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Supplementary Notes for Chapter 5

A1: Einstein's tensor in the weak field approximation

A1.1: The linearised Riemann Christoffel tensor

In Minkowski spacetime the Christoffel symbols are all identically zero. This reduces the Riemann Christoffel tensor to

$$R_{\alpha\beta\gamma\delta} \;=\; g_{\alpha\mu}\,R^{\mu}_{\beta\gamma\delta} \;=\; g_{\alpha\mu}\Gamma^{\mu}_{\beta\delta,\gamma} \;-\; g_{\alpha\mu}\Gamma^{\mu}_{\beta\gamma,\delta}$$

Substituting for the Christoffel symbols in terms of the metric and its derivatives

$$R_{\alpha\beta\gamma\delta} \;=\; g_{\alpha\mu}\,\frac{g^{\mu\sigma}}{2}(g_{\sigma\beta,\delta\gamma} + g_{\sigma\delta,\beta\gamma} - g_{\beta\delta,\sigma\gamma}) \;-\; g_{\alpha\mu}\,\frac{g^{\mu\sigma}}{2}(g_{\sigma\beta,\gamma\delta} + g_{\sigma\gamma,\beta\delta} - g_{\beta\gamma,\sigma\delta})$$

This reduces to

$$R_{\alpha\beta\gamma\delta} \;=\; \frac{1}{2}(g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma})$$

If $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ then

$$R_{\alpha\beta\gamma\delta} \;=\; \frac{1}{2}(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma})$$

A1.2: The linearised Ricci tensor and curvature scalar

Contracting the Riemann Christoffel tensor it then follows that

$$R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu} = \Gamma^{\sigma}_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\mu\sigma,\nu}$$

To first order this reduces to

$$R_{\mu\nu} = \frac{1}{2}\eta^{\sigma\alpha}(h_{\alpha\nu,\mu\sigma} + h_{\mu\sigma,\alpha\nu} - h_{\mu\nu,\alpha\sigma} - h_{\alpha\sigma,\mu\nu})$$

Since the partial derivatives of $\eta^{\sigma\alpha}$ are zero, we can write this as

$$R_{\mu\nu} = \frac{1}{2}\left[(\eta^{\sigma\alpha}\, h_{\alpha\nu})_{,\mu\sigma} + (\eta^{\sigma\alpha}\, h_{\mu\sigma})_{,\alpha\nu} - \eta^{\sigma\alpha} h_{\mu\nu,\alpha,\sigma} - (\eta^{\sigma\alpha}\, h_{\alpha\sigma})_{,\mu\nu}\right]$$

which further reduces to

$$R_{\mu\nu} = \frac{1}{2}\left[(h^{\sigma}_{\nu})_{,\mu\sigma} + \left(h^{\alpha}_{\mu}\right)_{,\nu\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu}\right]$$

Thus the curvature scalar R is given by

$$R = \eta^{\alpha\beta} R_{\alpha\beta} = \frac{1}{2}\eta^{\alpha\beta}\left[\left(h^{\sigma}_{\beta}\right)_{,\alpha\sigma} + (h^{\sigma}_{\alpha})_{,\beta\sigma} - h_{\alpha\beta,\sigma}{}^{,\sigma} - h_{,\alpha\beta}\right]$$

A1.3: The linearised Einstein tensor

Combining our results for the Ricci tensor and curvature scalar we find

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}\left[(h^{\sigma}_{\nu})_{,\mu\sigma} + \left(h^{\alpha}_{\mu}\right)_{,\nu\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu}\right] \\ &\quad -\frac{1}{4}\eta_{\mu\nu}\,\eta^{\alpha\beta}\left[\left(h^{\sigma}_{\beta}\right)_{,\alpha\sigma} + (h^{\sigma}_{\alpha})_{,\beta\sigma} - h_{\alpha\beta,\sigma}{}^{,\sigma} - h_{,\alpha\beta}\right] \end{aligned}$$

