# Astronomy A3/A4M, Physics P4M

## Gravitation and Relativity II: Example Sheet 3

1. Show that, if the **Lorentz gauge condition** holds

$$\overline{h}^{\mu\alpha}{}_{,\alpha} = 0$$

then it also follows that

$$\left(\overline{h}^{\alpha}_{\mu}\right)_{,\alpha} = 0$$

where $\overline{h}^{\alpha}_{\mu} = \eta^{\alpha\beta}\overline{h}_{\beta\mu} = \eta_{\sigma\mu}\overline{h}^{\alpha\sigma}$. Show, further that, if

$$\overline{h}_{\mu\nu} = \mathrm{Re}\left[A_{\mu\nu}\,\exp\left(ik_\alpha x^\alpha\right)\right]$$

then

$$A_{\mu\alpha}\,k^\alpha = 0$$

i.e. the amplitude components of a gravitational wave must be orthogonal to the wave vector, **k**.

Moreover, if the wave is travelling in the is travelling in the positive $z$-direction, such that

$$k^t = \omega\,, \quad k^x = k^y = 0\,, \quad k^z = \omega$$

and

$$k_t = -\omega\,, \quad k_x = k_y = 0\,, \quad k_z = \omega$$

show that

$$A_{\alpha z} = 0 \quad \text{for all } \alpha$$

given also that

$$A_{\alpha t} = 0 \quad \text{for all } \alpha$$

and given the Transverse – Traceless gauge condition

$$A^\mu_\mu = \eta^{\mu\nu}\,A_{\mu\nu} = 0$$

Show that

$$\overline{h}^{TT}_{\mu\nu} = A^{TT}_{\mu\nu}\cos\left[\omega(t-z)\right]$$

where

$$A^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Show that, in the transverse – traceless gauge, if the (re-scaled) components of the metric perturbation satisfy

$$\overline{h}^{TT}_{\mu\nu} = A^{TT}_{\mu\nu}\cos\left[\omega(t-z)\right]$$

where the $A^{TT}_{\mu\nu}$ are constants, then the (unscaled) components, $h^{TT}_{\mu\nu}$, of the metric perturbation satisfy

$$h^{TT}_{\mu\nu} = B^{TT}_{\mu\nu}\cos\left[\omega(t-z)\right]$$

where the $B^{TT}_{\mu\nu}$ are constants. You should recall that

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

and

$$h \equiv h^{\alpha}_{\alpha} = \eta^{\alpha\beta} h_{\alpha\beta}$$

3. The metric perturbation for a particular nearly flat spacetime can be written in the form

$$h_{yz} = A \sin\omega(t-x), \qquad \text{all other} \;\; h_{\mu\nu} = 0$$

where $A$ and $\omega$ are constants and $|A| << 1$. Calculate the components of the Riemann–Christoffel tensor for this metric, and show that they are not all zero; i.e. that the spacetime is not flat.

Show, further, that if the metric perturbation for another nearly flat spacetime can be written in the form

$$h'_{yz} = A \sin\omega(t-x), \qquad h'_{tt} = 2B(x-t),$$

$$h'_{tx} = -B(x-t), \qquad \text{all other} \;\; h'_{\mu\nu} = 0$$

where $|B| << 1$, then the components of the Riemann–Christoffel tensor for this metric are identical to the previous one. Can you, therefore, find a small coordinate change, $\xi_\mu$, (i.e. a *gauge transformation*) such that

$$h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$$

4. One example of a background Lorentz transformation is a 45° rotation of the $x$ and $y$ axes in the $x-y$ plane. Show that, under such a rotation from $(x, y)$ to $(x', y')$, it follows that

$$h^{TT}_{x'y'} = h^{TT}_{xx}$$

and

$$h^{TT}_{x'x'} = -h^{TT}_{xy}$$

i.e. one can transform from one polarisation state of gravitational radiation to the other via a background Lorentz transformation.

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