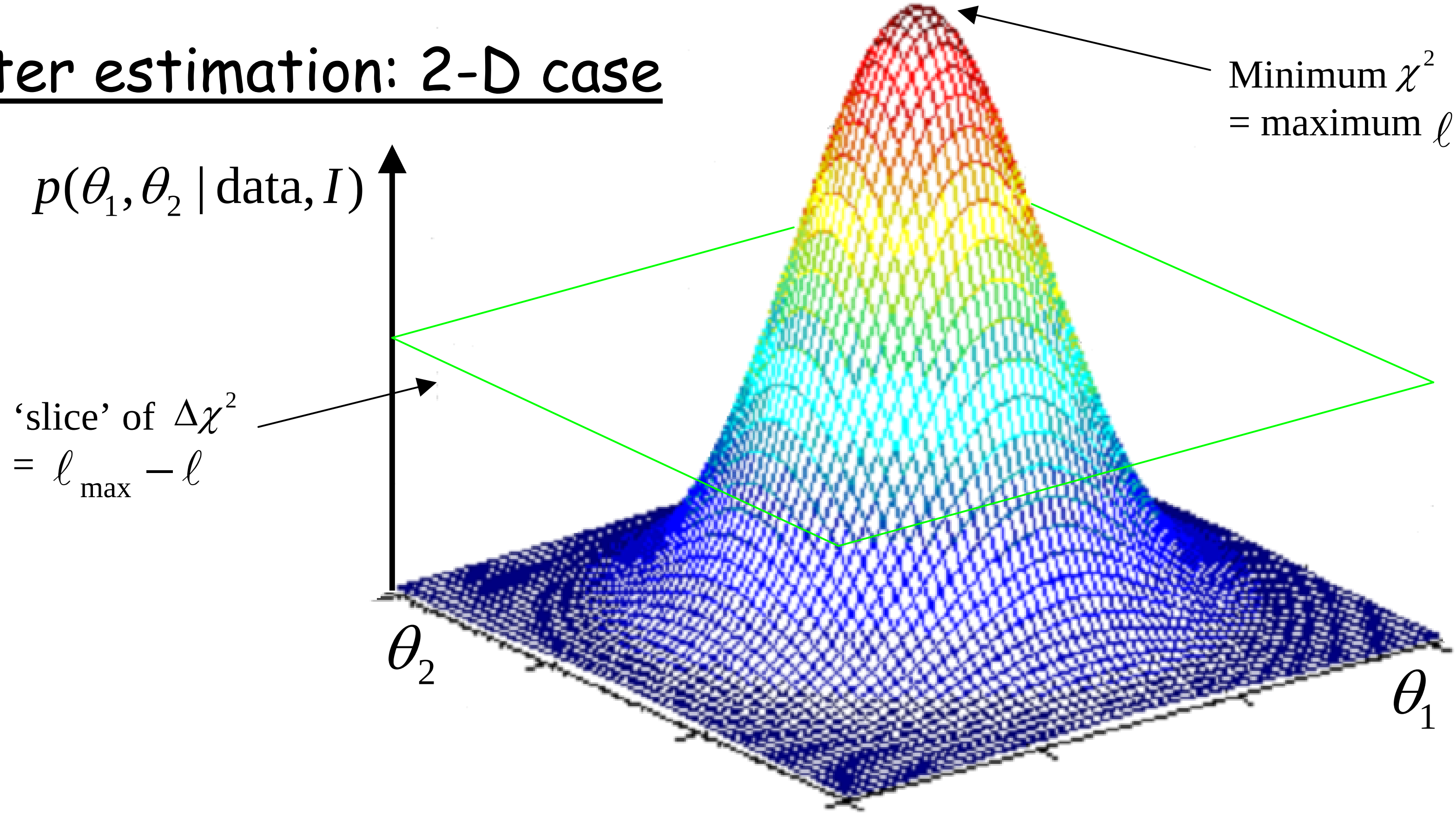

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Fisher information matrix

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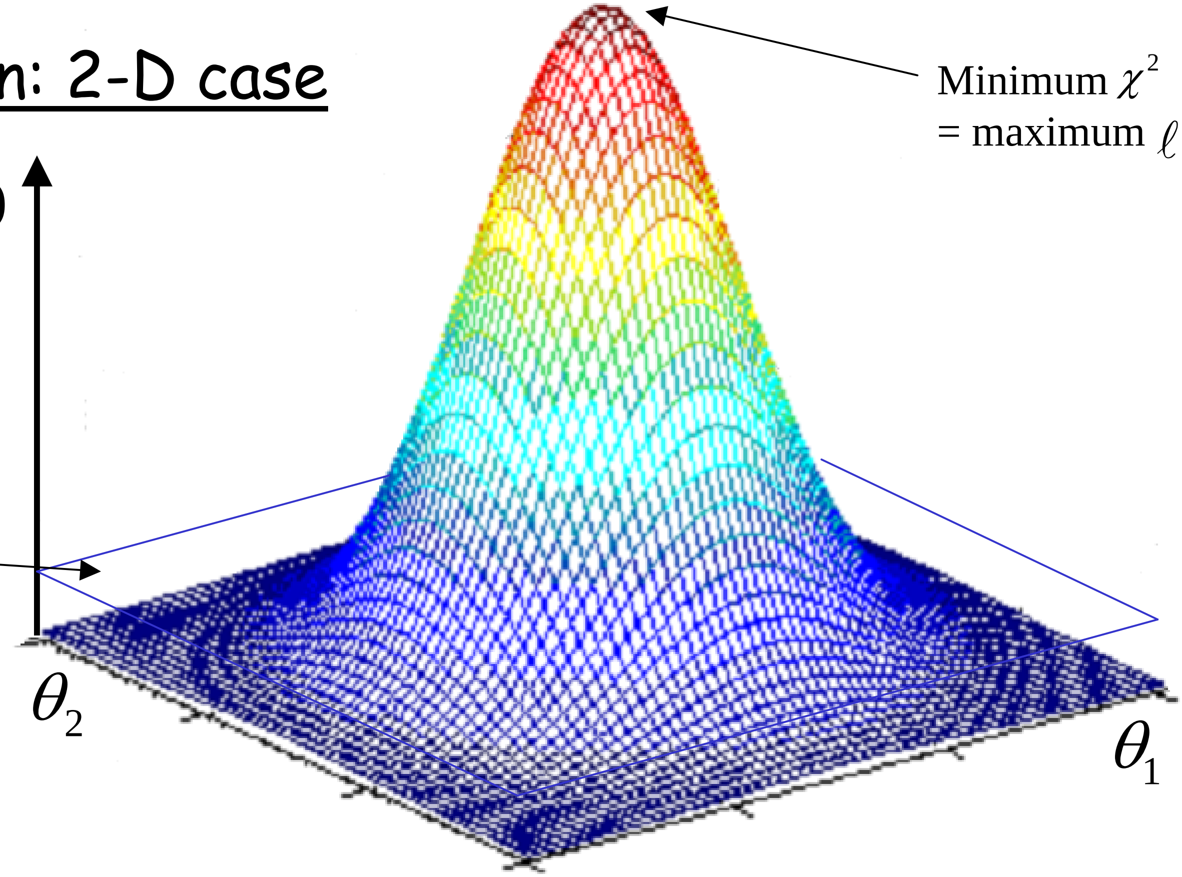
Parameter estimation: 2-D case



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Fisher information matrix

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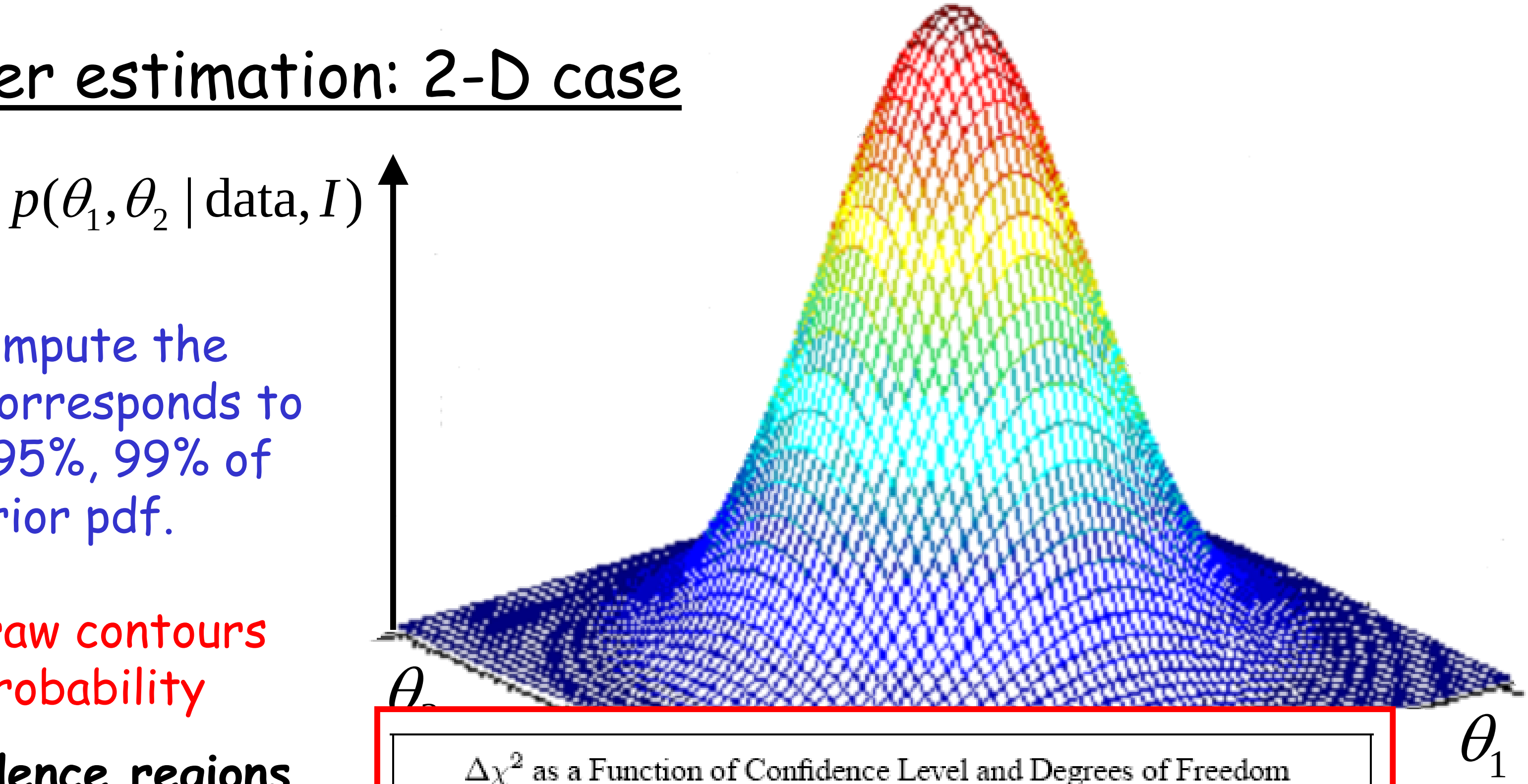
Parameter estimation: 2-D case



We can compute the $\Delta\chi^2$ that corresponds to e.g. 68%, 95%, 99% of the posterior pdf.

We can draw contours of equal probability

$\Rightarrow$ **Confidence regions for the parameters**

Extends easily to N parameters – or *degrees of freedom*

$\Delta\chi^2$ as a Function of Confidence Level and Degrees of Freedom

	ν					
p	1	2	3	4	5	6
68.3%	1.00	2.30	3.53	4.72	5.89	7.04
90%	2.71	4.61	6.25	7.78	9.24	10.6
95.4%	4.00	6.17	8.02	9.70	11.3	12.8
99%	6.63	9.21	11.3	13.3	15.1	16.8
99.73%	9.00	11.8	14.2	16.3	18.2	20.1
99.99%	15.1	18.4	21.1	23.5	25.7	27.8