To see that this expression reduces to equation 5.29 it is easiest to work backwards from that equation. We have, from equation 5.29

$$\begin{aligned} \text{LHS} &= \frac{1}{2}\left[h_{\mu\alpha,\nu}{}^{,\alpha} + h_{\nu\alpha,\mu}{}^{,\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{,\beta}\right)\right] \\ &= \frac{1}{2}\left[\eta^{\alpha\sigma}\, h_{\mu\alpha,\nu\sigma} + \eta^{\alpha\sigma}\, h_{\nu\alpha,\mu\sigma} - \eta^{\alpha\sigma}\, h_{\mu\nu,\alpha\sigma} - h_{,\mu\nu} - \eta_{\mu\nu}\eta^{\alpha\gamma}\eta^{\beta\sigma}\, h_{\alpha\beta,\gamma\sigma} + \eta_{\mu\nu}\eta^{\beta\alpha}\, h_{,\beta\alpha}\right] \\ &= \frac{1}{2}\left[(\eta^{\alpha\sigma} h_{\mu\alpha})_{,\nu\sigma} + (\eta^{\alpha\sigma} h_{\nu\alpha})_{,\mu\sigma} - h_{\mu\nu,\sigma}{}^{,\sigma} - h_{,\mu\nu} - \eta_{\mu\nu}\eta^{\alpha\gamma}\left(\eta^{\beta\sigma} h_{\alpha\beta}\right)_{,\gamma\sigma} + \eta_{\mu\nu}\eta^{\alpha\beta} h_{,\alpha\beta}\right] \\ &= \frac{1}{2}\left[(h^{\sigma}_{\nu})_{,\mu\sigma} + \left(h^{\sigma}_{\mu}\right)_{,\nu\sigma} - h_{\mu\nu,\sigma}{}^{,\sigma} - h_{,\mu\nu} - \eta_{\mu\nu}\eta^{\alpha\gamma}\,(h^{\sigma}_{\alpha})_{,\gamma\sigma} + \eta_{\mu\nu}\eta^{\alpha\beta} h_{,\alpha\beta}\right] \end{aligned}$$

The first four bracketed terms match the first four terms in the expression for the Einstein tensor given above. If we now consider the remaining four terms in the above expression for the Einstein tensor, then since $\eta^{\alpha\beta} = \eta^{\beta\alpha}$

$$-\frac{1}{4}\eta_{\mu\nu}\eta^{\alpha\beta}\left[\left(h^{\sigma}_{\beta}\right)_{,\alpha\sigma} + (h^{\sigma}_{\alpha})_{,\beta\sigma}\right] = -\frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\,(h^{\sigma}_{\alpha})_{,\beta\sigma}$$

Also

$$\frac{1}{4}\eta_{\mu\nu}\eta^{\alpha\beta}\left[h_{\alpha\beta,\sigma}{}^{,\sigma} + h_{,\alpha\beta}\right] = \frac{1}{4}\eta_{\mu\nu}\eta^{\alpha\beta}\left[\eta^{\sigma\gamma} h_{\alpha\beta,\sigma\gamma} + h_{,\alpha\beta}\right]$$

$$
\begin{aligned}
&= \frac{1}{4}\eta_{\mu\nu}\eta^{\sigma\gamma}\left(\eta^{\alpha\beta}h_{\alpha\beta}\right)_{,\sigma\gamma} + \frac{1}{4}\eta_{\mu\nu}\eta^{\alpha\beta}h_{,\alpha\beta} \\
&= \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{,\alpha\beta}
\end{aligned}
$$

Thus, apart from some permutation of repeated indices, we see that the remaining four terms of our expression for the Einstein tensor match exactly the final two terms of equation 5.29. This establishes that equation 5.29 is indeed the correct expression for $G_{\mu\nu}$.

