Astronomy A3/A4M, Physics P4M

Gravitation and Relativity II: Example Sheet 2

1. Assuming the equation of motion of a test mass orbiting a star in the Schwarzschild form given by equation (3.11) of your notes:-

$$\left(\frac{dr}{d\tau}\right)^2 = k^2 - 1 - \frac{h^2}{r^2} + \frac{2M}{r}\left(1 + \frac{h^2}{r^2}\right)$$

show that the values of h and k for a circular orbit of coordinate radius a are given by

$$h^2 = \frac{Ma}{\left(1 - \frac{3M}{a}\right)}$$

and

$$k^2 = \frac{\left(1 - \frac{2M}{a}\right)^2}{\left(1 - \frac{3M}{a}\right)}$$

Why are there no circular orbits for $a < 3M$?

2. Using equation (3.52), show that a photon could be moving in a circular orbit about a Schwarzschild black hole (i.e. a static black hole with exterior metric given by the Schwarzschild metric). Find the coordinate radius of the orbit, and the corresponding value of the angular momentum, h.

3. Verify that equation (4.26) of your notes may be re-written in the form of equation (4.29), i.e.

$$\frac{d}{dr}\left[r\left(1 - e^{-\lambda}\right)\right] = 8\pi\rho r^2$$

4. Using equation (4.44) to eliminate ν' from equation (4.35), derive the Oppenheimer-Volkoff equation, i.e.

$$\frac{dP}{dr} = -\frac{(\rho + P)(4\pi P r^3 + m)}{r(r - 2m)}$$

and show that this equation reduces to the classical Newtonian equation of hydrostatic equilibrium in the weak-field limit.

5. Verify, using the method of partial fractions or otherwise, that equation (4.51)

$$\frac{dP}{(\rho_0 + P)(\rho_0 + 3P)} = -\frac{4\pi r dr}{3\left(1 - \frac{8\pi\rho_0 r^2}{3}\right)}$$

reduces to equation (4.52)

$$\frac{1}{2\rho_0}\left[\frac{3dP}{(\rho_0 + 3P)} - \frac{dP}{(\rho_0 + P)}\right] = -\frac{4\pi}{3}\frac{r dr}{\left(1 - \frac{8\pi\rho_0 r^2}{3}\right)}$$

Integrating both sides, and applying the boundary conditions that $P = P_0$ for $r = 0$, and $P =$ for $r = R$, show that the central pressure for a spherically symmetric star of constant density is given by

$$P_0 = \frac{\rho_0\left[1 - \left(1 - \frac{2M}{R}\right)^{1/2}\right]}{3\left(1 - \frac{2M}{R}\right)^{1/2} - 1}$$

Show also that

$$P_0 \rightarrow \infty \quad \text{when} \quad \frac{M}{R} \rightarrow \frac{4}{9}$$

6. Prove that, if we are considering *Background Lorentz Transformations* – i.e. transformations for which the weak-field metric perturbations, $h_{\mu\nu}$, transform as if they are a $(0,2)$ tensor on a Background Minkowski spacetime, then the linearised Riemann-Christoffel tensor reduces to the form of (equation 5.22)

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}\right)$$

7. Show, further, that if we carry out a gauge transformation of the weak-field metric perturbations

$$h'_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$$

where

$$|\xi^{\alpha}{}_{,\beta}| << 1 \quad \text{for all} \quad \alpha, \beta$$

then to first order the linearised Riemann-Christoffel tensor in this new coordinate system is unchanged, i.e.

$$R_{\alpha\beta\gamma\delta} = R'_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}\right)$$

8. Consider the hypersurface

$$t^2 - x^2 - y^2 - z^2 = T^2$$

in Minkowski spacetime, where T is a constant. Show that the interval between two neighbouring points on this hypersurface is spacelike. Show further that all inertial observers that were present at the spatial origin at time $t = 0$ will cross this hypersurface a proper time, T, later – regardless of their velocity.

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