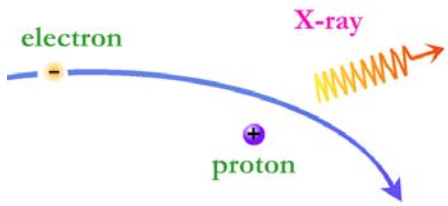
2. Bremsstrahlung and Atomic Line Sources

2.1 Source emission measure and line ratios

X-ray bremsstrahlung continuum (free-free) emission produced in hot, **tenuous** gases. (compare with black body).



Bremsstrahlung = 'braking radiation'

Emissivity proportional to number density of 'beam' and target particles

$$\boxed{j \propto n_P^2} \qquad (2.1)$$

11. Thermal Bremsstrahlung Revisited

The differential emissivity is, then,

$$\frac{dJ}{d\varepsilon} = \frac{Q_0\, m_e\, c^2}{\varepsilon} \int_V n_P(\vec{r}) \int_\varepsilon^\infty \frac{F(E,T)}{E}\, dE\, dV$$

$$= \frac{2\, n_P^{\,2}\, V\, Q_0\, m_e\, c^2}{\varepsilon} \int_\varepsilon^\infty \sqrt{\frac{2}{\pi\, m}}\, \frac{E}{E}\, \frac{1}{(kT)^{3/2}} \exp\left[-{E}/{kT}\right] dE$$

Put $z = E / kT$ and integrate:-

$$\boxed{\frac{dJ}{d\varepsilon} = 2\left(\frac{2}{\pi\, m_e}\right)^{1/2} \frac{n_P^{\,2}\, V\, Q_0\, m_e\, c^2}{(kT)^{1/2}\, \varepsilon} \exp\left[-{\varepsilon}/{kT}\right]}$$

In HEA1 we introduced the concept of the **emission measure function** – which is a measure of how much of the plasma is at temperature T

We showed (see Section 11)

$$\boxed{\frac{dJ}{d\varepsilon} = 2\left(\frac{2}{\pi\, m_e}\right)^{1/2} \frac{Q_0\, m_e\, c^2}{k^{1/2}\varepsilon} \int_0^\infty \xi(T)\, \frac{e^{-\varepsilon/kT}\, dT}{T^{1/2}}} \qquad (2.2)$$

Where

$$\boxed{\xi(T) = \frac{d}{dT}\int_V n_P(\vec{r})^2\, dV} \qquad (2.3)$$

Source emission measure function

In fact we considered the uniform temperature, density case first.

Suppose we have a plasma with uniform temperature $T = T_0$

Then

$$\boxed{\xi(T) = \xi_0 \times \delta(T - T_0)} \qquad (2.4)$$

Independent of temperature

Dirac delta function

And

$$\frac{dJ}{d\varepsilon} = 2\left(\frac{2}{\pi\, m_e}\right)^{1/2} \frac{Q_0\, m_e\, c^2}{k^{1/2}\varepsilon}\; \xi_0\, \frac{e^{-\varepsilon/kT_0}}{T_0^{1/2}} \qquad (2.5)$$

Matching up with HEA1, Sect. 11, we see that

$$\boxed{\xi_0 = n_P{}^2\, V} \qquad (2.6)$$

Or, if the proton number density is *not* constant

$$\boxed{\xi_0 = \int_V n_P(\vec{r})^2\, dV} \qquad (2.7)$$

In general, then, we can write

$$\frac{dJ}{d\varepsilon} = \int_0^\infty \xi(T)\, K(\varepsilon, T)\, dT \qquad (2.8)$$

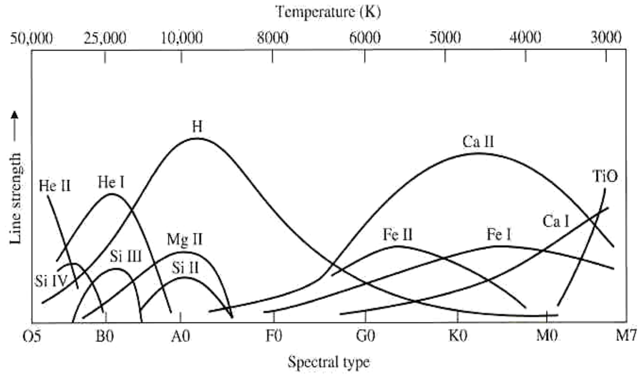
And for an **isothermal** plasma

$$\frac{dJ}{d\varepsilon} = \xi_0\, K(\varepsilon, T_0) \qquad (2.9)$$

The same principles and relations hold for **atomic line** (bound-bound) emission, but the emissivity will usually be a strongly peaked function of temperature – i.e. there will be a temperature at which the energy of thermal plasma is optimal to excite the ion involved and produce the line emission.