A1.4: Linearised Einstein tensor in barred form

Equation 5.31 is also easiest to establish in reverse – i.e. we start with equation 5.31 and show that its terms can be rewritten in a manner that reduces to equation 5.29. Consider in turn each of the four bracketed terms on the right hand side of equation 5.31.

$$
\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} = \eta^{\alpha\sigma}\overline{h}_{\mu\nu,\alpha\sigma} = \eta^{\alpha\sigma}\left[h_{\mu\nu,\alpha\sigma} - \frac{1}{2}\eta_{\nu\mu}\, h_{,\alpha\sigma}\right]
$$

$$
\eta_{\mu\nu}\,\overline{h}_{\alpha\beta}{}^{,\alpha\beta} = \eta_{\mu\nu}\,\eta^{\alpha\sigma}\,\eta^{\beta\sigma}\,\overline{h}_{\alpha\beta,\gamma\sigma} = \eta_{\mu\nu}\,\eta^{\alpha\sigma}\,\eta^{\beta\sigma}\left[h_{\alpha\beta,\gamma\sigma} - \frac{1}{2}\eta_{\alpha\beta}\, h_{,\gamma\sigma}\right]
$$

$$
\overline{h}_{\mu\alpha,\nu}{}^{,\alpha} = \eta^{\alpha\sigma}\,\overline{h}_{\mu\alpha,\nu\sigma} = \eta^{\alpha\sigma}\left[h_{\mu\alpha,\nu\sigma} - \frac{1}{2}\eta_{\mu\alpha}\, h_{,\nu\sigma}\right]
$$

$$
\overline{h}_{\nu\alpha,\mu}{}^{,\alpha} = \eta^{\alpha\sigma}\,\overline{h}_{\nu\alpha,\mu\sigma} = \eta^{\alpha\sigma}\left[h_{\nu\alpha,\mu\sigma} - \frac{1}{2}\eta_{\nu\alpha}\, h_{,\mu\sigma}\right]
$$

Hence we can write equation 5.31 as

$$
\begin{aligned}
G_{\mu\nu} \;=\; & -\frac{1}{2}\eta^{\alpha\sigma}\, h_{\mu\nu,\alpha\sigma} + \frac{1}{4}\eta^{\alpha\sigma}\,\eta_{\mu\nu}\, h_{,\alpha\sigma} \\
& -\frac{1}{2}\eta_{\mu\nu}\,\eta^{\alpha\gamma}\,\eta^{\beta\sigma}\, h_{\alpha\beta,\gamma\sigma} + \frac{1}{4}\eta_{\mu\nu}\,\eta^{\alpha\gamma}\,\eta^{\beta\sigma}\,\eta\alpha\beta\, h_{,\gamma\sigma} \\
& +\frac{1}{2}\eta^{\alpha\sigma}\, h_{\mu\alpha,\nu\sigma} - \frac{1}{4}\eta^{\alpha\sigma}\,\eta_{\mu\alpha}\, h_{,\nu\sigma} \\
& +\frac{1}{2}\eta^{\alpha\sigma}\, h_{\nu\alpha,\mu\sigma} - \frac{1}{4}\eta^{\alpha\sigma}\,\eta_{\nu\alpha}\, h_{,\mu\sigma}
\end{aligned}
$$

$$= \; -\frac{1}{2}\eta^{\alpha\sigma}\, h_{\mu\nu,\alpha\sigma} \;-\; \frac{1}{2}\eta_{\mu\nu}\,\eta^{\alpha\gamma}\,\eta^{\beta\sigma}\, h_{\alpha\beta,\gamma\sigma} \;+\; \frac{1}{2}\eta^{\alpha\sigma}\, h_{\mu\alpha,\nu\sigma}$$
$$+\frac{1}{2}\eta^{\alpha\sigma}\, h_{\nu\alpha,\mu\sigma} \;+\; \frac{1}{2}\eta^{\alpha\sigma}\,\eta_{\mu\nu}\, h_{,\alpha\sigma} \;-\; \frac{1}{2}h_{,\mu\nu}$$

Comparing with our expression for $G_{\mu\nu}$, we see that – after changing some repeated indices and using the fact that the Minkowski metric is symmetric – the above expression is identical. This establishes that equation 5.31 is indeed the correct expression for the $G_{\mu\nu}$ in terms of $\overline{h}_{\mu\nu}$.

