Astronomy A3/A4M, Physics P4M

Gravitation and Relativity II: Example Sheet 1

1. Verify equations (2.13) and (2.14) of your notes – i.e. show that the geodesic equation

$$\frac{d^2x^\nu}{dp^2} + \Gamma^\nu_{\alpha\beta}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} = 0$$

where p is an affine parameter, may be re-written as

$$\frac{d}{dp}\left(g_{\lambda\nu}\frac{dx^\nu}{dp}\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp} = 0$$

and, further, for an orthogonal metric simplifies to

$$\frac{d}{dp}\left(g_{\lambda\lambda}\frac{dx^\lambda}{dp}\right) - \frac{1}{2}\frac{\partial g_{\nu\nu}}{\partial x^\lambda}\left(\frac{dx^\nu}{dp}\right)^2 = 0$$

2. Verify equations (2.24) of your notes – i.e. show that the Christoffel symbols for a static, spherically symmetric spacetime with interval

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2\left(d\theta^2 + sin^2\theta\, d\phi^2\right)$$

are given by

$$\begin{aligned}
\Gamma^t_{rt} &= \tfrac{1}{2}\nu' & \qquad \Gamma^\theta_{r\theta} &= \tfrac{1}{r} \\
\Gamma^r_{tt} &= \tfrac{1}{2}\nu' e^{\nu-\lambda} & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^r_{rr} &= \tfrac{1}{2}\lambda' & \Gamma^\phi_{r\phi} &= \tfrac{1}{r} \\
\Gamma^r_{\theta\theta} &= -re^{-\lambda} & \Gamma^\phi_{\theta\phi} &= \cot\theta \\
\Gamma^r_{\phi\phi} &= -re^{-\lambda}\sin^2\theta & &
\end{aligned}$$

and all others are zero.

3. Given the results for the Christoffel symbols in Q.2, show that the components of the Ricci tensor in its fully covariant form are given by, for this spacetime

$$R_{tt} = \frac{1}{2}e^{\nu-\lambda}\left(\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\nu'\lambda' + \frac{2}{r}\nu'\right)$$

$$R_{rr} = -\frac{1}{2}\left(\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\nu'\lambda' - \frac{2}{r}\lambda'\right)$$

$$R_{\theta\theta} = 1 - e^{-\lambda}\left[1 + \frac{r}{2}\left(\nu' - \lambda'\right)\right]$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta$$

and all others are zero.

4. Given the results of Q.3, show that the curvature invariant, R, for a static, spherically symmetric spacetime is given by

$$R = -e^{-\lambda}\left[\left(\nu'' + \frac{1}{2}v'^2 - \frac{1}{2}\nu'\lambda'\right) + \frac{\nu' - \lambda'}{r}\right] + \frac{2}{r^2}\left[1 - e^{-\lambda}\left(1 + \frac{(\nu' - \lambda')r}{2}\right)\right]$$

Hence show that the covariant components of the Einstein tensor are given by

$$G_{tt} = \frac{e^\nu}{r^2}\left[1 + e^{-\lambda}\left(r\lambda' - 1\right)\right]$$

$$G_{rr} = \frac{\nu'}{r} - \frac{e^{\lambda}}{r^2}\left(1 - e^{-\lambda}\right)$$

$$G_{\theta\theta} = r^2 e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\nu' - \lambda'}{2r}\right]$$

$$G_{\phi\phi} = \sin^2\theta \, G_{\theta\theta}$$

and all others are zero.

5. Given the General Relativistic equation of motion for a material particle

$$\left(\frac{dr}{d\tau}\right)^2 = k^2 - 1 - \frac{h^2}{r^2} + \frac{2M}{r}\left(1 + \frac{h^2}{r^2}\right)$$

show that, changing variables from r to $u = 1/r$ and τ to ϕ, this equation may be written as

$$h^2\left(\frac{du}{d\phi}\right)^2 = \left(k^2 - 1\right) - h^2u^2 + 2Mu\left(1 + h^2u^2\right)$$

which in turn reduces to

$$\frac{d^2u}{d\phi^2} = -u + \frac{M}{h^2} + 3Mu^2$$

6. Show that, for the Earth's orbit, the ratio of the second and third terms on the right hand side of the above equation, (3.20), satisfy

$$\frac{3Mu^2}{M/h^2} \simeq 3 \times 10^{-8}$$

as stated in equation (3.21) of your notes.

7. Verify that the particular integrals given in equations (3.28) – (3.30) are indeed solutions of the General Relativistic equation, (3.26), for a planetary orbit. Hence, verify that the GR orbit for Mercury (ignoring all other perturbations) predicts an advance of perihelion of approximately 43 arc seconds per century.

8. Show that the geodesic equation for a photon propagating in the Schwarzschild metric may be written as

$$\left(\frac{dr}{d\lambda}\right)^2 = k^2 - \frac{h^2}{r^2} + \frac{2Mh^2}{r^3}$$

where λ is an affine parameter. Hence show that, after changing variables from r to $u = 1/r$ and λ to ϕ, this reduces to

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2$$

9. By carrying out the appropriate substitutions in equation (3.68) of your notes, verify that in the two gravitational lensing regimes of interest (i.e. distances on cosmological and galactic scales respectively) the size of the angular Einstein radius may be written as

$$\theta_E \simeq 3\text{''}\sqrt{\frac{M}{10^{12}M_\odot}\,\frac{10^9\,\mathrm{pc}}{D_S}\,\frac{(1-x)}{x}}$$

and

$$\theta_E \simeq 0.9\,\mathrm{mas}\sqrt{\frac{M}{M_\odot}\,\frac{10\,\mathrm{kpc}}{D_S}\,\frac{(1-x)}{x}}$$

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