Different lines will peak at different temperatures

We saw this in A1Y stellar astrophysics as a graph of line strength vs temperature



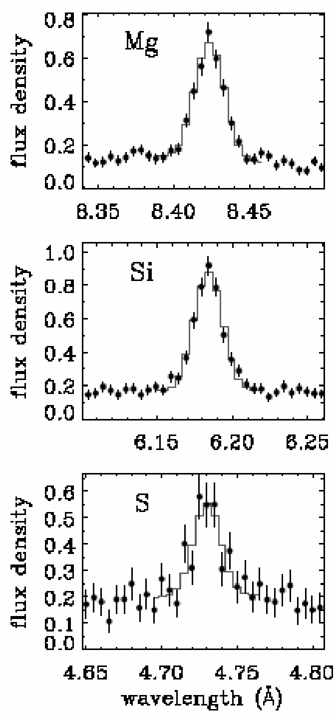
The ratio(s) of emissivity for two (or more) lines can be used as a diagnostic of the temperature and density of the plasma, since e.g. the ratio for two lines may be a monotonic function of temperature.

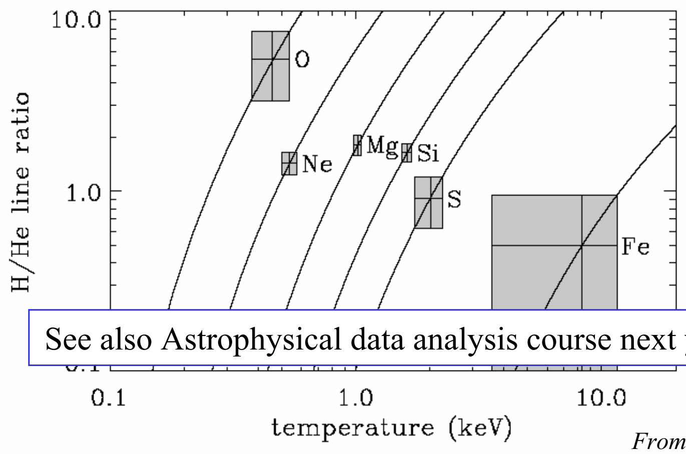
For example

Line 1 at energy ε_1 : $\frac{dJ}{d\varepsilon} = \xi_0\, K_1(\varepsilon_1, T_0)$

Line 2 at energy ε_2 : $\frac{dJ}{d\varepsilon} = \xi_0\, K_2(\varepsilon_2, T_0)$

Ratio may be monotonic function of temperature





See also Astrophysical data analysis course next year

From Mauche (2001)

2.2 Limitations on thermal bremsstrahlung spectra

In HEA1 we saw that for a general thermal source

$$\boxed{\frac{dJ}{d\varepsilon} = 2\left(\frac{2}{\pi m_e}\right)^{1/2} \frac{Q_0\, m_e\, c^2}{k^{1/2}\varepsilon} \int_0^\infty \xi(T)\, \frac{e^{-\varepsilon/kT}\, dT}{T^{1/2}}} \qquad (2.10)$$

But that if $\quad \xi(T) \propto T^{-\alpha} \quad \Rightarrow \quad \dfrac{dJ}{d\varepsilon} \propto \varepsilon^{-\beta}$

i.e. A power law form for the source emission measure can make a thermal source mimic a non-thermal source with a power law emissivity

Does this result generalise? By finding an appropriate form for $\xi(T)$ *can we make a thermal source mimic* *any* *form for* $\dfrac{dJ}{d\varepsilon}$ *?*

The answer is no, because if $\xi(T)$ is to be physically meaningful, this imposes certain constraints on the form which $\dfrac{dJ}{d\varepsilon}$ must take.

