Chapter 7

GR and Cosmology

In this final chapter we discuss some applications of general relativity to cosmology, providing a rigorous derivation of some results that are quoted and then discussed in detail within the honours cosmology course. We begin with a derivation of the general form of the spacetime metric which is consistent with the **cosmological principle**. We will then go on to use this metric to describe cosmological events.

7.1 The Robertson-Walker metric

In A1Y cosmology (and indeed in the Honours course too) the concept of the cosmological principle was introduced. Loosely speaking this states that, although the Universe is evolving, at an given time it appears **isotropic** and **homogeneous** on large scales. In other words, provided we sample sample a large enough volume of the Universe to be truly representative, then things look pretty much the same everywhere and in all directions. The observational evidence in support of the cosmological principle comes chiefly from the cosmic microwave background radiation and galaxy redshift surveys, as discussed in the A1Y and honours cosmology courses.

Of course what we learn from the study of relativity is that the rigid Newtonian concepts of space and time are rather misleading and poorly defined, and so we need to consider how to formulate the cosmological principle within the framework of a fully covariant theory of general relativity. Essentially we want to translate the physical idea underpinning the cosmological principle – isotropy and homogeneity of the Universe – into a mathematical statement about the geometry of the Universe, as embodied by its metric. More specifically we want to derive the general form of the spacetime metric which is consistent with the cosmological principle. This metric is known as the **Robertson-Walker metric**.

7.1.1 Spherically symmetric spacetimes

Let the coordinates of our four-dimensional spacetime manifold have labels $\{x^0, x^1, x^2, x^3\}$. Then the interval between neighbouring events is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{7.1}$$

where, as usual, the summation convention runs over $0, 1, 2, 3$ for both indices μ and ν. In the language of GR, then, the cosmological principle means that we can slice up the four-dimensional geometry of spacetime by identifying a coordinate (let it be x^0 for reasons that are, or will soon become, obvious) such that hypersurfaces of constant x^0 have the following properties:

1. they are spherically symmetric (this is what we mean by isotropy within a GR framework)

2. they have the same value of the Ricci curvature scalar, R, at every point on the hypersurface (this is what we mean by homogeneity within a GR framework)

What considerations should influence our choice of the remaining hypersurface coordinates $\{x^1, x^2, x^3\}$? A sensible choice is **comoving coordinates**, which we first introduced in A1Y cosmology. In such a coordinate system, although the proper distance between observers changes as the background Universe expands, our comoving coordinates *also* expand in perfect step with the background. Thus, if we ignore any motions of galaxies *within* the hypersurfaces of constant x^0 (what we termed in A1Y cosmology galaxy **peculiar motions**)[1], the comoving coordinates of each galaxy will remain fixed.

We now formally identify the x^0 coordinate as the proper time experienced by an observer in a galaxy – i.e. the time measured by a clock not moving relative to that observer, as would be the case for a comoving coordinate system.

Consider two galaxies with comoving coordinates $\{x^1, x^2, x^3\}$ and $\{x^1 + dx^1, x^2 + dx^2, x^3 + dx^3\}$. Suppose at some time t_0 the hypersurface of constant t_0 has line element

$$d\ell^2 \quad = \quad g_{ij}(t_0)dx^i dx^j \tag{7.2}$$

(i.e. this is just the interval from equation (7.1) with $dx^0 = 0$).

Suppose that at some later time, t_1 say, the hyperfsurface line element is given by

$$d\ell^2 \quad = \quad g_{ij}(t_1)dx^i dx^j \tag{7.3}$$

[1] In the interests of expediency we are being a little sloppy here, since strictly speaking we would consider neighbouring surfaces of *different* x^0 in order to define these galaxy motions

Since the galaxies have constant comoving coordinates, the change in the proper distance between the galaxies due to the expansion of the Universe between t_0 and t_1 must be described by the change in the metric coefficients g_{ij} between t_0 and t_1. In general the g_{ij}'s would depend not just on x^0 (i.e. t_0 or t_1) but would also be functions of the spatial coordinates x^1, x^2 and x^3. However, because of isotropy, the g_{ij}'s must all change at the same rate for $i = 1, 2, 3$ or else the expansion of the Universe would have a preferred direction. This means that the metric coefficients must depend only on x^0 (which we will henceforth label simply as t). Thus we can write the line element as

$$d\ell^2 = a^2(t) h_{ij} dx^i dx^j \tag{7.4}$$

where $a(t)$ is an overall scale factor (known generally as the **cosmic scale factor**) that depends only on time t, and h_{ij} are the metric coefficients for the comoving coordinates on the three-dimensional spatial hypersurface at constant t. Note that the h_{ij} do *not* depend on t, since they describe the metric coefficients for comoving coordinates.

