Chapter 2

Static Models with Spherical Symmetry

2.1 Orthogonal metrics

In almost all astrophysical situations that we will consider in GR-II, we can generally work with the metric tensor in **orthogonal** form. This means that, in a particular coordinate system, the components, $g_{\alpha\beta}$, of the metric tensor satisfy

$$g_{\alpha\beta} = 0 \quad \text{for all } \alpha \neq \beta \tag{2.1}$$

This implies that there are no 'cross terms' in the expression for the invariant interval; i.e. we can write

$$ds^2 = g_{\alpha\alpha}(dx^\alpha)^2 \tag{2.2}$$

Note that the components of a metric will not be orthogonal in *any* coordinate system. We can prove this as follows.

Suppose that $g_{\alpha\beta}$ are the metric components in a particular coordinate system, such that eq. (2.1) holds. Let $g'_{\mu\nu}$ denote the metric components in another coordinate system. Since $\mathbf{g}$ is a $(0,2)$ tensor, we have

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \tag{2.3}$$

Eq. (2.1) implies that

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\alpha}{\partial x'^\nu} g_{\alpha\alpha} \tag{2.4}$$

but it does **not** follow that

$$g'_{\mu\nu} = 0 \quad \text{for all } \mu' \neq \nu' \tag{2.5}$$

In fact the question of whether the metric components are orthogonal in a particular coordinate system is closely related to the question of whether the basis vectors are orthogonal in that coordinate system. (Although we often choose a coordinate basis with this property, we do not *have* to make such a choice). Consider, for example, a coordinate system with basis vectors $\{\vec{e}_i\}$, and two vectors $\vec{A} = A^i\vec{e}_i$ and $\vec{B} = B^i\vec{e}_i$.

The scalar product $\vec{A}\cdot\vec{B}$ was defined in GR-I as

$$\vec{A}\cdot\vec{B} = (A^i\vec{e}_i)\cdot(B^j\vec{e}_j) = A^iB^j(\vec{e}_i\cdot\vec{e}_j) \tag{2.6}$$

from which it follows that

$$g_{ij} = \vec{e}_i\cdot\vec{e}_j \tag{2.7}$$

Thus, in particular, $g_{ij} = 0$ if and only if $\vec{e}_i$ and $\vec{e}_j$ are orthogonal.

2.1.1 Contravariant components for an orthogonal metric

It is generally in our interests to *choose* a coordinate system in which the metric coefficients are orthogonal in form. This is because it simplifies the expressions for certain tensors and other geometrical objects in which we are interested.

For example, the contravariant metric components are also orthogonal, and the diagonal terms are simply given by the reciprocal of the covariant diagonal terms. We can prove this as follows. We know that

$$g^{\alpha\beta} g_{\alpha\gamma} = \delta^{\beta}_{\gamma} \tag{2.8}$$

But from eq. (2.1) it follows that

$$g^{\gamma\beta} g_{\gamma\gamma} = \delta^{\beta}_{\gamma} \tag{2.9}$$

i.e.

$$g^{\gamma\beta} = 0 \quad \text{if} \gamma \neq \beta \tag{2.10}$$

and

$$g^{\gamma\gamma} = 1/g_{\gamma\gamma} \tag{2.11}$$

2.1.2 Christoffel symbols for an orthogonal metric

One can show that the Christoffel symbols also take a simple form for an orthogonal metric:-

$$\begin{aligned} \Gamma^{\lambda}_{\mu\nu} &= 0 \quad \text{for } \lambda,\ \mu,\ \nu \text{ all different} \\ \Gamma^{\lambda}_{\lambda\mu} &= \Gamma^{\lambda}_{\mu\lambda} = g_{\lambda\lambda,\mu}/2g_{\lambda\lambda} \\ \Gamma^{\lambda}_{\mu\mu} &= -g_{\mu\mu,\lambda}/2g_{\lambda\lambda} \\ \Gamma^{\lambda}_{\lambda\lambda} &= g_{\lambda\lambda,\lambda}/2g_{\lambda\lambda} \end{aligned} \tag{2.12}$$

(Note: the summation convention does not apply in these equations). The proof of these results is left as an exercise; see Examples Sheet I-6.