Let's write

$$\boxed{\mathrm{H}(\varepsilon) \equiv \frac{dL}{d\varepsilon} = \varepsilon\frac{dJ}{d\varepsilon} = C \int_0^\infty \xi(T)\, \frac{e^{-\varepsilon/kT}\, dT}{T^{1/2}}} \qquad (2.11)$$

Positive constant

Then, we can show that

$$\frac{d\mathrm{H}}{d\varepsilon} = -\frac{C}{k}\int_0^\infty \xi(T)\, \frac{e^{-\varepsilon/kT}\, dT}{T^{3/2}} \qquad (2.12)$$

and

$$\frac{d^n\mathrm{H}}{d\varepsilon^n} = (-1)^n \frac{C}{k^n}\int_0^\infty \xi(T)\, \frac{e^{-\varepsilon/kT}\, dT}{T^{n+1/2}} \qquad (2.13)$$

(see example sheets)

This means that, for $\xi(T) \geq 0$

$$\boxed{\frac{d\mathrm{H}}{d\varepsilon} \leq 0 \qquad \frac{d^2\mathrm{H}}{d\varepsilon^2} \geq 0 \qquad \frac{d^3\mathrm{H}}{d\varepsilon^3} \leq 0}$$

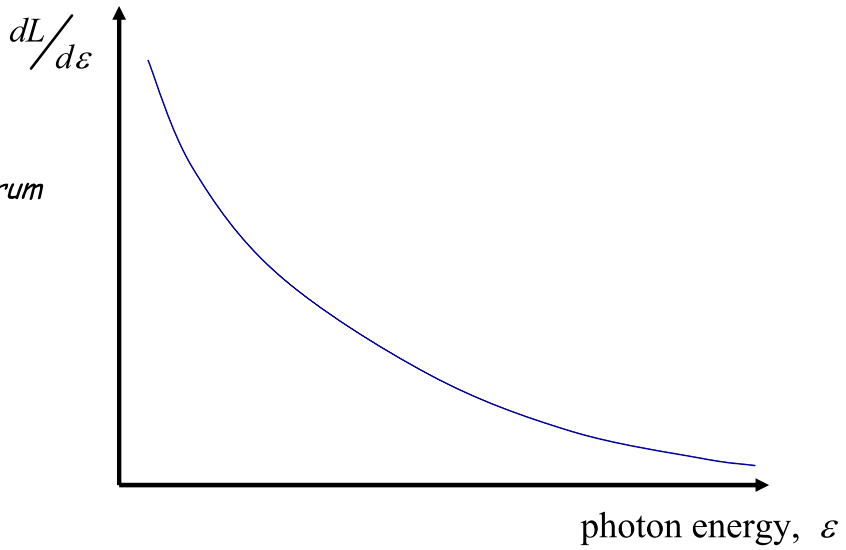
and so on...

Any source with observed differential luminosity that doesn't satisfy these conditions ***cannot*** be 100% thermal.

Example:

Exponentially decaying spectrum

This shape satisfies the derivative constraints



We can contrast this with a *non-thermal* spectrum.

We saw in HEA1

$$\frac{dj}{d\varepsilon} = n_P \int_{\varepsilon}^{\infty} F(E) \frac{dQ_B}{d\varepsilon}(\varepsilon, E)\, dE \tag{2.14}$$

And taking

$$\frac{dQ_B}{d\varepsilon}(\varepsilon, E) = \frac{Q_0\, m_e c^2}{\varepsilon\, E}$$

Implies that

$$\frac{dj}{d\varepsilon} = n_P \frac{Q_0\, m_e\, c^2}{\varepsilon} \int_{\varepsilon}^{\infty} \frac{F(E)}{E}\, dE \tag{2.15}$$

This relates to issues addressed in HEA1

11d. Interpreting Energy Spectra

Consider a non-thermal source, homogeneous plasma

$$\frac{dj}{d\varepsilon} = n_P \frac{Q_0\, m_e\, c^2}{\varepsilon} \int_{\varepsilon}^{\infty} \frac{F(E)}{E}\, dE \quad \text{photons m}^{-3}\,\text{s}^{-1}\,\text{keV}^{-1}$$

$$\boxed{\frac{dJ}{d\varepsilon} = \frac{n_P V\, Q_0\, m_e\, c^2}{\varepsilon} \int_{\varepsilon}^{\infty} \frac{F(E)}{E}\, dE \quad \text{photons s}^{-1}\,\text{keV}^{-1}}$$