The full spacetime interval of equation (7.1) can therefore be written as

$$ds^2 = -dt^2 + g_{0i} dt dx^i + a^2(t) h_{ij} dx^i dx^j \tag{7.5}$$

We can use symmetry arguments, similar to those of Section 2.3, to show that

$$g_{0i} = \vec{e}_0 \cdot \vec{e}_i = 0 \tag{7.6}$$

so that

$$ds^2 = -dt^2 + a^2(t) h_{ij} dx^i dx^j \tag{7.7}$$

We know that $g_{00} = -1$ because we have identified t as the proper time, τ, along a line with $dx^i = 0$ and

$$d\tau = \sqrt{-ds^2} \tag{7.8}$$

What form should the h_{ij} take? Since they are independent of t, and should describe a spatial hypersurface which is spherically symmetric, we can make use of the results which we already derived in Chapter 2. There we showed that the metric of a static, spherically symmetric spacetime took the form of equation (2.23)

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{7.9}$$

where $\lambda \equiv \lambda(r)$. In the present case, then, the metric of our spherically symmetric spacetime takes the form

$$ds^2 \quad = \quad -dt^2 + a^2(t) \left[e^\lambda dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right] \tag{7.10}$$

(Note that the g_{00} component appears in many textbooks as $-c^2$, but recall that we are generally working with geometrised units in which $c = 1$).

We can determine the function $\lambda(r)$ by using our requirement that spatial hypersurfaces of constant t have constant curvature. For the line element

$$d\ell^2 = e^\lambda dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{7.11}$$

(i.e. for simplicity taking $a(t) = 1$) we can write down an expression for the curvature scalar, R, using results which we derived in Chapter 2. Firstly the Christoffel symbols are the same as those derived in equations (2.24), but with $\nu' = \nu'' = 0$. Thus

$$
\begin{array}{rclrcl}
\Gamma^r_{rr} & = & \frac{1}{2}\lambda' & \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} & = & \frac{1}{r} \\
\Gamma^r_{\theta\theta} & = & -re^{-\lambda} & \Gamma^\theta_{\phi\phi} & = & -\sin\theta\cos\theta \\
\Gamma^r_{\phi\phi} & = & -re^{-\lambda}\sin^2\theta & \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} & = & \frac{1}{r} \\
\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} & = & \cot\theta & & &
\end{array}
$$

$$
\text{All others zero} \tag{7.12}
$$

Similarly, adapting the results of equations (2.27) – (2.29), the only non-zero components of the Ricci tensor are

$$
R_{rr} = -\frac{\lambda'}{r} \tag{7.13}
$$

$$
R_{\theta\theta} = 1 - e^{-\lambda}\left(1 - \lambda'\right) \tag{7.14}
$$

$$
R_{\phi\phi} = R_{\theta\theta}\sin^2\theta \tag{7.15}
$$

Since both the metric and Ricci tensors are orthogonal, the curvature scalar is given by

$$
R = g^{ij}R_{ij} = g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \tag{7.16}
$$

and adapting the result of equation (4.11) we find that

$$
R = 2\left[\frac{1}{r^2} + \frac{\lambda' e^{-\lambda}}{r} - \frac{e^{-\lambda}}{r^2}\right] \tag{7.17}
$$

or equivalently

$$
R = \frac{2}{r^2}\left[1 - e^{-\lambda}\left(1 - r\lambda'\right)\right] \tag{7.18}
$$

A little further algebra (see Examples Sheet II.6) shows that equation (7.18) can be re-written as

$$
R = \frac{2}{r^2}\left[1 - \frac{d}{dr}\left(re^{-\lambda}\right)\right] \tag{7.19}
$$

Requiring that the curvature is equal to the same constant everywhere on the spatial hypersurface implies that

$$\frac{d}{dr}\left(re^{-\lambda}\right) = 1 - \frac{1}{2}Cr^2 \tag{7.20}$$

where C is a constant (equal to the value of the curvature scalar). Integrating equation (7.20)

$$re^{-\lambda} = r - \frac{1}{6}Cr^3 + A \tag{7.21}$$

or

$$e^{-\lambda} = 1 - \frac{1}{6}Cr^2 + \frac{A}{r} \tag{7.22}$$

where A is another constant.