From Numerical Recipes

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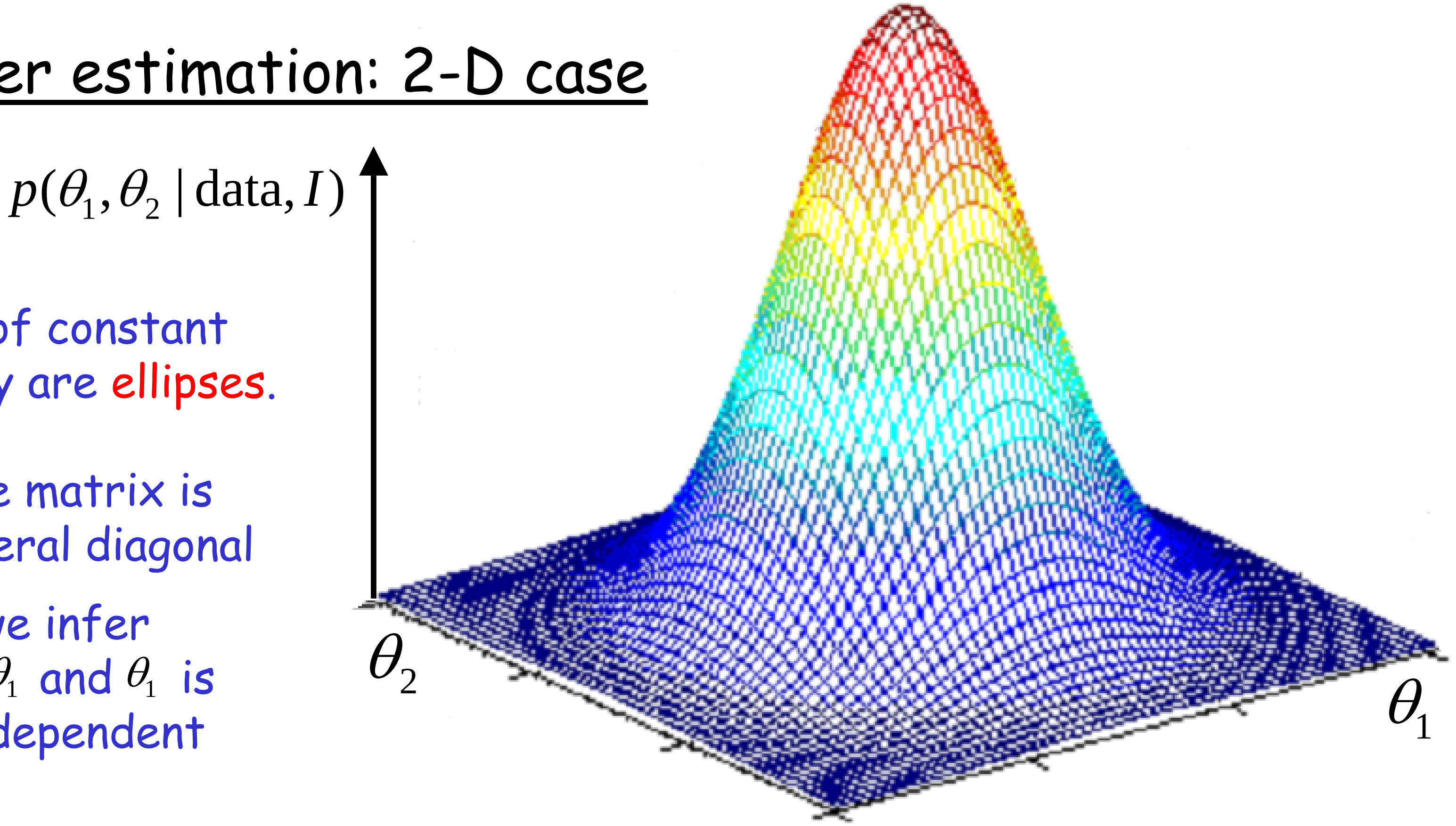
Parameter estimation: 2-D case



Contours of constant probability are ellipses.

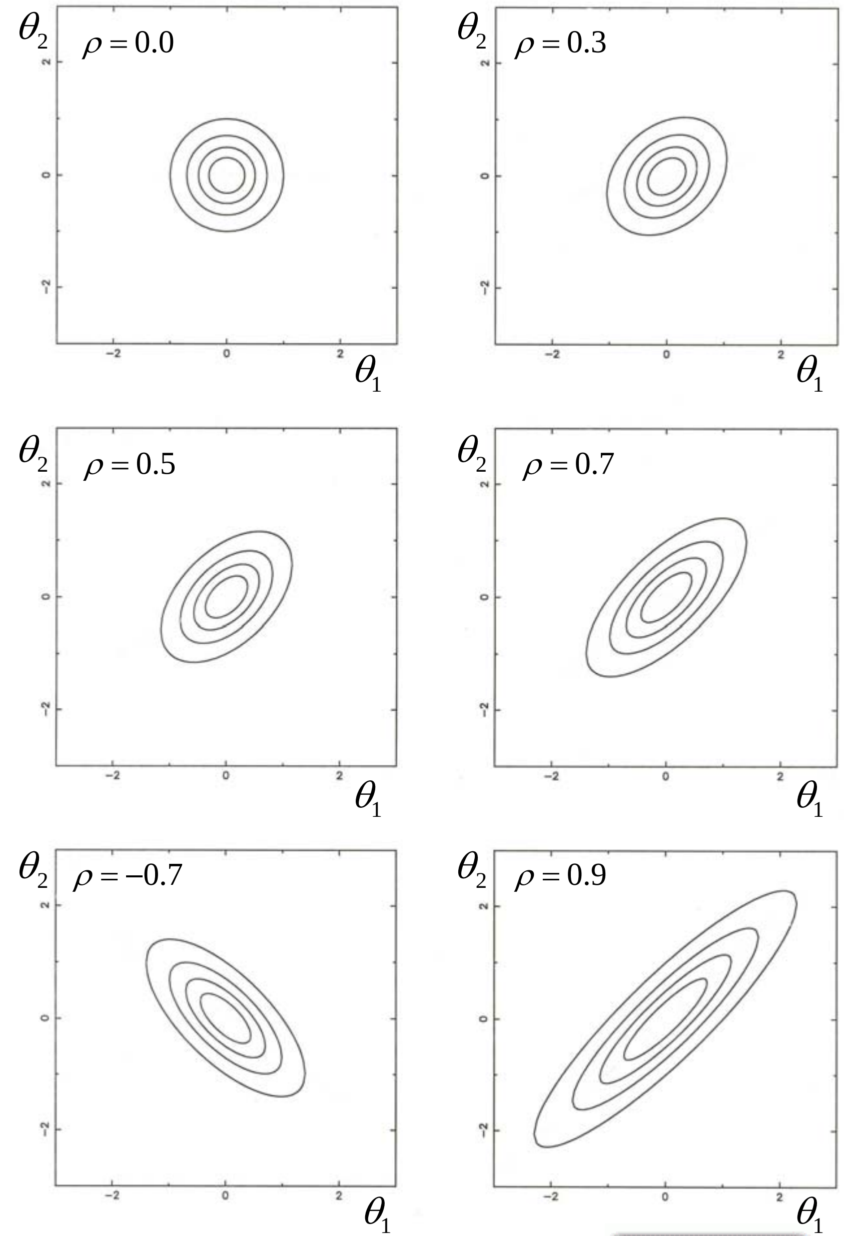
Covariance matrix is *not* in general diagonal

$\Rightarrow$ What we infer about θ_1 and θ_1 is *not* independent

Parameter estimation: 2-D case

Can define *correlation coefficient*

$$\rho = \frac{\mathrm{cov}[\theta_1, \theta_2]}{\sqrt{\mathrm{var}[\theta_1]}\,\sqrt{\mathrm{var}[\theta_2]}} \qquad -1 \le \rho \le 1$$



Covariance matrix becomes less diagonal

$\Rightarrow$ $|\rho|$ increases

$\Rightarrow$ isoprobability contours elongate

Very important if we are interested only in *one* parameter

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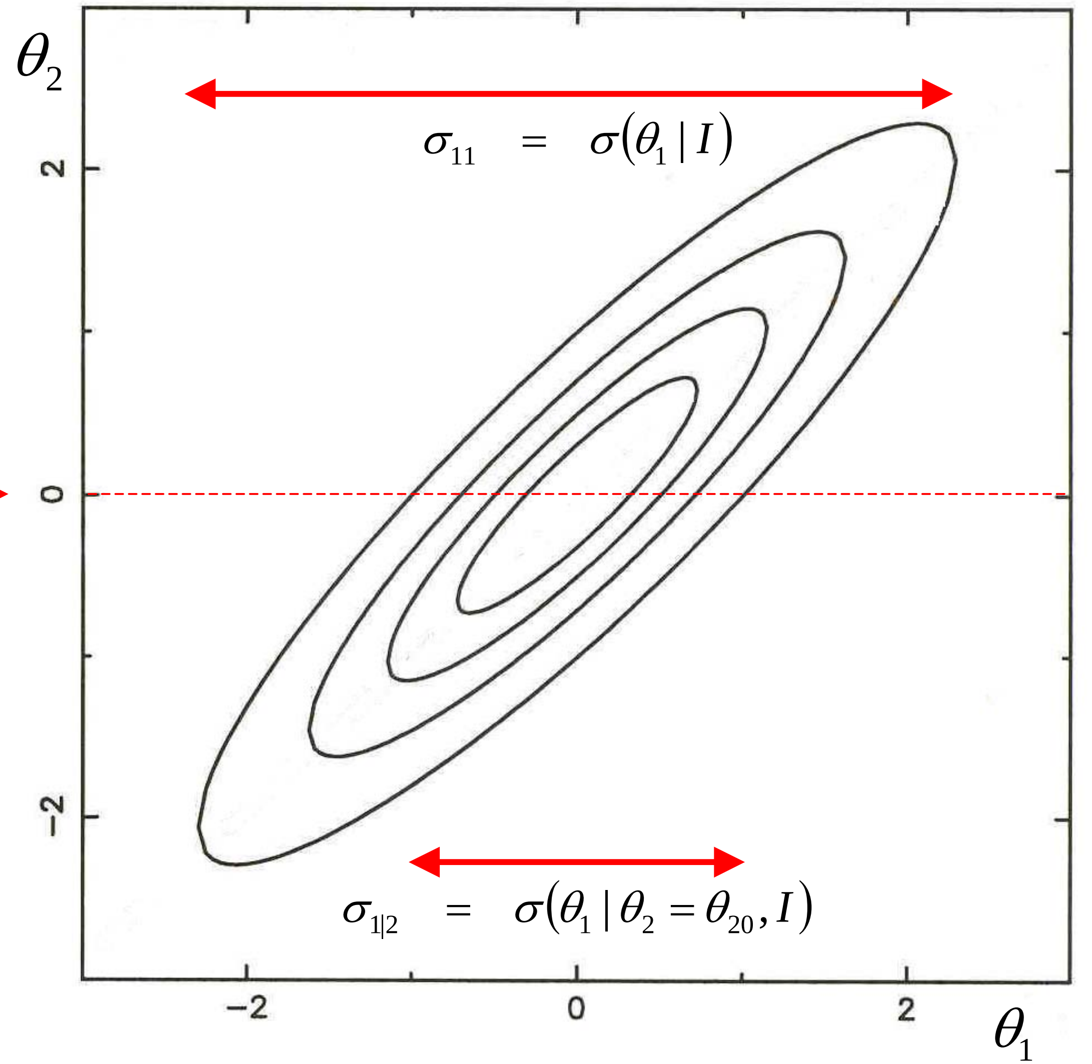
Parameter estimation: 2-D case

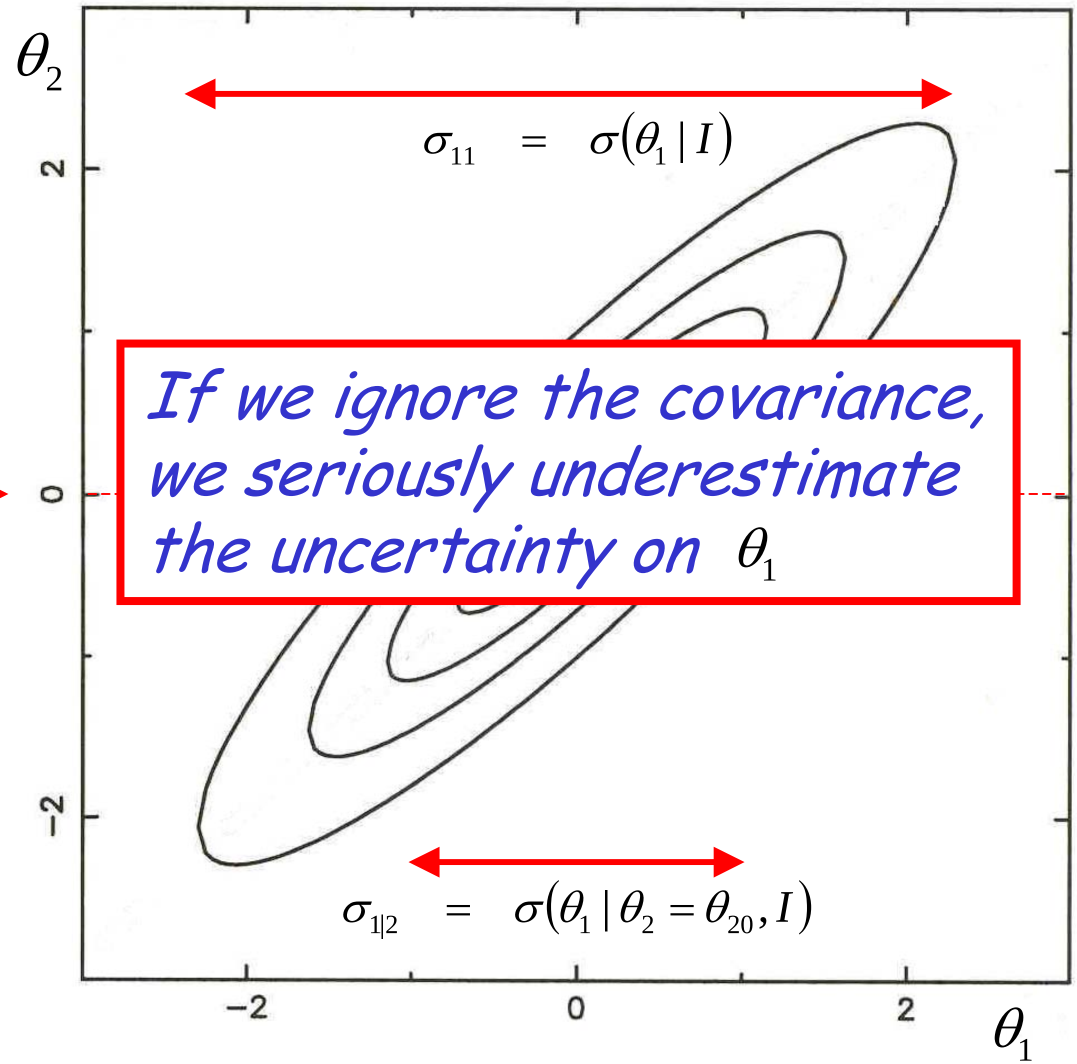
'Best-fit' value of θ_2 , found from

$$\left.\frac{\partial \ell}{\partial \theta_j}\right|_{\theta_j=\theta_{0j}} = 0$$