A2: The Lorentz Gauge Condition

First we show that, if $\overline{h}^{\mu\alpha}{}_{,\alpha} = 0$, the final three terms on the right hand side of equation 5.31 vanish. Consider the bracketed terms in turn

$$\begin{aligned}
\overline{h}_{\alpha\beta}{}^{,\alpha\beta} &= \left(\eta_{\alpha\gamma}\,\eta_{\beta\sigma}\,\overline{h}^{\gamma\sigma}\right)^{,\alpha\beta} \\
&= \left(\eta_{\alpha\gamma}\,\eta_{\beta\sigma}\,\eta^{\alpha\tau}\,\eta^{\beta\epsilon}\,\overline{h}^{\gamma\sigma}{}_{,\tau\epsilon}\right) \\
&= \delta^{\tau}_{\gamma}\,\delta^{\epsilon}_{\sigma}\,\overline{h}^{\gamma\sigma}{}_{,\tau\epsilon} \\
&= \overline{h}^{\gamma\sigma}{}_{,\gamma\sigma} \quad = \quad \left(\overline{h}^{\gamma\sigma}{}_{,\sigma}\right)_{,\gamma} \quad = \quad 0
\end{aligned}$$

$$\begin{aligned}
\overline{h}_{\mu\alpha,\nu}{}^{,\alpha} &= \eta_{\mu\gamma}\,\eta_{\alpha\sigma}\,\left(\overline{h}^{\gamma\sigma}{}_{,\nu}\right)^{,\alpha} \\
&= \eta_{\mu\gamma}\,\eta_{\alpha\sigma}\,\eta^{\alpha\tau}\,\overline{h}^{\gamma\sigma}{}_{,\nu\tau} \\
&= \eta_{\mu\gamma}\,\delta^{\tau}_{\sigma}\,\overline{h}^{\gamma\sigma}{}_{,\nu\tau} \\
&= \eta_{\mu\gamma}\,\overline{h}^{\gamma\sigma}{}_{,\nu\sigma} \quad = \quad \eta_{\mu\gamma}\left(\overline{h}^{\gamma\sigma}{}_{,\sigma}\right)_{,\nu} \quad = \quad 0
\end{aligned}$$

$$\begin{aligned}
\overline{h}_{\nu\alpha,\mu}{}^{,\alpha} &= \eta_{\nu\gamma}\,\eta_{\alpha\sigma}\,\left(\overline{h}^{\gamma\sigma}{}_{,\mu}\right)^{,\alpha} \\
&= \eta_{\nu\gamma}\,\eta_{\alpha\gamma}\,\eta^{\alpha\tau}\,\overline{h}^{\gamma\sigma}{}_{,\mu\tau} \\
&\ \eta_{\nu\gamma}\,\delta^{\tau}_{\sigma}\,\overline{h}^{\gamma\sigma}{}_{,\mu\tau} \\
&\ \eta_{\nu\gamma}\,\overline{h}^{\gamma\sigma}{}_{,\mu\sigma} \quad = \quad \eta_{\nu\gamma}\left(\overline{h}^{\gamma\sigma}{}_{,\sigma}\right)_{,\mu} \quad = \quad 0
\end{aligned}$$

So we see that the final three terms do indeed equal zero provided $\overline{h}^{\mu\alpha}{}_{,\alpha} = 0$.

Finally we establish the equation which must be solved in order that the Lorentz gauge condition $\overline{h}^{\mu\alpha}{}_{,\alpha} = 0$ is satisfied.