Since spacetime is locally flat we require that $e^{\lambda} \rightarrow 1$ as $r \rightarrow 0$. Hence it also follows that $e^{-\lambda} \rightarrow 1$ as $r \rightarrow 0$. This implies that $A = 0$. Re-writing $C/6$ as a new constant k, finally gives us

$$e^{\lambda} = \frac{1}{1 - kr^2} \tag{7.23}$$

which means that the metric of the full four-dimensional spacetime may be written as

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{7.24}$$

This is the most general form of the spacetime metric which has the property that its spatial hypersurfaces are isotropic and homogenous – in concordance with the cosmological principle. The metric is known as the **Robertson-Walker metric**. The values $k = 0$, $k > 0$ and $k < 0$ correspond to spatial hypersurfaces which have zero, positive and negative curvature respectively. (We can see this by applying the formula for the intrinsic curvature, first introduced in A2 Relativity and Gravitation, to the line element of the spatial hypersurface. See also Examples Sheet II.6). This should not be surprising, since k is proportional to the constant, C, equal to the value of the curvature scalar, R, everywhere on the spatial hypersurface.

In fact we need only ever consider three values of k: $k = 0$, $k = +1$ and $k = -1$. This is because the line element for any other positive (negative) value of k can be reduced to the line element for $k = +1$ ($k = -1$) by a simple re-scaling of the cosmic scale factor, $a(t)$. (See Examples Sheet II.6). The values $k = 0$, $k = +1$, and $k = -1$ are said to represent Universes with flat, closed and open geometries respectively.

7.2 Friedmann-Lemaitre Models

Our next task will be to explore cosmologically interesting model Universes which are described by the Robertson-Walker metric. These are known as Friedmann–Lemaitre models, after the researchers who first studied them. The steps required to derive these models are as follows

- Compute the Einstein tensor for the Robertson-Walker metric (by first computing its Christoffel symbols and hence its Riemann-Christoffel tensor)
- Assume a form for the energy-momentum tensor which is consistent with the cosmological principle
- Apply Einstein's equations to connect the Einstein and energy-momentum tensors
- Solve Einstein's equations to derive expressions for the time evolution of the cosmic scale factor and other important physical quantities

7.2.1 Christoffel symbols for the Robertson-Walker metric

Since the Robertson-Walker metric is orthogonal, the contravariant metric components are just the inverse of the covariant metric components. Thus

$$g^{tt} = -1; \quad g^{rr} = \frac{1 - kr^2}{a^2(t)}; \quad g^{\theta\theta} = \frac{1}{a^2(t)r^2}; \quad g^{\phi\phi} = \frac{1}{a^2(t)r^2 \sin^2\theta} \tag{7.25}$$

The orthogonality of the metric means that we can also use the results of equations (2.12) to simplify the calculation of its Christoffel symbols. Indeed equations (7.12) already give us the Christoffel symbols for all terms not involving t, although those expressions were given in terms of the function $\lambda(r)$. Note, however, that

$$\lambda = \ln\left(\frac{1}{1 - kr^2}\right) = -\ln\left(1 - kr^2\right) \tag{7.26}$$

so that

$$\lambda' = \frac{2kr}{1 - kr^2} \tag{7.27}$$

It is then straightforward, although somewhat laborious, to show (see Examples Sheet II.6) that the Christoffel symbols for the Robertson-Walker metric are