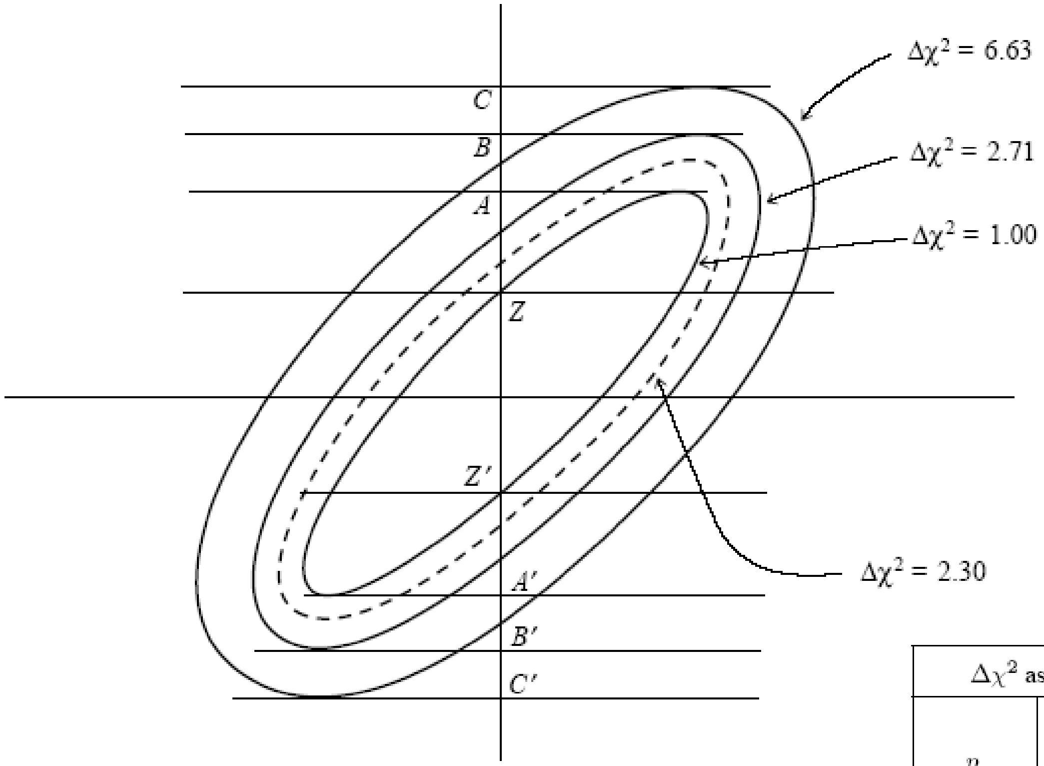


Parameter estimation: 2-D case

'Best-fit' value of θ_2 , found from $\left.\frac{\partial \ell}{\partial \theta_j}\right|_{\theta_j=\theta_{0j}} = 0$



Parameter estimation: 2-D case



From Numerical Recipes

$\Delta\chi^2$ as a Function of Confidence Level and Degrees of Freedom						
	ν					
p	1	2	3	4	5	6
68.3%	1.00	2.30	3.53	4.72	5.89	7.04
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Parameter estimation: 2-D case

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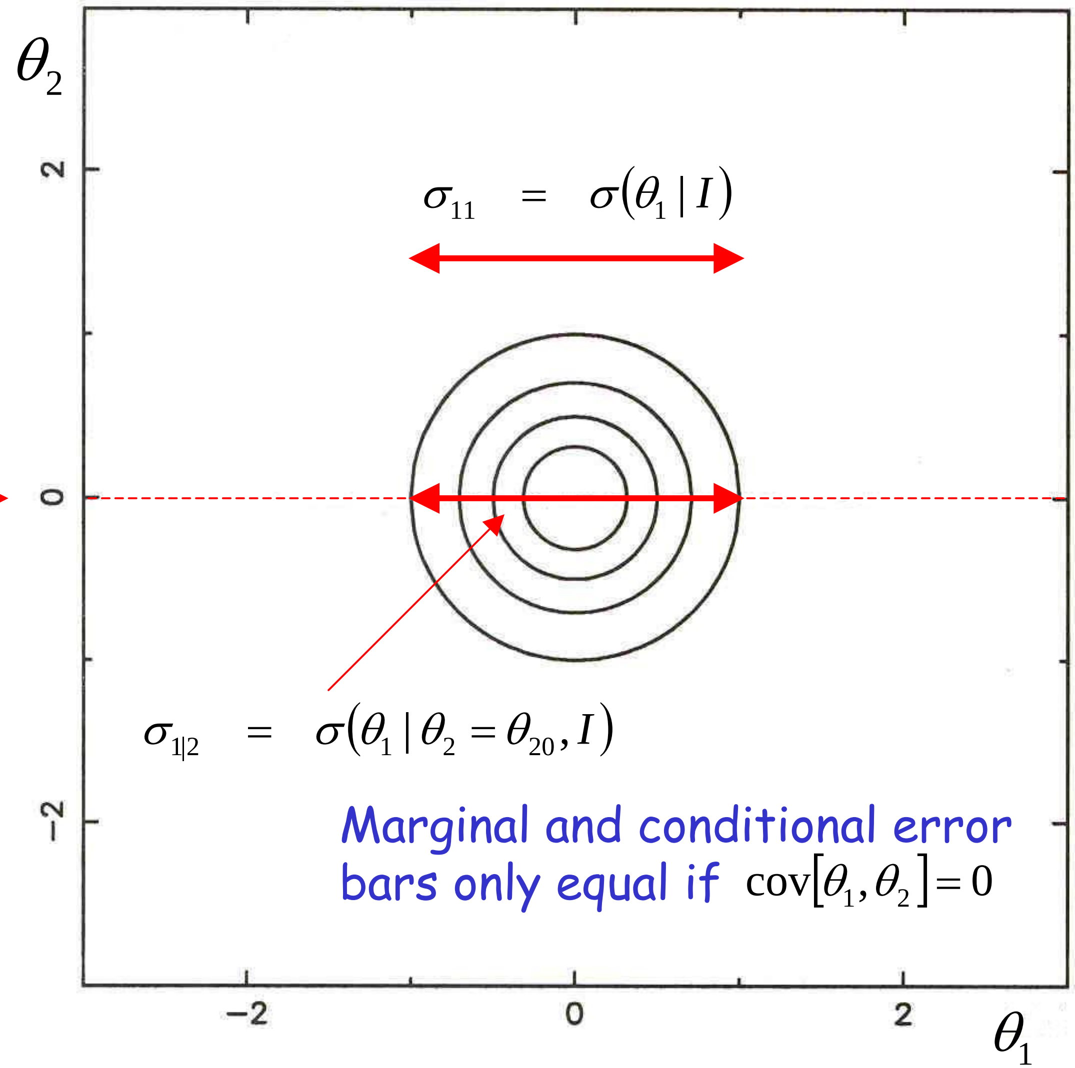


$$\sigma_{11} \;=\; \sigma(\theta_1 \mid I)$$

$$\sigma_{1|2} \;=\; \sigma(\theta_1 \mid \theta_2 = \theta_{20}, I)$$

Marginal and conditional error bars only equal if $\mathrm{cov}[\theta_1, \theta_2] = 0$

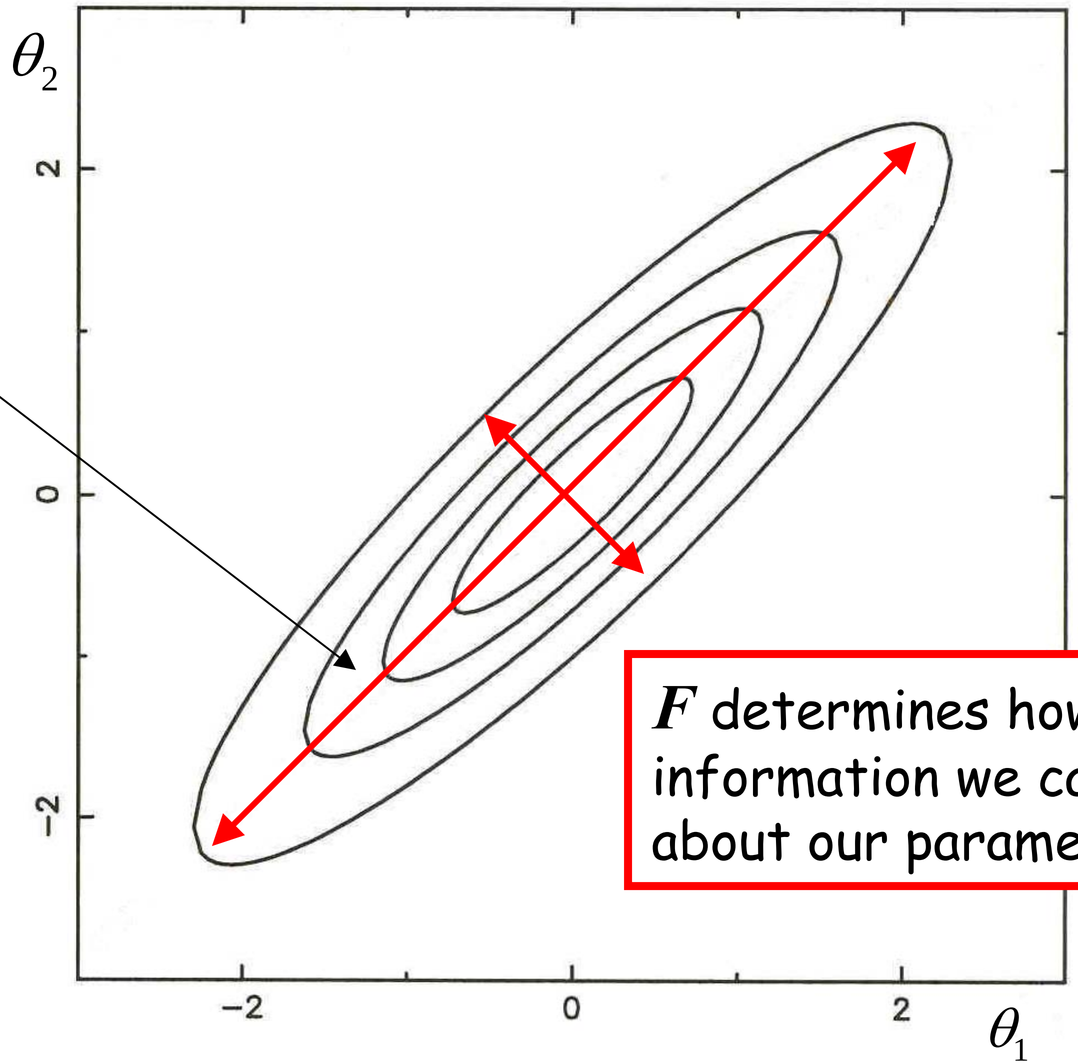
Parameter estimation: 2-D case

Linear combination of θ_1 and θ_2 well constrained by data

Length of axes determined by the **eigenvalues** of the Fisher information matrix

$$F_{ij} = \frac{\partial^2 \ell}{\partial\theta_i \partial\theta_j} = \left[-\sigma_{ij}^2\right]^{-1}$$

$$\boxed{\boldsymbol{F}\,\boldsymbol{\theta} = \lambda\,\boldsymbol{\theta}}$$



F determines how much information we can learn about our parameters

Direction of axes are the *eigenvectors* of ***F***

Parameter estimation: Gaussian approximation

Taylor expand $\ell(\theta_1, \theta_2)$ around θ_{0j} :

$$
\boxed{
\begin{aligned}
\ell(\theta_1,\theta_2) \;=\; & \ell(\theta_{01},\theta_{02}) \;+\; \cancel{\left.\frac{\partial \ell}{\partial \theta_1}\right|_{\theta_j=\theta_{0j}} (\theta_1-\theta_{01})} \;+\; \cancel{\left.\frac{\partial \ell}{\partial \theta_2}\right|_{\theta_j=\theta_{0j}} (\theta_2-\theta_{02})} \;+ \\
& \frac{1}{2}\left[\left.\frac{\partial^2 \ell}{\partial \theta_1^{\,2}}\right|_{\theta_j=\theta_{0j}} (\theta_1-\theta_{01})^2 + \left.\frac{\partial^2 \ell}{\partial \theta_2^{\,2}}\right|_{\theta_j=\theta_{0j}} (\theta_2-\theta_{02})^2 + 2\left.\frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_2}\right|_{\theta_j=\theta_{0j}} (\theta_1-\theta_{01})(\theta_2-\theta_{02}) \right] + \ldots
\end{aligned}
}
$$

$$p(\theta_1, \theta_2 \mid \text{data}, \; I) \;=\; \exp\left[\ell(\theta_1, \theta_2)\right]$$

$$= \exp\left[-\tfrac{1}{2} Q\right]$$

← Gaussian approximation

Is the Gaussian approximation a good idea?