Suppose we begin with arbitrary metric perturbation components $h^{(\text{old})}_{\mu\nu} \neq 0$. We define

$$h^{(\text{LG})}_{\mu\nu} \;=\; h^{(\text{old})}_{\mu\nu} \,-\, \xi_{\mu,\nu} \,-\, \xi_{\nu,\mu}$$

where the components ξ^μ are to be determined. We can also define

$$\begin{aligned}
\overline{h}^{(\text{LG})}_{\mu\nu} \;&=\; h^{(\text{LG})}_{\mu\nu} \,-\, \frac{1}{2}\eta_{\mu\nu}\,\eta^{\alpha\beta}\,h^{(\text{LG})}_{\alpha\beta} \\
&=\; h^{(\text{old})}_{\mu\nu} \,-\, \xi_{\mu,\nu} \,-\, \xi_{\nu,\mu} \,-\, \frac{1}{2}\eta_{\mu\nu}\left[\eta^{\alpha\beta}\left(h^{(\text{old})}_{\alpha\beta} \,-\, \xi_{\alpha,\beta} \,-\, \xi_{\beta,\alpha}\right)\right] \\
&=\; h^{(\text{old})}_{\mu\nu} \,-\, \frac{1}{2}\eta_{\mu\nu}h^{(\text{old})} \,-\, \xi_{\mu,\nu} \,-\, \xi_{\nu,\mu} \,+\, \eta_{\mu\nu}\,\eta^{\alpha\beta}\,\xi_{\alpha,\beta} \\
&=\; \overline{h}^{(\text{old})}_{\mu\nu} \,-\, \xi_{\mu,\nu} \,-\, \xi_{\nu,\mu} \,+\, \eta_{\mu\nu}\,\xi^{\beta}{}_{,\beta}
\end{aligned}$$

Now

$$\begin{aligned}
\overline{h}^{(\text{LG})\,\mu\nu} \;&=\; \eta^{\mu\alpha}\,\eta^{\nu\beta}\left[\overline{h}^{(\text{old})}_{\alpha\beta} \,-\, \xi_{\alpha,\beta} \,-\, \xi_{\beta,\alpha} \,+\, \eta_{\alpha\beta}\,\xi^{\sigma}{}_{,\sigma}\right] \\
&=\; \overline{h}^{(\text{old})\,\mu\nu} \,-\, \eta^{\mu\alpha}\,\eta^{\nu\beta}\,\xi_{\alpha,\beta} \,-\, \eta^{\mu\alpha}\,\eta^{\nu\beta}\,\xi_{\beta,\alpha} \,+\, \eta^{\mu\nu}\,\xi^{\sigma}{}_{,\sigma}
\end{aligned}$$

And

$$\overline{h}^{(\text{LG})\,\mu\nu}{}_{,\nu} \;=\; \overline{h}^{(\text{old})\,\mu\nu}{}_{,\nu} \,-\, \eta^{\nu\beta}\,\xi^{\mu}{}_{,\beta\nu} \,-\, \eta^{\mu\alpha}\,\xi^{s}igma_{,\sigma\alpha} \,+\, \eta^{\mu\nu}\,\xi^{\sigma}{}_{,\sigma\nu}$$

i.e.

$$\overline{h}^{(\text{LG})\,\mu\nu}{}_{,\nu} \;=\; \overline{h}^{(\text{old})\,\mu\nu}{}_{,\nu} \,-\, \eta^{\nu\beta}\,\xi^{\mu}{}_{,\nu\beta}$$

So we can ensure that $\overline{h}^{(\text{LG})\,\mu\nu}{}_{,\nu} = 0$ provided we can find gauge components ξ^μ satisfying

$$\overline{h}^{(\text{old})\,\mu\nu}{}_{,\nu} \;=\; \eta^{\nu\beta}\,\xi^{\mu}{}_{,\nu\beta} \;=\; \left(-\frac{\partial^2}{\partial t^2} \,+\, \nabla^2\right)\xi^{\mu}$$

We can always solve this equation for well-behaved metrics using standard methods for finding particular solutions of second order partial differential equations.