11d. Interpreting Energy Spectra

The integral $\int_{\varepsilon}^{\infty} \frac{F(E)}{E}\, dE$ is a function of photon energy, ε

We define $$G(\varepsilon) = \int_{\varepsilon}^{\infty} \frac{F(E)}{E}\, dE$$

Then $$G(\varepsilon + d\varepsilon) = \int_{\varepsilon + d\varepsilon}^{\infty} \frac{F(E)}{E}\, dE$$

So that $$G(\varepsilon + d\varepsilon) - G(\varepsilon) = \int_{\varepsilon + d\varepsilon}^{\infty} \frac{F(E)}{E}\, dE - \int_{\varepsilon}^{\infty} \frac{F(E)}{E}\, dE$$

$$= -\int_{\varepsilon}^{\varepsilon + d\varepsilon} \frac{F(E)}{E}\, dE$$

11d. Interpreting Energy Spectra

For small $d\varepsilon$ we write $G(\varepsilon + d\varepsilon) - G(\varepsilon) = -\left[\dfrac{F(E)}{E}\right]_{E=\varepsilon} d\varepsilon$

so that, $\dfrac{dG}{d\varepsilon} = -\left[\dfrac{F(E)}{E}\right]_{E=\varepsilon}$ or $F(E) = -E\left[\dfrac{dG}{d\varepsilon}\right]_{\varepsilon=E}$

Now, since $\dfrac{dJ}{d\varepsilon} = \dfrac{n_P V Q_0 m_e c^2}{\varepsilon}\displaystyle\int_{\varepsilon}^{\infty}\dfrac{F(E)}{E}\,dE = \dfrac{n_P V Q_0 m_e c^2}{\varepsilon} G(\varepsilon)$

it follows that $G(\varepsilon) = \dfrac{1}{n_P V Q_0 m_e c^2}\,\varepsilon\,\dfrac{dJ}{d\varepsilon}$

Must be negative

and so $F(E) = -\dfrac{E}{n_P V Q_0 m_e c^2}\left[\dfrac{dJ}{d\varepsilon} + \varepsilon\dfrac{d^2J}{d\varepsilon^2}\right]_{\varepsilon=E}$

So, because $F(E) \geq 0$ we must have

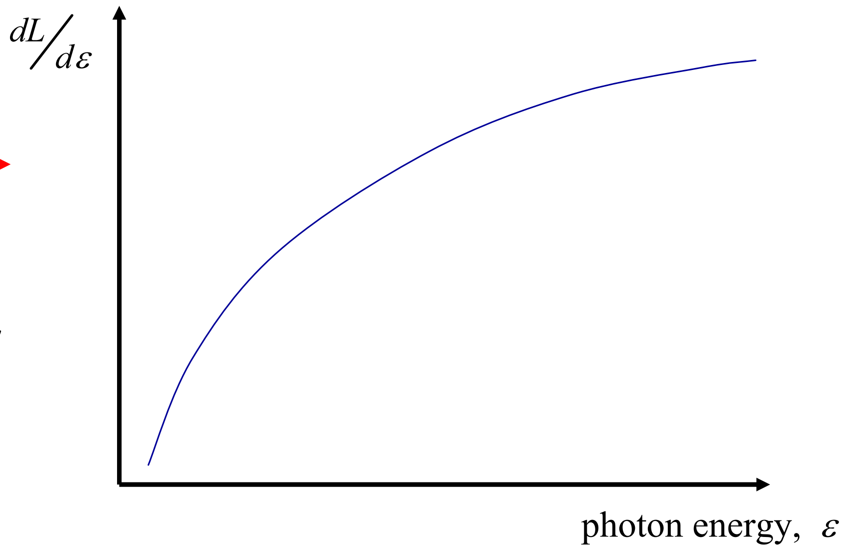
$$\boxed{\frac{d}{d\varepsilon}\left[\varepsilon\,\frac{dJ}{d\varepsilon}\right] \leq 0} \qquad (2.16)$$

For a non-thermal source to be physically meaningful.

Providing a source satisfies this single condition, however, otherwise it can display any differential luminosity and still be consistent with a source of non-thermal bremsstrahlung.