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Parameter estimation: Gaussian approximation

Taylor expand $\ell(\theta_1,\theta_2)$ around θ_{0j} :

$$\ell(\theta_1,\theta_2) = \ell(\theta_{01},\theta_{02}) + \cancel{\left.\frac{\partial \ell}{\partial \theta_1}\right|_{\theta_j=\theta_{0j}} (\theta_1-\theta_{01})} + \cancel{\left.\frac{\partial \ell}{\partial \theta_2}\right|_{\theta_j=\theta_{0j}} (\theta_2-\theta_{02})} +$$

$$\frac{1}{2}\left[\left.\frac{\partial^2 \ell}{\partial \theta_1^{\,2}}\right|_{\theta_j=\theta_{0j}} (\theta_1-\theta_{01})^2 + \left.\frac{\partial^2 \ell}{\partial \theta_2^{\,2}}\right|_{\theta_j=\theta_{0j}} (\theta_2-\theta_{02})^2 + 2\left.\frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_2}\right|_{\theta_j=\theta_{0j}} (\theta_1-\theta_{01})(\theta_2-\theta_{02}) \right] + \dots$$

Is the Gaussian approximation a good idea?

- Greatly simplifies calculations – only need to compute the elements of the Fisher matrix (covariance matrix)
- Nowadays much better to compute **full posterior** pdf. Not too hard with present-day computers, even for large N

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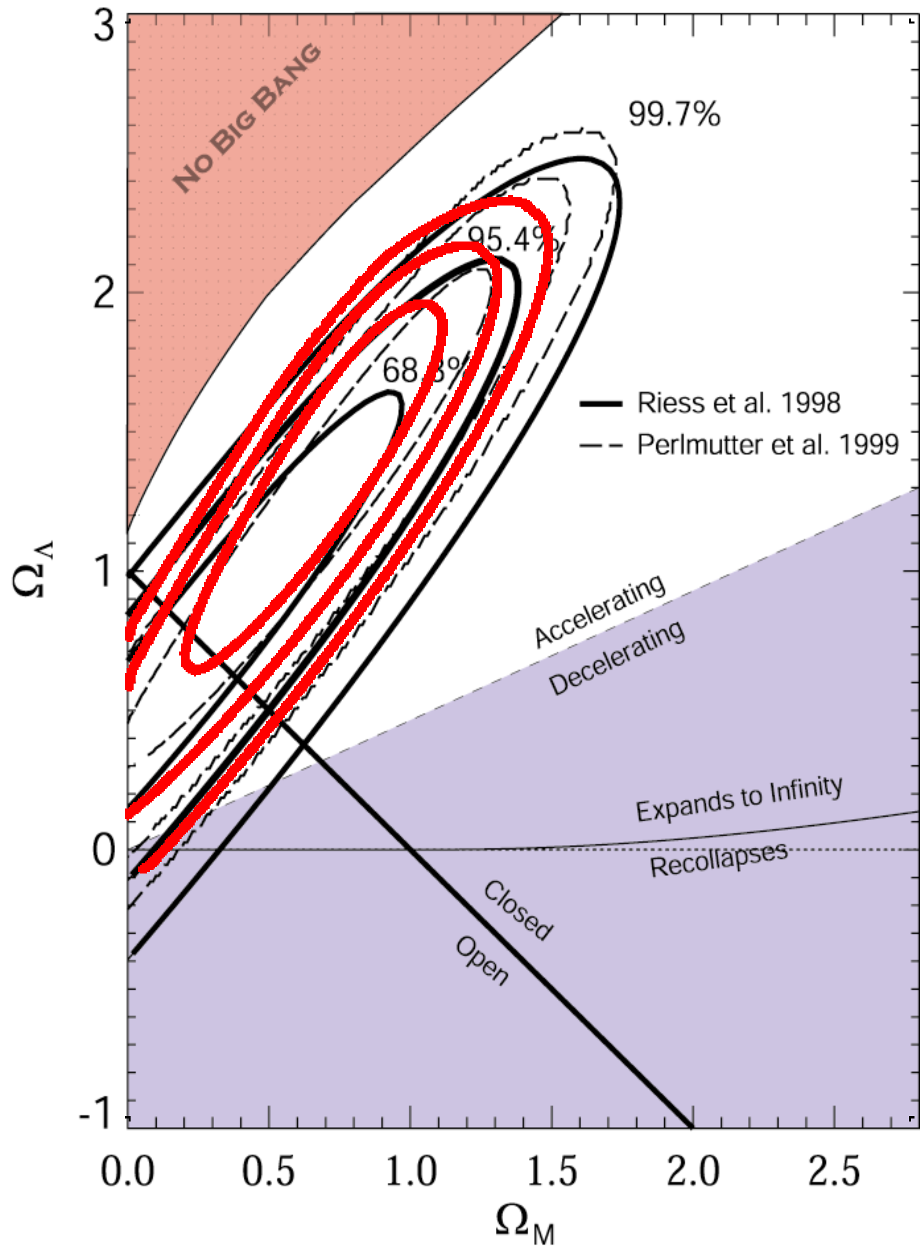
Markov Chain Monte Carlo Methods



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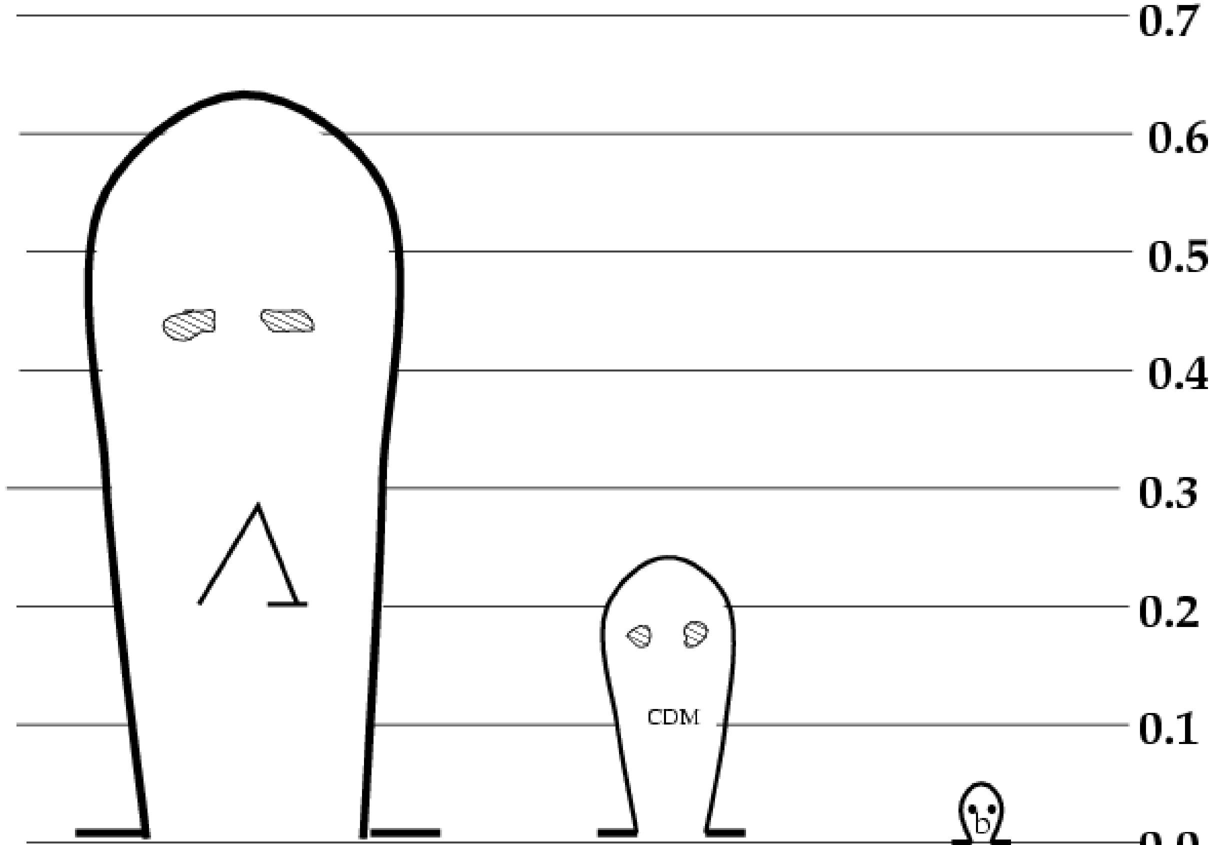
Λ CDM

Figure 3. A line up of cosmological culprits Ω_Λ is the big shot controlling the Universe. He's going to make it blow up. Ω_{CDM} would like to make the Universe collapse but can't compete with Ω_Λ. Ω_b just follows Ω_{CDM} around. Like all dangerous criminals, one can never be sure of Ω_Λ until he is behind bars. The CMB police is being beefed up. Hundreds of heroic CMB observers are now planning his capture.

From Lineweaver (1998)

Cosmology's Most Wanted



Ω_Λ	Ω_{CDM}	Ω_b
cosmological constant energy of the vacuum He never clumps His evil plan is to blow up the Universe	cold dark matter He likes to clump but has never been detected directly His evil plan is to make the Universe collapse	normal baryonic matter a pawn in the cosmic game who just follows CDM around. He thinks he's a complex life form but is really just a bunch of hydrogen

General Relativity:-

Geometry $\longleftrightarrow$ matter / energy

> *"Spacetime tells matter how to move and matter tells spacetime how to curve"*

Einstein's Field Equations

$$\underbrace{G^{\mu\nu}}_{\text{Einstein tensor}} = \underbrace{R^{\mu\nu}}_{\text{Ricci tensor}} - \frac{1}{2}\underbrace{g^{\mu\nu}}_{\text{Metric tensor}}\underbrace{R}_{\text{Curvature scalar}} = 8\pi G \underbrace{T^{\mu\nu}}_{\text{Energy-momentum tensor of gravitating mass-energy}}$$

General Relativity:-

Geometry $\longleftrightarrow$ matter / energy

> *"Spacetime tells matter how to move and matter tells spacetime how to curve"*

Einstein's Field Equations

Given $g^{\mu\nu}$ can compute $R^{\mu\nu}$ and R ;

These are *generated* by $T^{\mu\nu}$

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Treat Universe as a *perfect fluid*

$$\boxed{T^{\mu\nu} = (\underbrace{\rho}_{\text{Density}} + \underbrace{P}_{\text{Pressure}})\underbrace{u^{\mu}u^{\nu}}_{\text{Four-velocity}} + Pg^{\mu\nu}}$$

N.B. $c = 1$

Solve to give *Friedmann's Equations*

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3P)$$

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Einstein originally sought *static solution* i.e. :-

$$\dot{R} = 0 \quad \text{for all} \quad t$$

But if $\rho, P \geq 0$ can't have $\ddot{R} = 0$

However, GR actually gives

Covariant derivative

$$G^{\mu\nu}{}_{;\nu} = T^{\mu\nu}{}_{;\nu} = 0$$

Can add a constant times $g^{\mu\nu}$ to $G^{\mu\nu}$

Einstein's cosmological constant

$$\boxed{G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + g^{\mu\nu} \Lambda}$$

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Friedmann's Equations now give:-

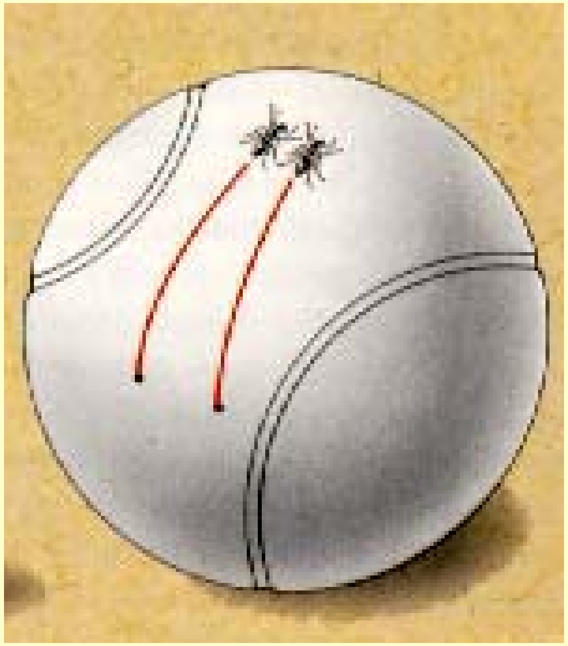
$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - \frac{k}{R^2}$$