2.1.3 Geodesic equations for an orthogonal metric

We can find another (and potentially more useful) form of the geodesic equation (1.26).

For affine parameter, p, this takes the form

$$\frac{d}{dp}\left(g_{\lambda\nu}\frac{dx^\nu}{dp}\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp} = 0 \tag{2.13}$$

For an orthogonal metric this reduces further to

$$\frac{d}{dp}\left(g_{\lambda\lambda}\frac{dx^\lambda}{dp}\right) - \frac{1}{2}\frac{\partial g_{\mu\mu}}{\partial x^\lambda}\left(\frac{dx^\mu}{dp}\right)^2 = 0 \tag{2.14}$$

The proof of equations (2.13) and (2.14) is left as an exercise; see Examples Sheet II.1.

2.2 Geometrised units

In GR-I we adopted units in which the speed of light, $c = 1$. This effectively means that we are measuring time in units of **length** – specifically, the distance travelled by light in that time. Thus

$$1 \text{ second} \equiv 3 \times 10^8 \text{ m}$$

At the end of GR-I we derived the weak field limit of Einstein's equations, which allowed us to determine the constant, k, in equation (1.52) in terms of the gravitational constant, G. Now, in SI units

$$G \simeq 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$$

but the Newton is a composite SI unit; in fact

$$1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}$$

so that

$$G \simeq 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$$

Replacing our unit of time with the unit of length defined above, we obtain

$$G \simeq 7.41 \times 10^{-28}\,\mathrm{m}\,\mathrm{kg}^{-1}$$

In GR-II we will, in turn, find it useful to set $G = 1$; this effectively means that we will measure *mass* also in units of length. This approach is often referred to as **geometrised units**. It follows that, in these new units

$$1\ \mathrm{kg} = 7.41 \times 10^{-28}\ \mathrm{m}$$

In summary, our geometrised units take the form

Unit of length:	1 m		
Unit of time:	1 m	$\equiv$	3.33×10^{-9} s
Unit of mass:	1 m	$\equiv$	1.34×10^{27} kg

2.3 Spherically symmetric metrics

A large part of GR-I was concerned with formulating Einstein's equations – having first to develop a lot of necessary mathematical machinery to make this possible. In GR-II we now face the (perhaps more interesting) challenge of trying to solve Einstein's equations. This is far from trivial, however, and indeed exact solutions have been obtained for a (fairly small) number of special cases only. Fortunately however, these cases include some examples which are excellent approximations to real astrophysical situations.

Here we consider the case of a **spherically symmetric solution**, which we will then use to model the spacetime outside and inside a star. What exactly do we mean by spherical symmetry in curved spacetime, however? In the flat Minkowski spacetime of SR we can change our spatial variables to the usual polar coordinates, to give an invariant interval

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{2.15}$$

Thus, surfaces of constant r and t have the geometry of a 2-sphere, with interval

$$d\ell^2 = r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{2.16}$$

We can also use this property to define spherical symmetry in a curved spacetime:

A spacetime is spherically symmetric if every point in the spacetime lies on a 2-D surface which is a 2-sphere

If we label the coordinates of our spacetime by (r', t, θ, ϕ) – the reasons for our choice of labels will become clear shortly – then every point in a spherically symmetric spacetime lies on a 2-D surface which is a 2-sphere, with interval given by

$$d\ell^2 = f(r', t) \left[d\theta^2 + \sin^2\theta \, d\phi^2\right] \tag{2.17}$$

where $\sqrt{f(r', t)}$ is the **radius of curvature** of the 2-sphere.

Unlike flat spacetime, where transforming to ordinary polar coordinates immediately reveals the radius of curvature to be simply our radial polar coordinate, r, in curved spacetime no such trivial relation always exists between the angular coordinates of the 2-D sphere and the remaining two coordinates at each point in spacetime.