Example:

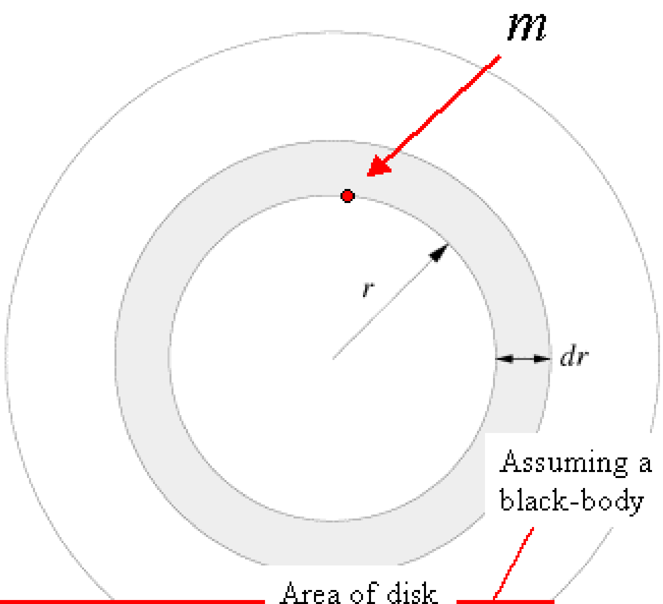
Not *consistent with non-thermal (or indeed thermal) bremsstrahlung!*



Evidence Supporting the Unified Scheme

4. Hot accretion disk explains 'blue bump' in quasar continuum

= UV excess from hot accretion disk



In time t (say), energy radiated from ring between r and $r + dr$:

$$dE = dL_{\text{ring}}\, t = \left(\tfrac{dE}{dr}\right)dr$$

$$= \frac{d}{dr}\left(-\frac{GMm}{2r}\right)dr = \frac{GMm}{2r^2}dr$$

In time t, mass passing through ring is $m = \dot{M}\, t$

$$\boxed{\Rightarrow \quad dL_{\text{ring}} = \frac{GM\dot{M}}{2r^2}\,dr = \overbrace{4\pi r dr}^{\text{Area of disk}}\sigma T^4} \qquad \textbf{(2.14)}$$

Evidence Supporting the Unified Scheme

4. Hot accretion disk explains 'blue bump' in quasar continuum

= UV excess from hot accretion disk

Re-arranging

$$\boxed{T = \left(\frac{GM\dot{M}}{8\pi\sigma r^3}\right)^{1/4}} \qquad \textbf{(2.15)}$$

Substituting in eq. (2.14), with

$$M = 4\times10^{8} M_{\text{Sun}} = 8\times10^{38}\ \text{kg}$$

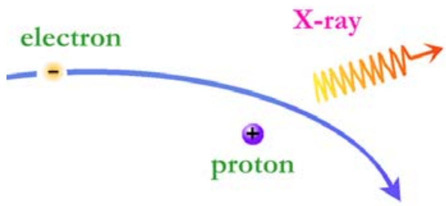
$$\dot{M} = 2.8\times10^{23}\ \text{kgs}^{-1}$$

$$r \sim 50 R_S$$

$$\Rightarrow \quad \boxed{T \sim 1.6\times10^{4}\ \text{K}} \qquad \textbf{(2.16)}$$

2.3 Inefficiency of non-thermal bremsstrahlung

In a non-thermal source, fast electrons of energy E emit X-ray bremsstrahlung due to short-range electron-proton (or electron-ion) interactions.



The electrons *also* lose some of their energy, via long-range (coulomb) interactions, to heating the 'cold' background plasma (for which $kT << E$).

This makes non-thermal bremsstrahlung an inefficient source of radiation

Rate of energy loss to heating in a cold plasma by an electron of energy E

$$\left(\frac{dE}{dt}\right)_{\text{Heating}} = -2\pi\left(\frac{e^2}{4\pi\,\varepsilon_0}\right)^2 \frac{\Lambda(E)}{E}\, n_P\, \upsilon(E) \qquad (2.17)$$

Slowly varying logarithmic function of energy E , $\Lambda(E) \sim 20$

Rate of bremsstrahlung radiation by a **single** electron is

$$\frac{dj}{d\varepsilon} = n_P\, \upsilon\, \frac{dQ_B}{d\varepsilon}$$

(Compare HEA1, Section 7)

(2.18)