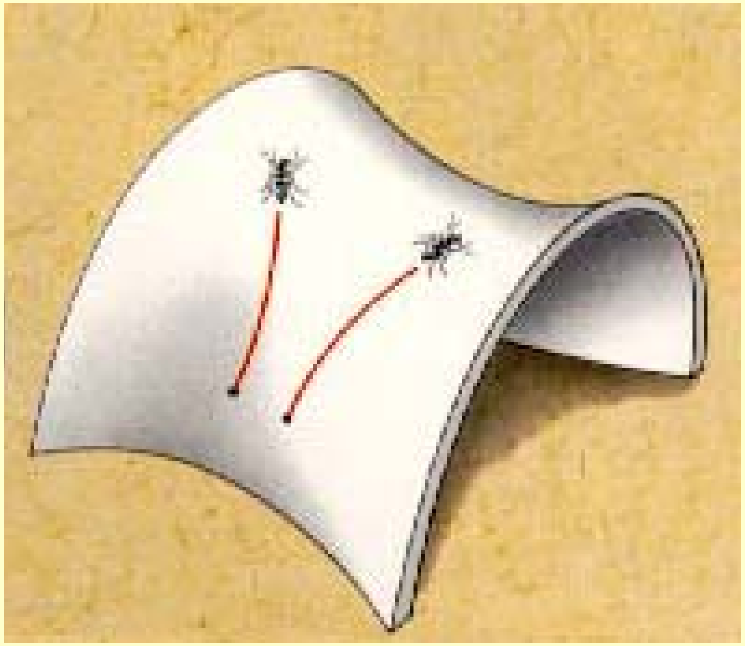
$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + 3P\right) + \frac{\Lambda}{3}$$

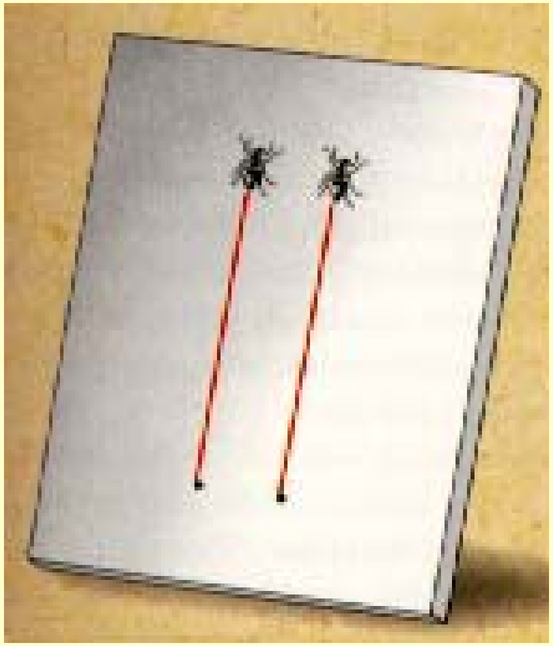
Can tune Λ to give $\dot{R} = 0 \text{ for all } t$ but *unstable*

(and *Hubble expansion* made idea redundant)

But Lambda term could still be non-zero anyway !


Closed


Open


Flat

$$k = \text{curvature constant} = \begin{cases} -1, & \text{open} \\ 0, & \text{flat} \\ +1, & \text{closed} \end{cases}$$

Einstein's greatest blunder?

Re-expressing Friedmann's Equations:-

For $\Lambda = 0$

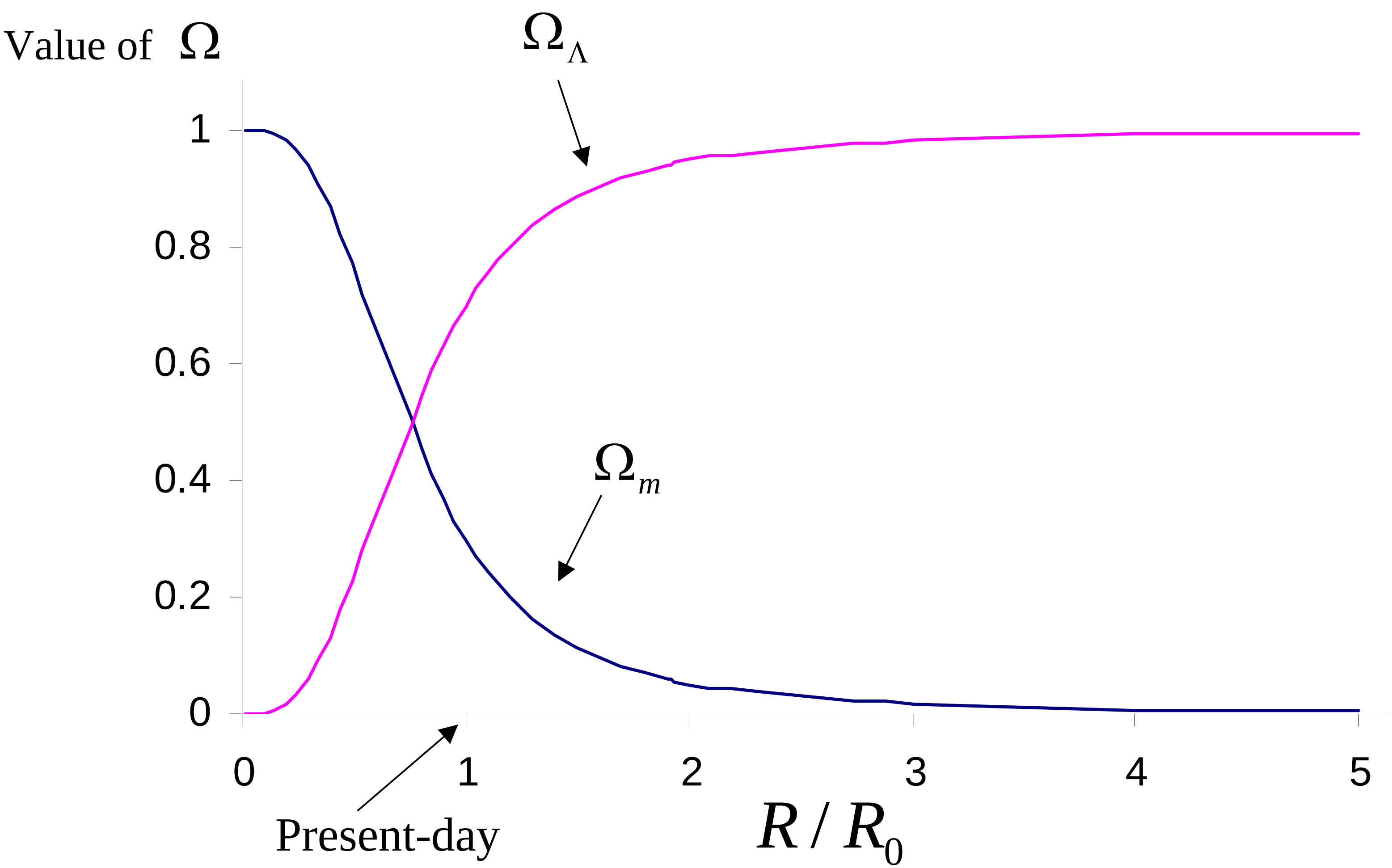
$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2} \quad \Rightarrow \quad k = 0 \Leftrightarrow \rho = \left[\frac{8\pi G}{3H^2}\right]^{-1} = \rho_{\text{crit}}$$

Define

$$\Omega_m = \rho \big/ \rho_{crit} = \frac{8\pi G\rho}{3H^2} \qquad \Omega_\Lambda = \frac{\Lambda}{3H^2} \qquad \Omega_k = -\frac{k}{R^2 H^2}$$

It follows that, at *any time*

$$\boxed{\Omega_m + \Omega_\Lambda + \Omega_k = 1}$$



If the Concordance Model is right, we live at a special epoch. Why?...

Hubble diagram of distant supernovae

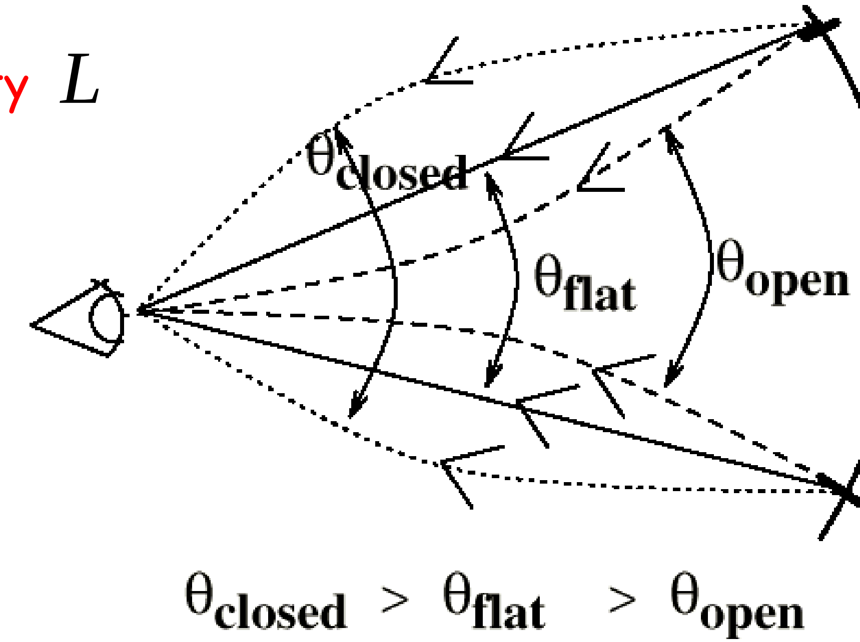
Consider an object of intrinsic luminosity L

from which we observe a flux $\mathfrak{I}$

Define the Luminosity Distance via:-

$$\boxed{\mathfrak{I} = \frac{L}{4\pi d_L^{\,2}}}$$

Distance required to give observed flux *if* Universe has a flat geometry



Actual distance depends on true geometry, and expansion history of the Universe

$$\boxed{d_{\mathrm{L}} \equiv d_{\mathrm{L}}(z;\Omega_m,\Omega_\Lambda) = (1+z)^2 d_{\mathrm{ang}}(z;\Omega_m,\Omega_\Lambda)}$$

Adapted from Schmidt (2002)

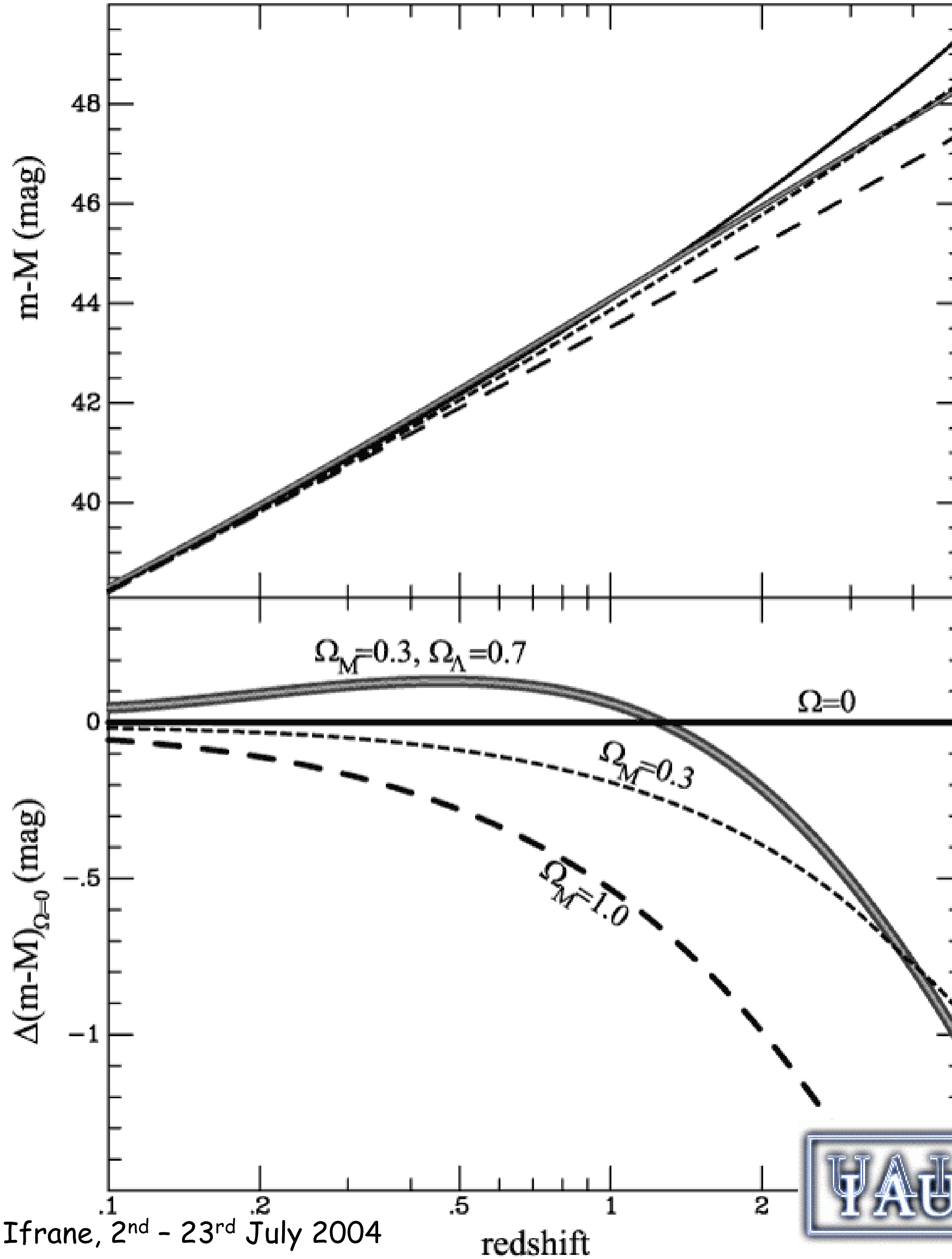
Distance Modulus

$$(m - M)_{mag} = 5\log\left[\frac{d_L}{Mpc}\right] + 25$$

Fractional distance change
$\cong \frac{1}{2}$(mag change)