$$\begin{array}{ccccccc}
\Gamma^t_{rr} & = & \frac{a\dot{a}}{1-kr^2} & & \Gamma^t_{\theta\theta} & = & a\dot{a}r^2 \\
\Gamma^t_{\phi\phi} & = & a\dot{a}r^2\sin^2\theta & & \Gamma^r_{rr} & = & \frac{kr}{1-kr^2} \\
\Gamma^r_{rt} = \Gamma^r_{tr} & = & \frac{\dot{a}}{a} & & \Gamma^r_{\theta\theta} & = & -r\left(1 - kr^2\right) \\
\Gamma^r_{\phi\phi} & = & -r\left(1 - kr^2\right)\sin^2\theta & & \Gamma^\theta_{\phi\phi} & = & -\sin\theta\cos\theta \\
\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} & = & \cot\theta & & \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} & = & \frac{1}{r} \\
\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} & = & \frac{1}{r} & & \Gamma^\theta_{\theta t} = \Gamma^\theta_{t\theta} & = & \frac{\dot{a}}{a} \\
\Gamma^\phi_{\phi t} = \Gamma^\phi_{t\phi} & = & \frac{\dot{a}}{a} & & & &
\end{array}$$

$$\text{All others zero} \tag{7.28}$$

7.2.2 The Ricci tensor for the Robertson-Walker metric

Using equation (1.43), which expresses the Ricci tensor as a contraction of the Riemann-Christoffel tensor, we see that e.g.

$$R_{tt} = R^{\sigma}_{t\sigma t} = R^{t}_{ttt} + R^{r}_{trt} + R^{\theta}_{t\theta t} + R^{\phi}_{t\phi t} \tag{7.29}$$

The first term on the right hand side is given by

$$R^{t}_{ttt} = \Gamma^{\sigma}_{tt}\Gamma^{t}_{\sigma t} - \Gamma^{\sigma}_{tt}\Gamma^{t}_{\sigma t} + \Gamma^{t}_{tt,t} - \Gamma^{t}_{tt,t} \tag{7.30}$$

each term of which, from equations (7.28), can be seen to be zero. After evaluation of the remaining terms on the right hand side of equation (7.29) it is straightforward to show that

$$R_{tt} = -3\,\frac{\ddot{a}}{a} \tag{7.31}$$

Similar calculations then give (see Examples Sheet II.6)

$$R_{rr} = \frac{2\dot{a}^2 + 2k + a\ddot{a}}{1 - kr^2} \tag{7.32}$$

$$R_{\theta\theta} = r^2\left[a\ddot{a} + 2\dot{a}^2 + 2k\right] \tag{7.33}$$

$$R_{\phi\phi} = R_{\theta\theta}\,\sin^2\theta \tag{7.34}$$

7.2.3 The Einstein tensor for the Robertson-Walker metric

Combining equation (7.25) and equations (7.31) – (7.34), and after some algebraic tidying up, it follows that the curvature scalar, $R = R_{\mu\nu}g^{\mu\nu}$, is given by

$$R = 6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right] \tag{7.35}$$

We can now evaluate the components of the Einstein tensor, in its fully covariant form, as we originally defined in equation (4.12), namely

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{7.36}$$

Since both the Ricci and metric tensors are orthogonal, so too is the Einstein tensor and its non-zero components are found to be (see Examples Sheet II.6)

$$G_{tt} = \frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} \tag{7.37}$$

$$G_{rr} = -\frac{a^2}{1-kr^2}\left[\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right] \tag{7.38}$$

$$G_{\theta\theta} = -r^2 a^2 \left[\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right] \tag{7.39}$$

$$G_{\phi\phi} = G_{\theta\theta} \sin^2\theta \tag{7.40}$$

7.2.4 The energy-momentum tensor for the cosmic fluid

To proceed further we need to assume an appropriate form for the energy-momentum tensor. It is customary to treat the cosmological matter and energy fields as a **perfect fluid**. From GR-I, the general expression for the components of the energy-momentum tensor, in fully covariant form, is

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu} \tag{7.41}$$

with $c = 1$. Here ρ and P denote the density and pressure of the fluid, and u_μ denotes the (covariant components of the) four-velocity of a fluid element. In order to be consistent with the cosmological principle, which requires that the Universe be homogeneous, ρ and P must be functions only of t.