Power radiated by a single electron is

$$\left(\frac{dE}{dt}\right)_{\text{Radiation}} = -L = -\int \frac{dL}{d\varepsilon} d\varepsilon \approx -\varepsilon\frac{dL}{d\varepsilon} \approx -\varepsilon^2 \frac{dj}{d\varepsilon}$$

(2.19)

Taking $\varepsilon \sim E$

$$\left(\frac{dE}{dt}\right)_{\text{Radiation}} \approx -E^2\, n_P\, \upsilon(E) \times \frac{Q_0\, m_e c^2}{E^2} \qquad (2.20)$$

7. Reaction cross section

7.1 Incident Flux = number of beam particles crossing per unit area of the target per unit time

$$\boxed{F = n\,\upsilon} \quad \mathrm{m^{-2}\,s^{-1}}$$

7.2 Reaction Rate = number of *interactions* per unit time

$$\boxed{R \propto F\, N_{\mathrm{T}}}$$

7. Reaction cross section

Assuming that the incident flux of beam particles is independent of $\vec{r}$ then

$$\boxed{J = N_{\mathrm{T}}\, F\, Q} \quad \mathrm{s^{-1}}$$

Differential emissivity of photons with energy $\varepsilon \longrightarrow \varepsilon + d\varepsilon$ can be written as $dj = \dfrac{dj}{d\varepsilon}\, d\varepsilon$

$$\boxed{\frac{dj(\varepsilon)}{d\varepsilon} = n_{\mathrm{T}}\, F\, \frac{dQ}{d\varepsilon}} \quad \mathrm{m^{-3}\,s^{-1}\,keV^{-1}}$$

Differential emissivity of photons with energy, ε per unit energy range per unit volume

Differential cross section

Comparing equations (2.17) and (2.20), we get a **bremsstrahlung emission efficiency** which tells us the fraction of the energy loss rate that is in radiation

$$\eta = \frac{dE/dt_{\text{radiation}}}{dE/dt_{\text{heating}}} = \frac{1}{2\pi}\left(\frac{4\pi\varepsilon_0}{e^2}\right)^2 \frac{E}{\Lambda} \times Q_0 m_e c^2 \qquad (2.21)$$

Simplifying, by substituting (see HEA1) $\left(\frac{4\pi\varepsilon_0}{e^2}\right) = \frac{1}{r_e m_e c^2}$ and $Q_0 = \frac{8}{3}\alpha r_e^2$

Classical elecrton radius

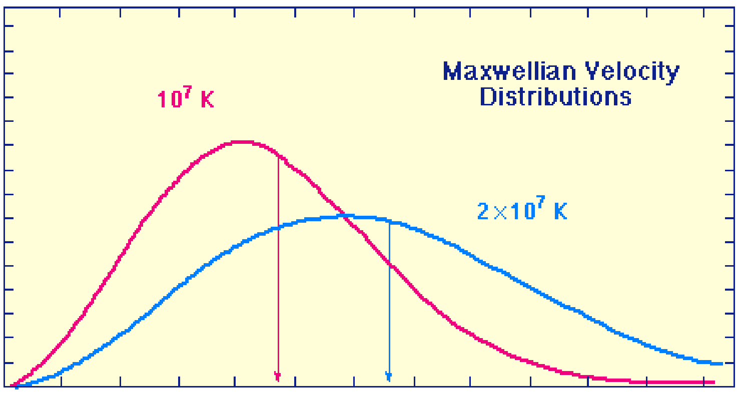
Fine structure constant ~ 1/137

gives, for $E = 50\,\text{keV}$

$$\eta = \frac{dE/dt_{\text{radiation}}}{dE/dt_{\text{heating}}} = \frac{4}{3\pi}\frac{\alpha}{\Lambda} \times \frac{E}{m_e c^2} \quad \sim \quad 10^{-5} \qquad (2.22)$$