We can, however, simply *define* a new radial coordinate, r, which satisfies

$$r^2 = f(r', t) \tag{2.18}$$

and we can 'line up' the origins of our 2-sphere coordinate systems, (θ, ϕ), for points in spacetime with different values of r.

Spherical symmetry also requires that any radial path in the space is **orthogonal** to the 2-D spheres on the which the points along that radial path lie, since otherwise this would allow us to define a preferred direction in the space. This implies that, in the metric of our spherically symmetric spacetime

$$g_{r\theta} = g_{r\phi} = 0 \tag{2.19}$$

Hence, we have restricted the form of the spacetime metric to

$$ds^2 = g_{tt}dt^2 + 2g_{tr}drdt + 2g_{t\theta}d\theta dt + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{2.20}$$

Consider now the curve $r =$ const., $\theta =$ const., $\phi =$ const. This is the wordline of a particle in the spacetime which has constant spatial coordinates. This curve must *also* be orthogonal to the 2-spheres on which each point on the curve lies, since otherwise we could define a preferred direction in *spacetime*. This means that we can further restrict our metric coefficients to be

$$g_{t\theta} = g_{t\phi} = 0 \tag{2.21}$$

So we can write the general form of the metric for a spherically symmetric spacetime as

$$ds^2 = g_{tt}dt^2 + 2g_{tr}drdt + g_{rr}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{2.22}$$

where g_{tt}, g_{tr} and g_{rr} are arbitrary functions of r and t.

2.3.1 The case of a static spacetime

We now consider the special case of a **static spherically symmetric spacetime** (which we henceforth refer to as S^4 for short). This means that we can find a time coordinate, t, for which

1. all metric components are independent of t
2. the metric is unchanged if we apply the transformation $t \rightarrow -t$ (i.e. if we undergo time reversal)

It is fairly easy to see that property (2) implies that $g_{tr} = 0$, which means that we can write the interval for S^4 as

$$ds^2 = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \tag{2.23}$$

which is orthogonal in form. Note that we have introduced the functions $\nu(r)$ and $\lambda(r)$ in replacing g_{tt} and g_{rr}. Since the exponential function is strictly positive for all r, this replacement is legitimate provided that $g_{tt} < 0$ and $g_{rr} > 0$ for all points in our spacetime. We will consider carefully later whether these conditions are satisfied, and under what circumstances they break down.

2.3.2 Christoffel symbols for S^4

Using equations (2.12) it is straightforward to calculate the Christoffel symbols for this metric (see Examples Sheet II.1). Denoting by dashes differentiation with respect to r, the Christoffel symbols are given by

$$
\begin{array}{lclclcl}
\Gamma^t_{rt} = \Gamma^t_{tr} & = & \frac{1}{2}\nu' & \quad & \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} & = & \frac{1}{r} \\
\Gamma^r_{tt} & = & \frac{1}{2}\nu' e^{\nu-\lambda} & & \Gamma^\theta_{\phi\phi} & = & -\sin\theta\cos\theta \\
\Gamma^r_{rr} & = & \frac{1}{2}\lambda' & & \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} & = & \frac{1}{r} \\
\Gamma^r_{\theta\theta} & = & -re^{-\lambda} & & \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} & = & \cot\theta \\
\Gamma^r_{\phi\phi} & = & -re^{-\lambda}\sin^2\theta & & & &
\end{array}
$$

$$\text{All others zero} \tag{2.24}$$

2.3.3 Ricci tensor for S^4

We can write the Ricci tensor (Eq. 1.36) as

$$R_{\lambda\nu} = \Gamma^\tau_{\lambda\nu}\Gamma^\sigma_{\tau\sigma} - \Gamma^\tau_{\lambda\sigma}\Gamma^\sigma_{\tau\nu} + \Gamma^\sigma_{\lambda\nu,\sigma} - \Gamma^\sigma_{\lambda\sigma,\nu} \tag{2.25}$$