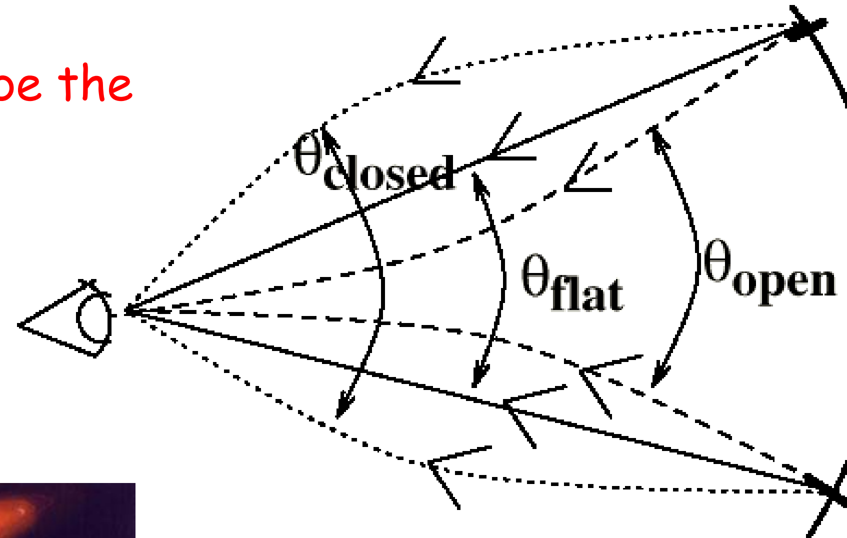
e.g.

0.1 mag difference is 5% distance difference



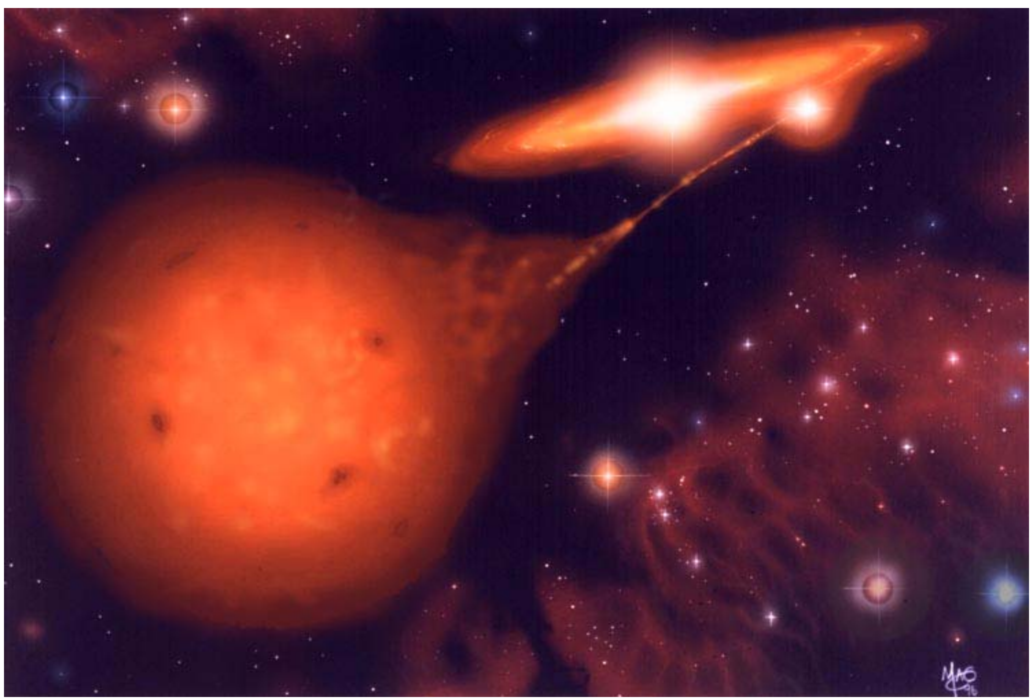
Hubble diagram of distant supernovae

We need a good standard candle, to probe the geometry of the Universe



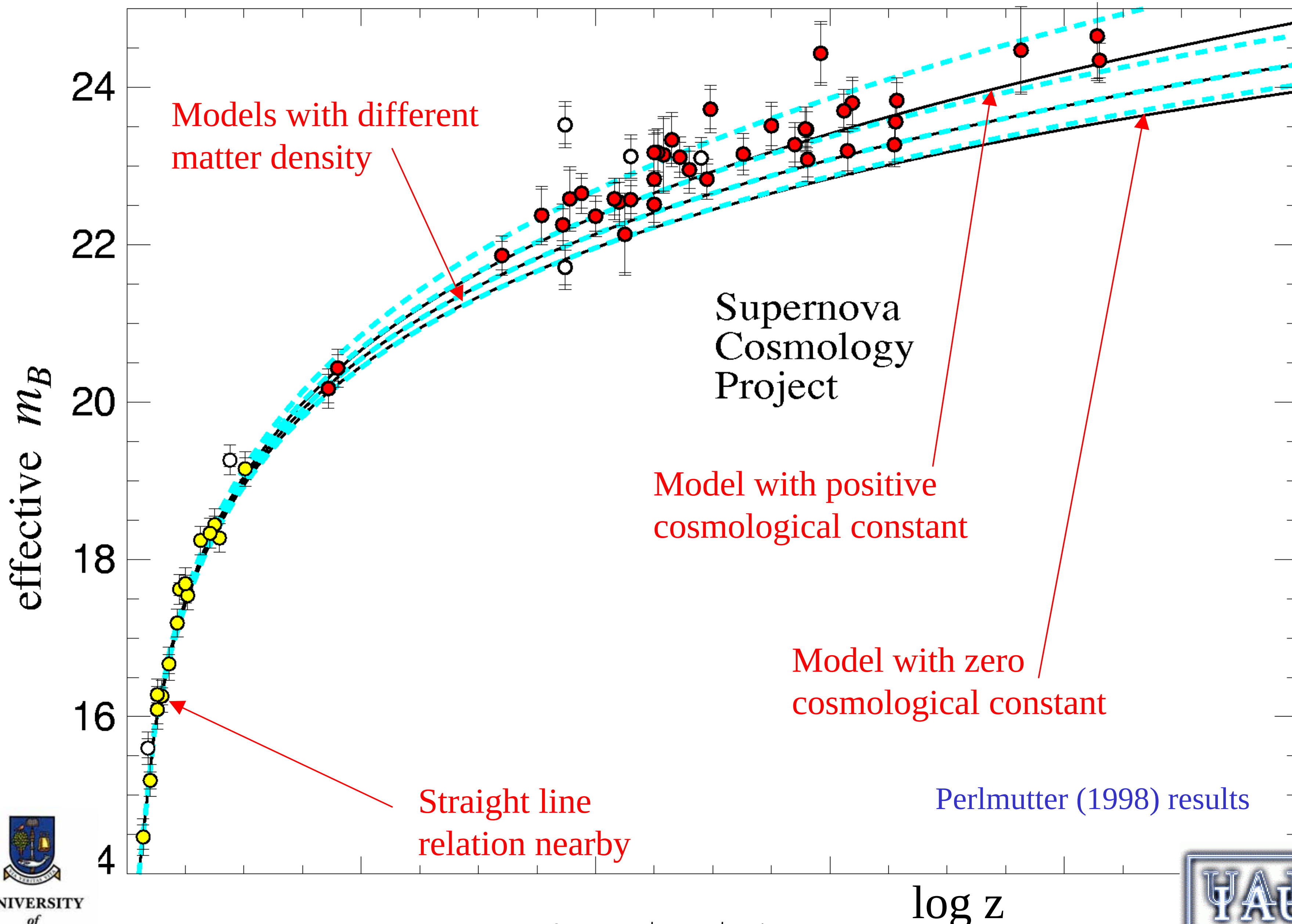
$\theta_{closed} > \theta_{flat} > \theta_{open}$

Type Ia Supernovae



- Visible to huge distances
- Small scatter in luminosity at peak brightness
- Observational programs to detect dozens (100s)
- Can also use `Light Curve Shape' to improve distance estimates

Hubble diagram of distant Type Ia supernovae



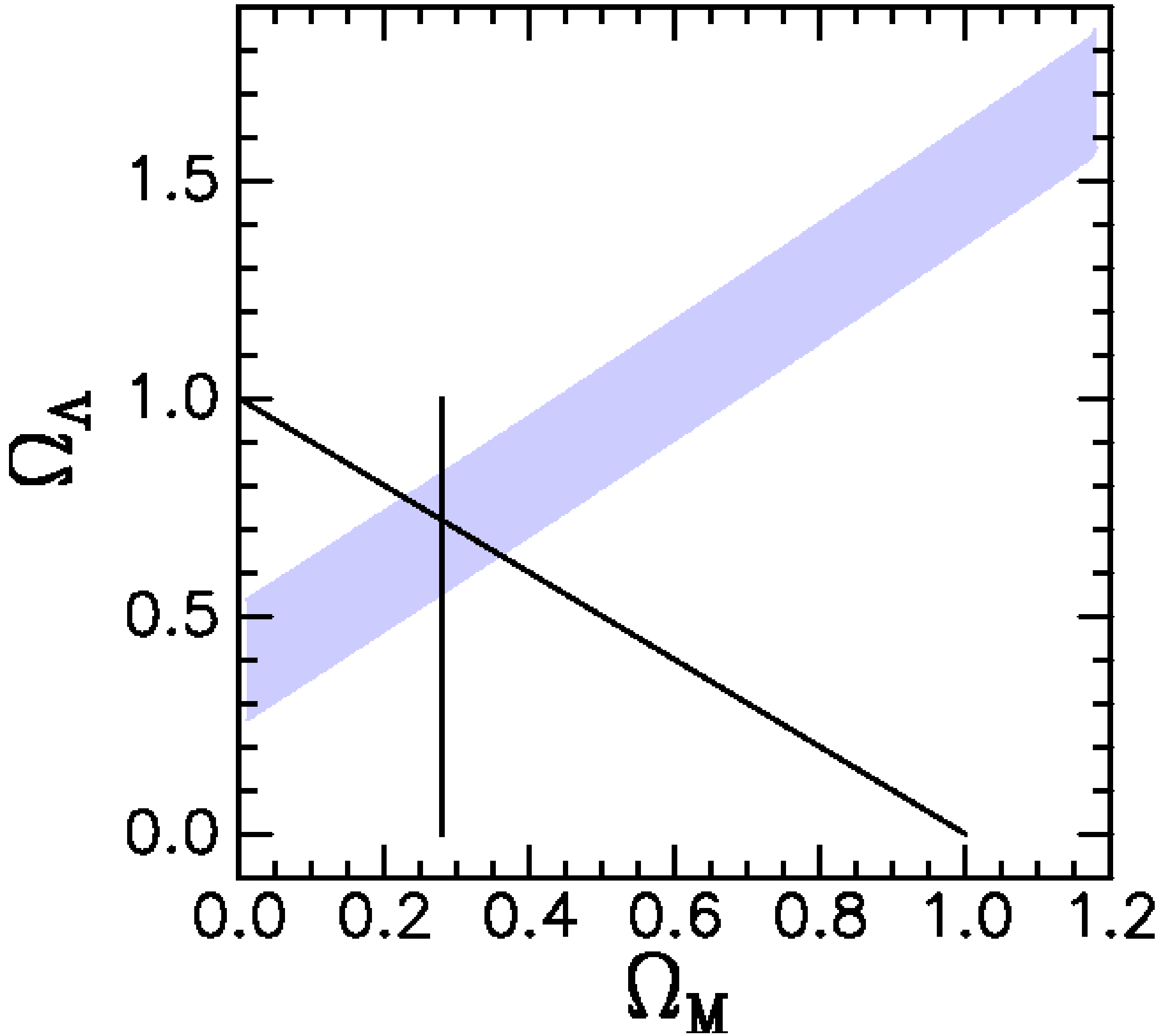
SNIa at z = 0.5

At low redshift, SN1a essentially measure the deceleration parameter

$$q_0 \equiv \frac{-R(t_0)R(t_0)}{\dot{R}^2(t_0)}$$

$$= \frac{1}{2}\sum_i \Omega_i (1+3w_i)$$

$$e.g.,\ q_0 = \frac{1}{2}\Omega_m - \Omega_\Lambda$$



Adapted from Schmidt (2002)

