

Astronomy A3/A4M, Physics P4M

Gravitation and Relativity II: Example Sheet 4

1. Following Section 5.5.1 of your GR-II notes, show that two test particles initially separated by ϵ in the y -direction, have a geodesic deviation vector which obeys the differential equations

$$\frac{\partial^2}{\partial t^2} \xi^x = \frac{1}{2} \epsilon \frac{\partial^2}{\partial t^2} h_{xy}^{TT}$$

and

$$\frac{\partial^2}{\partial t^2} \xi^y = -\frac{1}{2} \epsilon \frac{\partial^2}{\partial t^2} h_{xx}^{TT}$$

as they are disturbed by a plane gravitational wave propagating along the z -axis.

2. More generally, show that the geodesic deviation vector of two particles – one at the origin and the other initially at coordinates $x = \epsilon \cos \theta$, $y = \epsilon \sin \theta$ and $z = 0$ – as a gravitational wave propagates in the z -direction, has components ξ^x and ξ^y which obey the differential equations

$$\frac{\partial^2}{\partial t^2} \xi^x = \frac{1}{2} \epsilon \cos \theta \frac{\partial^2}{\partial t^2} h_{xx}^{TT} + \frac{1}{2} \epsilon \sin \theta \frac{\partial^2}{\partial t^2} h_{xy}^{TT}$$

and

$$\frac{\partial^2}{\partial t^2} \xi^y = \frac{1}{2} \epsilon \cos \theta \frac{\partial^2}{\partial t^2} h_{xy}^{TT} - \frac{1}{2} \epsilon \sin \theta \frac{\partial^2}{\partial t^2} h_{xx}^{TT}$$

3. Show further that

$$\xi^x = \epsilon \cos \theta + \frac{1}{2} \epsilon \cos \theta B_{xx}^{TT} \cos \omega t + \frac{1}{2} \epsilon \sin \theta B_{xy}^{TT} \cos \omega t$$

and

$$\xi^y = \epsilon \sin \theta + \frac{1}{2} \epsilon \cos \theta B_{xy}^{TT} \cos \omega t - \frac{1}{2} \epsilon \sin \theta B_{xx}^{TT} \cos \omega t$$

are solutions to the differential equations of Q.2

4. Does a similar analysis to that considered in Q.1 and Q.2, but now for two test particles – one at the origin and the other at $z = \epsilon$ – lead to the conclusion that the particles will be *unaffected* by the passage of the gravitational wave along the z -axis?

5. A particle is released from rest at coordinate radius $r = R_0$ and falls radially inwards towards a Schwarzschild black hole. Show that equation (6.8) of your lecture notes may be re-written in the form

$$\Delta \tau = \sqrt{\frac{R_0}{R_S}} \int_{R_S}^{R_0} \frac{r^{1/2} dr}{(R_0 - r)^{1/2}}$$

Hence, show that the proper time taken for the particle to reach the Event Horizon is given by

$$\Delta \tau = \left(\frac{R_0^3}{R_S} \right)^{1/2} \left(\frac{1}{2} \pi - \theta_0 + \sin \theta_0 \cos \theta_0 \right)$$

where

$$\theta_0 = \sin^{-1} \sqrt{\frac{R_S}{R_0}}$$