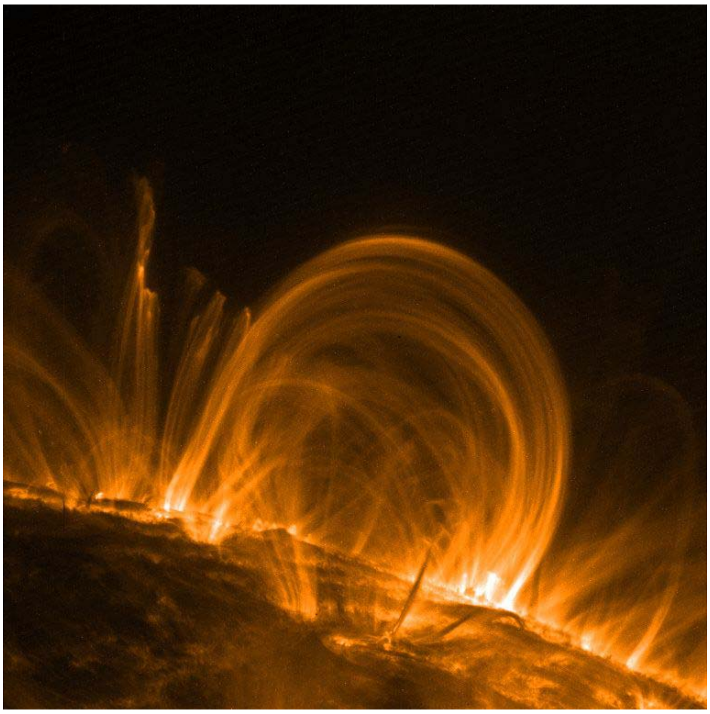
Why is thermal bremsstrahlung a so much more efficient source of radiation?

In a thermal source all electrons have E fairly close to kT, so dE/dt_{heating} is small.



2.4 Example (1): Hot solar corona magnetic loop

We are now obtaining solar data of sufficient angular resolution (at least at lower X-ray energies) to permit measurement of how the temperature varies with position, pixel by pixel, in a flare loop.

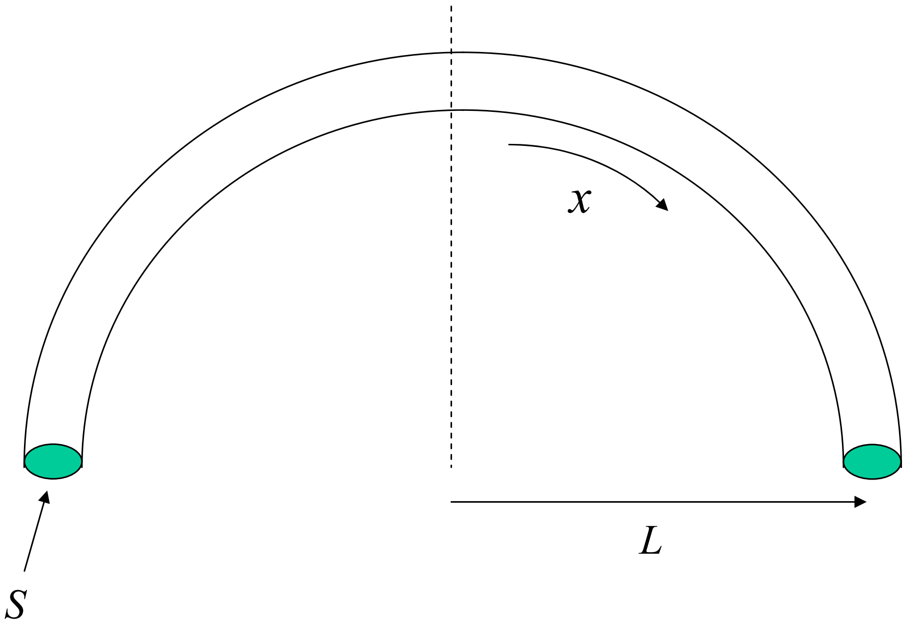


e.g. TRACE images: indicate that most of the heating occurs near to the base of the loop (confounding previous view of uniform heating along loop).

Suppose we only measure the emissivity from the whole of a solar loop. If we make certain assumptions, we can still estimate how temperature varies along the loop.

Assume symmetric loop, uniform area S

Assume $T = T(x)$ and $n = n(x)$ only.



Recall from HEA1 that

$$\boxed{\xi(T) = \int_{S_T} \frac{n_P(\vec{r})^2\, dS}{|\nabla T|}}$$

Surfaces of constant T

Here S_T surfaces are disks of area S

So

$$\boxed{\xi(T) = 2\, S\, n^2(x) \big/ |dT/dx|} \qquad (2.23)$$

For high X-ray temperatures, pressure scale height of the corona: $H \sim \frac{2kT}{m_P g} >> L$

So we can assume that the pressure is constant along the loop.