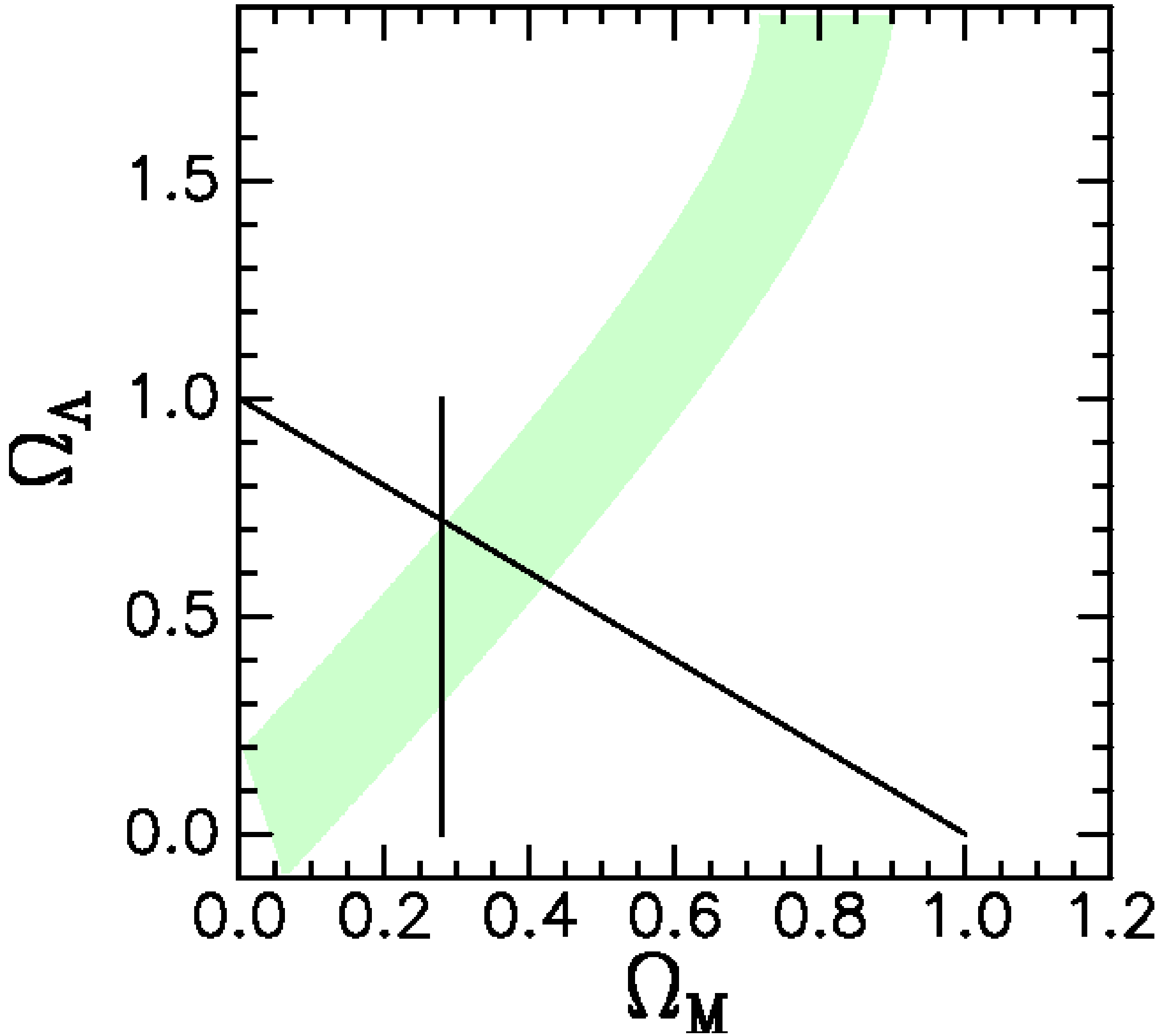
SNIa at z = 1.0

At low redshift, SN1a essentially measure the deceleration parameter

$$q_0 \equiv \frac{-\ddot{R}(t_0)R(t_0)}{\dot{R}^2(t_0)}$$

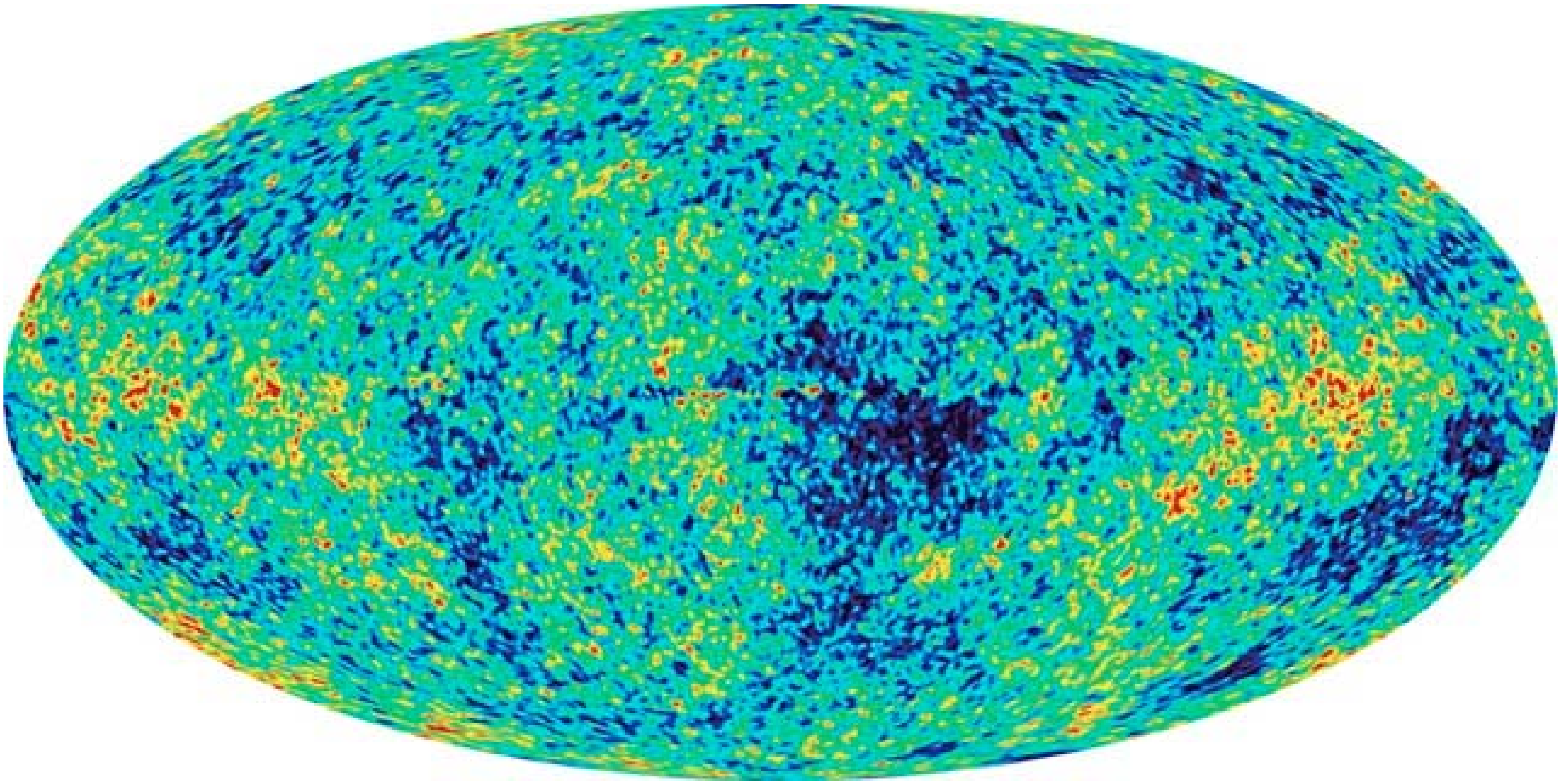
$$= \frac{1}{2}\sum_i \Omega_i (1+3w_i)$$

$$e.g.,\ q_0 = \frac{1}{2}\Omega_m - \Omega_\Lambda$$



Adapted from Schmidt (2002)

From Bennett et al (2003)



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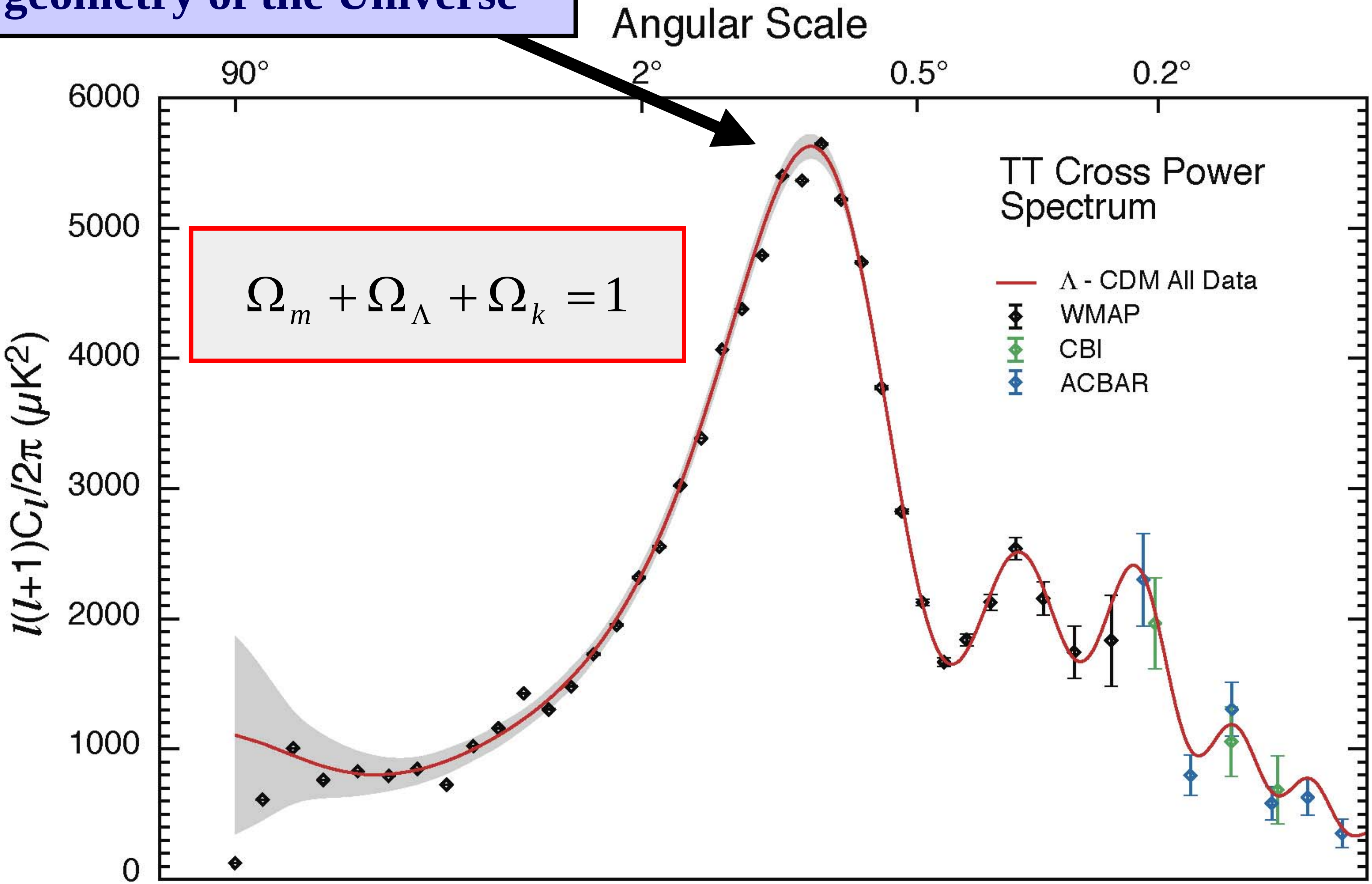
Position of first peak sensitive probe of the geometry of the Universe



$$\Omega_m + \Omega_\Lambda + \Omega_k = 1$$

Tegmark et al (1998)