For the case of a perfect fluid the energy-momentum tensor simplifies to

$$\mathbf{T} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{7.42}$$

Note that the perfect fluid approximation assumes that fluid elements do not exchange energy with their neighbours due to heat conduction or viscosity. (Such processes would produce the *shearing* of fluid elements, which would introduce non-zero off-diagonal terms in the energy-momentum tensor). This approximation breaks down when one considers e.g. the dynamics of hot gas in individual or interacting galaxies (see honours Galaxies II course) but it is a remarkably good approximation when one considers the behaviour of the 'background' cosmic fluid on large scales. Indeed, since we believe that the matter content of the Universe is dominated today by weakly interacting cold dark matter (see Galaxies II, A1Y cosmology and honours cosmology courses), we can also usually neglect the contribution of matter pressure and further approximate the cosmic fluid as *dust*. In recent years, however, it has become fashionable to reintroduce in a more general way the notion of cosmic pressure via the so-called **equation of state parameter**, w, in the equation

$$P = w\rho \tag{7.43}$$

Different values of w then correspond to different physical regimes with correspondingly different sources of pressure: e.g. dust is the case $w = 0$; a radiation dominated Universe (which one can think of as an ideal 'photon' gas, with $c = 1$) corresponds to $w = 1/3$. More exotic cosmological models, such as networks of cosmic strings, can also be incorporated by appropriate choice of w, and a wide range of models – with evocative names such as **dark energy**, **quintessence** and even **phantom energy** – in which w evolves with time has also been proposed. We return briefly later to the idea of the w parameter, and its relation to the so-called **cosmological constant**. Although this exciting topic lies mainly well beyond the scope of this (and indeed also the honours cosmology) course, it is a highly active area of current cosmology research.

7.2.5 Solutions of Einstein's equations

Having armed ourselves with the components of the Einstein tensor and the energy-momentum tensor, we can now apply equation (1.52) as the solution of Einstein's equations, i.e.

$$8\pi G T_{\mu\nu} = G_{\mu\nu} \tag{7.44}$$

Note that the G on the left hand side is the Newtonian gravitational constant (which we now include explicitly, as is common practice in most cosmology texts – even those which take $c = 1$, as we do here) and the factor of $8\pi G$ arises from requiring that the solution to Einstein's equations reduces to the Newtonian expression in the limit of a weak gravitational field.

If we first consider the 'tt' components of the Einstein and energy-momentum tensors, it follows from equations (7.37) and (7.44) that

$$8\pi G\rho = \frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} \tag{7.45}$$

Re-arranging, this can be written as

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{7.46}$$

where we have used the definition of the **Hubble parameter**, $H = \dot{a}/a$. Equation (7.46) is identical to the so-called **Friedmann equation** derived in A1Y cosmology from purely Newtonian considerations, but now re-derived rigorously within the framework of GR. This is essentially a differential equation for the time-evolution of the cosmic scale factor; recall from A1Y cosmology that we used the Friedmann equation to identify a **critical density** for the Universe that marked the division between models which would expand indefinitely and models which would expand and then contract to a 'Big Crunch'.

Of course within our rigorous GR framework we can make use of the other components of the Einstein and energy-momentum tensors to derive a further constraint equation for the cosmic scale factor; this is usually referred to as the second Friedmann equation and may be presented in different forms in different textbooks – depending on which component(s) of the tensors are used to derive it. Here we obtain our second equation by *contracting* the Einstein and energy-momentum tensors – i.e. by computing

$$\mathcal{T} = g^{\mu\nu} T_{\mu\nu} \qquad \text{and} \qquad \mathcal{G} = g^{\mu\nu} G_{\mu\nu} \tag{7.47}$$

(Here we have introduced the notation $\mathcal{T}$ and $\mathcal{G}$ to avoid confusion with e.g. temperature and with the Newtonian gravitational constant respectively).