But $P = n k T$ so $n(x) \propto T(x)^{-1}$

i.e. $\boxed{n(x) \propto n_0 T_0 / T(x)}$

and

$$\boxed{\xi(T) = \frac{2\, S\, n_0^2\, T_0^2}{T^2 \left| dT/dx \right|}} \qquad (2.24)$$

where $n_0 T_0$ are the density and temperature at loop centre

Remember that we saw in eq. (2.8), for a thermal source the emissivity and emission measure are related via

$$\frac{dJ}{d\varepsilon} = \int_0^\infty \xi(T)\, K(\varepsilon, T)\, dT$$

So if we can 'invert' this equation* to determine $\xi(T)$ from the observed $\frac{dJ}{d\varepsilon}$, then we can plug $\xi(T)$ into eq. (2.24) to obtain:

$$\boxed{T^2 \xi(T) \left| \frac{dT}{dx} \right| = 2\, S\, n_0^2\, T_0^2} \qquad (2.25)$$

And we can in principle solve this differential equation to get $T(x)$ and then use the results to test models of coronal loop heating.

* This is known as an inverse problem; for a thermal source we can solve for $\xi(T)$ via Laplace transforms, although the details lie beyond the scope of this course.

2.4 Example (2): Hot star wind

See HEA1 – Example sheet 2

1. The steady-state wind from a hot star consists of fully ionised hydrogen gas moving radially outwards with constant velocity v_0. By considering the mass per second passing through a spherical surface of radius r outside the photosphere (of radius R_*), show that $n_P(r)$, the number density of protons at radius r satisfies

$$\dot{M} = 4\pi r^2 v_0 n_P(r) m_P$$

where $\dot{M}$ is the mass loss rate of the star and m_P is the proton mass. (This is known as the *mass continuity equation*; see also Dr Woan's Stellar Atmospheres and Winds course).

2. Hence, show that the *source emission measure function*, $\xi(T)$, for the wind is given by

$$\xi(T) = \frac{\dot{M}^2}{4\pi r^2 v_0^2 m_P^2} \left| \frac{dr}{dT} \right|$$

3. If the temperature of the wind outside of the photosphere varies with radius according to the formula

$$T(r) = T_0 \left(\frac{r}{R_*} \right)^{-\alpha}$$

where α and T_0 are constants, derive an expression for $r(T)$, and hence show that $\xi(T) = 0$ for $T > T_0$ and

$$\xi(T) = \frac{\dot{M}^2}{4\pi\alpha\, v_0^2\, m_P^2\, T_0^{1/\alpha}\, R_*} T^{\frac{1}{\alpha}-1} \qquad \text{for } \mathrm{T} \leq \mathrm{T_0}$$

4. Determine the integrated source emission measure, Ξ, first by integrating $\xi(T)$ over temperature, i.e.

$$\Xi = \int_0^\infty \xi(T)dT$$

and then via the volume integral

$$\Xi = \int_V n_P^2\, dV$$

showing that these two expressions are equivalent. Thus verify the relation

$$\xi(T) = \frac{d}{dT} \int_V n_P^2\, dV$$

5. Assuming that the X-ray emission from the hot wind is thermal bremsstrahlung, show that the differential luminosity of the star is given by

$$\frac{dL}{d\epsilon} = 2 \left(\frac{2}{\pi m_e} \right)^{\frac{1}{2}} \frac{Q_0 m_e c^2\, \dot{M}^2}{k^{1/2}\, 4\pi\alpha\, v_0^2\, m_P^2\, T_0^{1/\alpha}\, R_*} \int_0^{T_0} T^{\frac{1}{\alpha}-\frac{3}{2}}\, e^{-\epsilon/kT} dT$$

6. Applying the substitution $x = \epsilon/kT$, or otherwise, show that the above expression may be reduced to

$$\frac{dL}{d\epsilon} \propto \epsilon^{\frac{1}{\alpha}-\frac{1}{2}} \int_{\frac{\epsilon}{kT_0}}^\infty x^{-\frac{1}{2}-\frac{1}{\alpha}} e^{-x} dx$$

7. Hence, explain why – for large X-ray photon energies – the shape of the differential photon luminosity is independent of α.