SNIa measure:-

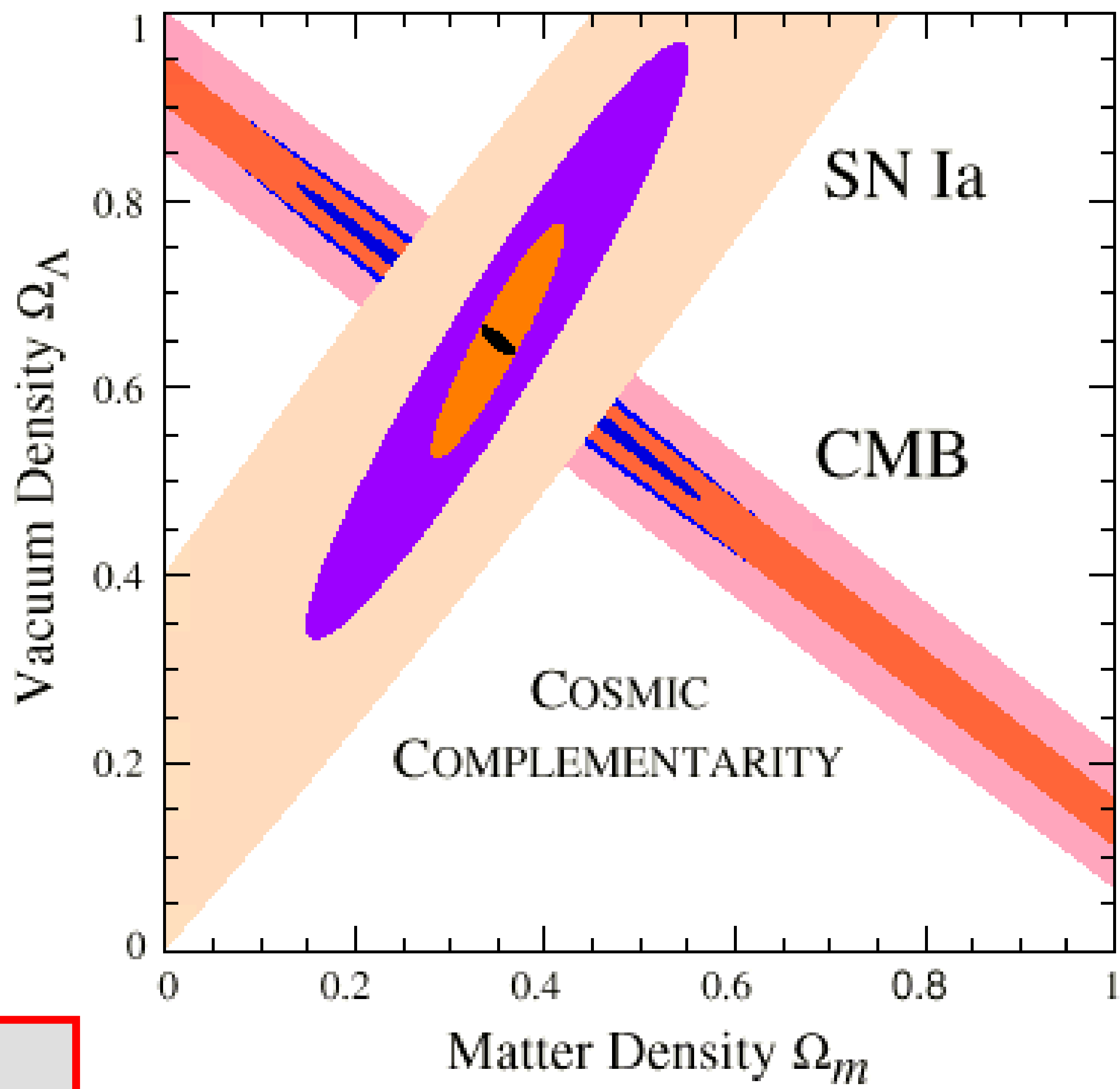
$$q_0 = \frac{1}{2}\Omega_m - \Omega_\Lambda$$

CMBR measures:-

$$\Omega_k = 1 - \Omega_m - \Omega_\Lambda$$

Together, can constrain:-

$$\boxed{\Omega_m, \Omega_\Lambda}$$



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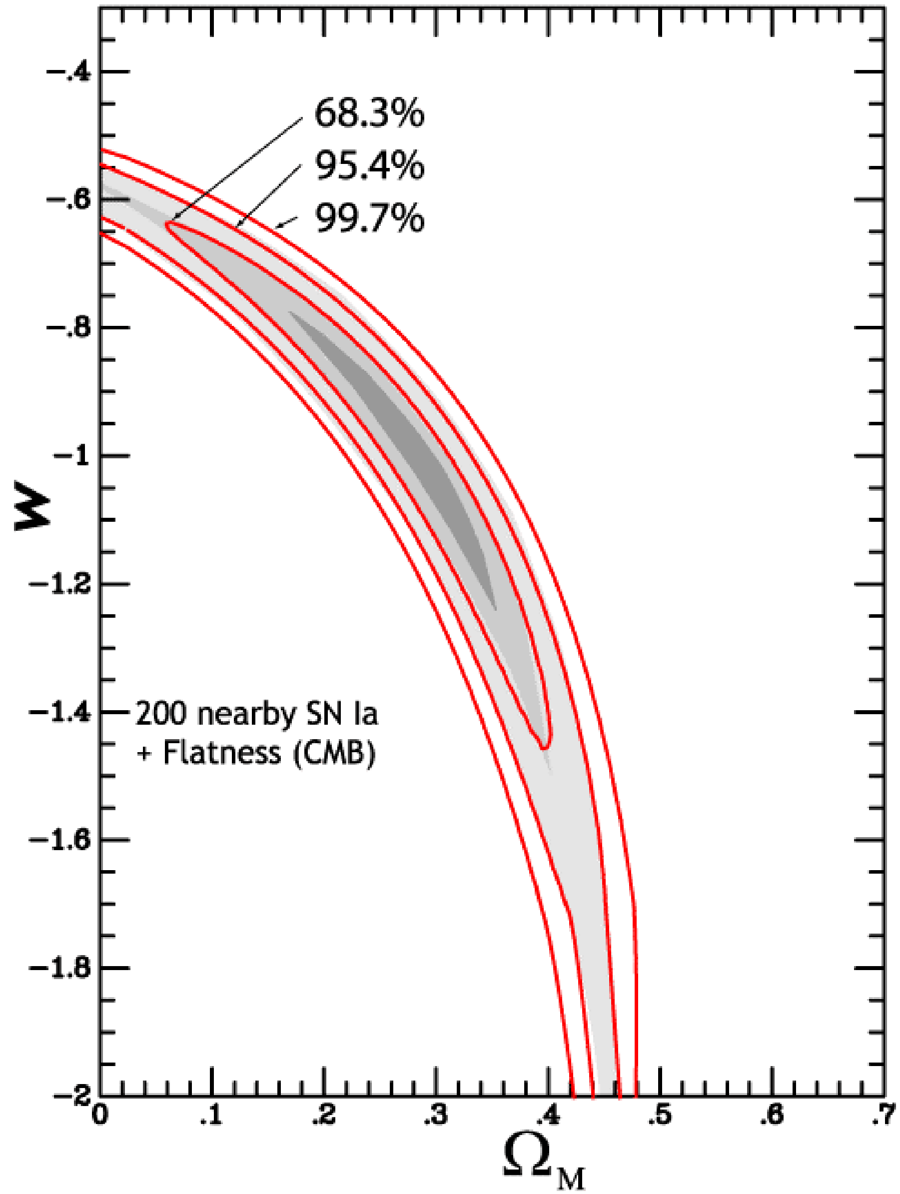
Can we distinguish a constant Λ term from quintessence?…

Not from current ground-based SN observations (combined with e.g. LSS)…

…or from future ground-based observations



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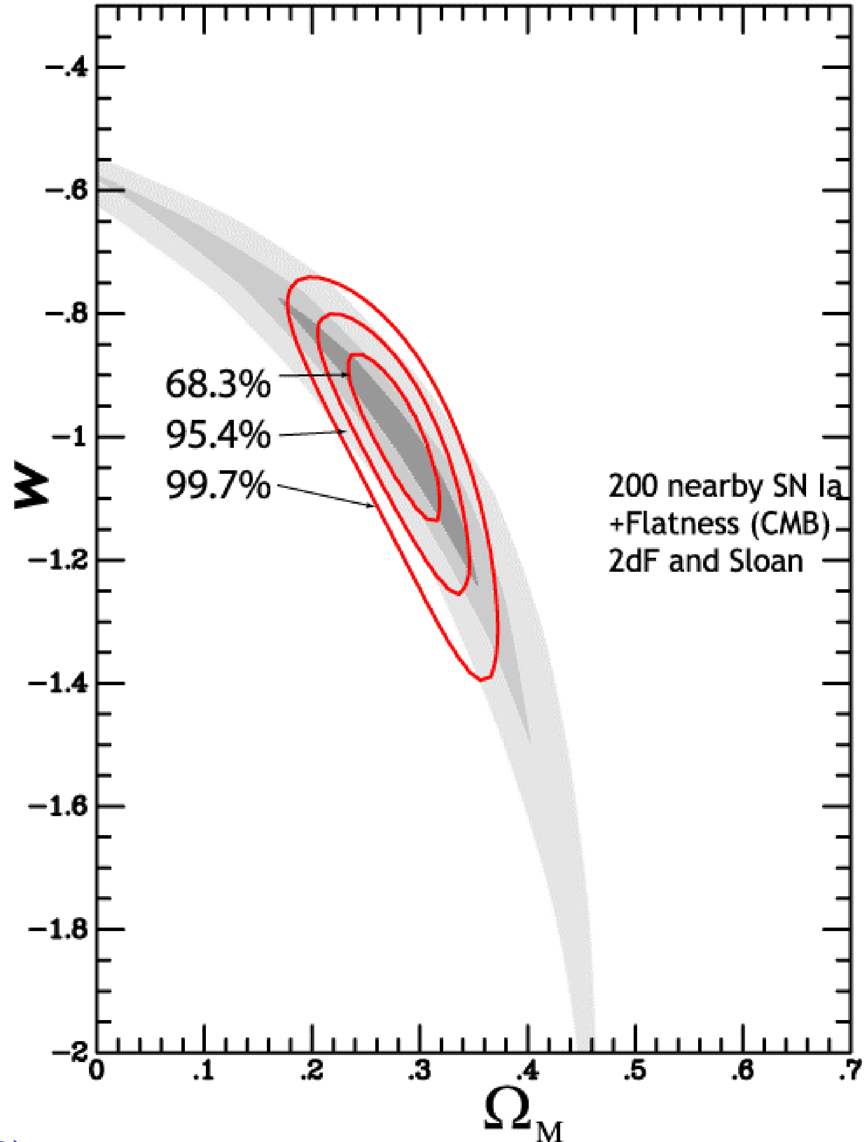

Adapted from Schmidt (2002)

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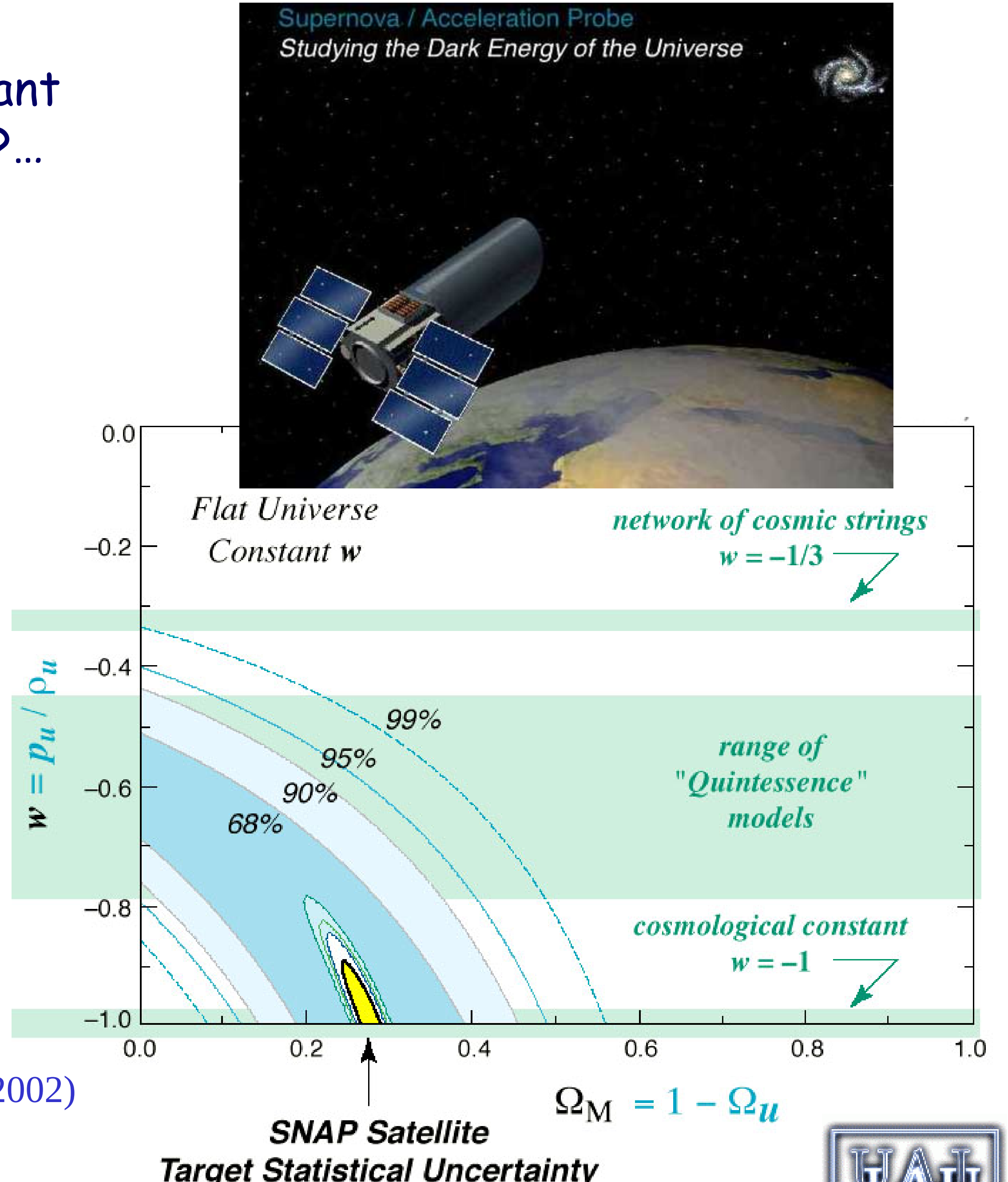
Main goal of the SNAP satellite (launch ~2010?)



Adapted from Schmidt (2002)