Once again, the orthogonality of the Einstein, energy-momentum and metric tensors considerably simplifies our calculations. It is easy to show (see Examples Sheet II.6) that

$$\mathcal{G} \quad = \quad -6\,\frac{\ddot{a}}{a} - 6\,\frac{\dot{a}^2}{a^2} - 6\,\frac{k}{a^2} \tag{7.48}$$

and

$$\mathcal{T} \quad = \quad 3P - \rho \tag{7.49}$$

From Einstein's equations, then

$$8\pi G\mathcal{T} \quad = \quad \mathcal{G} \tag{7.50}$$

or

$$8\pi G(3P - \rho) \quad = \quad -6\,\frac{\ddot{a}}{a} - 6\,\frac{\dot{a}^2}{a^2} - 6\,\frac{k}{a^2} \tag{7.51}$$

Adding together twice equation (7.45) and equation (7.51) eliminates the terms involving $\dot{a}$ and k to give

$$\frac{\ddot{a}}{a} \quad = \quad -\frac{4\pi G}{3}\,(\rho + 3P) \tag{7.52}$$

7.2.6 Einstein's cosmological constant

We can see from equation (7.52) that, for 'normal' matter with $\rho > 0$ and $P \geq 0$, it follows that we cannot have a static solution for $a(t)$, with $\dot{a} = \ddot{a} = 0$. It was essentially this fact which motivated Einstein to introduce his so-called *cosmological constant* since, prior to Hubble's discovery of the recession of the nebulae, the expectation was that the Universe was indeed static. Einstein introduced the cosmological constant, Λ, as an additional term in the Einstein tensor, i.e.

$$G_{\mu\nu} \quad = \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda \tag{7.53}$$

or

$$G^{\mu\nu} \quad = \quad R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + g^{\mu\nu}\Lambda \tag{7.54}$$

Since the covariant derivatives of the metric are zero, it still follows that

$$G^{\mu\nu}{}_{;\nu} \quad = \quad 0 \tag{7.55}$$

as before.

We can think of Λ as a modification to the large-scale geometry of the Universe, although nowadays it is generally more common to think of the cosmological constant instead as a modification of the energy-momentum tensor, rather than the Einstein tensor. In some sense both descriptions are equivalent, but the latter interpretation is perhaps more useful in helping to understand physically what the cosmological constant actually *is* – e.g. due to the zero-point energy of the vacuum.

Combining equation (7.54) with the earlier results of sections 7.2.3 and 7.2.4, it is straightforward to show that the Friedmann equations (7.46) and (7.52) now take the form

$$H^2 \quad = \quad \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{7.56}$$

and

$$\frac{\ddot{a}}{a} \quad = \quad -\frac{4}{3}\pi G\left(\rho + 3P\right) + \frac{\Lambda}{3} \tag{7.57}$$

The presence of the Λ terms means that a solution for $a(t)$ *can* be found with $\dot{a} = \ddot{a} = 0$. For example, if the Universe is assumed to be matter dominated with $P = 0$ (i.e. assuming a dust solution), then setting $\ddot{a} = 0$ in equation (7.57) implies that $\Lambda > 0$, since $\rho > 0$ for normal matter. This in turn implies that $k > 0$ if we set $H = 0$ in equation (7.56). Hence $k = 1$, and from equation (7.57)

$$\Lambda = 4\pi G\rho \tag{7.58}$$

and

$$a = \frac{1}{\sqrt{4\pi G\rho}} = \frac{1}{\sqrt{\Lambda}} \tag{7.59}$$

Equation (7.59) is known as the *Einstein solution* for the scale factor, and it does indeed satisfy what was in 1916 the apparently sensible requirement that the scale factor be constant, corresponding to a static Universe. However, it is quite easy to show (see Examples Sheet II.6) that this solution is inherently *unstable* – i.e. the slightest perturbation of the density would lead to rapid expansion or collapse. The Einstein solution is rather like the condition of a pencil balanced on its point: although the pencil is static, the slightest disturbance would cause it to topple over. This weakness of the Einstein solution – together with the even stronger physical evidence provided by Hubble's observations of the expansion of the Universe – prompted Einstein much later to describe his introduction of the cosmological constant as his 'greatest blunder'. Indeed one might argue that it is unfortunate that Einstein did not stick to the courage of his convictions and use the result of equation (7.52) to make the startling prediction that the Universe *was* expanding. The confirmation of such a prediction by Hubble would have been an even more spectacular vindication of GR than, say, the observations of light deflection carried out by Eddington during a total Solar eclipse.