Substituting the results of equations (2.24) into equation (2.25) we find that

$$R_{tt} = \frac{1}{2}e^{\nu-\lambda}\left(\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\nu'\lambda' + \frac{2}{r}\nu'\right) \tag{2.26}$$

$$R_{rr} = -\frac{1}{2}\left(\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\nu'\lambda' - \frac{2}{r}\lambda'\right) \tag{2.27}$$

$$R_{\theta\theta} = 1 - e^{-\lambda}\left[1 + \frac{r}{2}(\nu' - \lambda')\right] \tag{2.28}$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta \tag{2.29}$$

and all other terms of the Ricci tensor are identically zero. (See Examples Sheet II.1)

2.4 Derivation of the Schwarzschild metric

We now seek to apply the general S^4 metric to derive the spacetime exterior to a spherically symmetric star; this is known as the **Schwarzschild solution**. If the star is in an isolated region of space, then we can assume that all components of the Ricci tensor exterior to the star are identically zero. Hence the right hand side of equations (2.26) – (2.29) are all zero. Thus

$$e^{\lambda-\nu} R_{tt} + R_{rr} = \frac{\nu' + \lambda'}{r} = 0 \tag{2.30}$$

which in turn implies that

$$\nu + \lambda = \text{constant} \tag{2.31}$$

At large distances from the star we want the Schwarzschild metric to reduce to SR. Hence, as

$$r \to \infty \ , \quad e^{\nu} \to 1 \quad \text{and} \quad e^{\lambda} \to 1 \tag{2.32}$$

Thus, as

$$r \to \infty \ , \quad \nu \to 0 \quad \text{and} \quad \lambda \to 0 \tag{2.33}$$

which implies that

$$\nu + \lambda = 0 \tag{2.34}$$

so that

$$e^{\nu} = e^{-\lambda} \tag{2.35}$$

This allows us to eliminate ν from equation (2.28), giving

$$e^{-\lambda} \left(1 - \lambda' r\right) = 1 \tag{2.36}$$

i.e.

$$\frac{d}{dr} \left(r e^{-\lambda} \right) = 1 \tag{2.37}$$

which we can integrate to give

$$e^{\nu} = e^{-\lambda} = 1 + \frac{\alpha}{r} \tag{2.38}$$

where α is a constant.

To evaluate α, suppose we release a material 'test' particle (i.e. a particle of so little rest mass that it does not disturb the spacetime metric) from rest. Thus, initially

$$\frac{dx^j}{d\tau} = 0 \quad \text{for } j = 1, 2, 3 \tag{2.39}$$

where τ is **proper time**, and

$$\frac{dx^0}{d\tau} \equiv \frac{dt}{d\tau} \neq 0 \tag{2.40}$$

Applying equation (1.28) and after some reduction we see that

$$\frac{dt}{d\tau} = e^{-\nu/2} \tag{2.41}$$

We now apply the first of the geodesic differential equations (1.26). At the instant when the particle is released this reduces to

$$\frac{d^2r}{d\tau^2} + \Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 = 0 \tag{2.42}$$

Substituting from equations (2.24) and (2.41) we obtain finally

$$\frac{d^2r}{d\tau^2} = \frac{\alpha}{2r^2} \tag{2.43}$$

In the limit of a weak gravitational field this result must reduce to the prediction of Newtonian gravity, which predicts

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} \tag{2.44}$$

where M is the mass of the star. If we adopt convenient units such that the gravitational constant, $G = 1$ (see below), this means that

$$\alpha = -2M \tag{2.45}$$

We can now write down the invariant interval for the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{2.46}$$

2.5 The Schwarzschild radius and the event horizon

The quantity $2M$ in equation (2.46) is known as the **Schwarzschild radius**. The Sun (which has a mass of about 2×10^{30} kg) has a Schwarzschild radius of about 3 km. (Note: this formula agrees with that given in A2 relativity when since $c = G = 1$).

We can see from equation (2.46) that the metric 'misbehaves' when $r = 2M$, since $g_{tt} = 0$ and $g_{rr} \to \infty$. We refer to this surface as the **event horizon** of the Schwarzschild metric, and we will discuss its physical significance in greater detail later in the course when we consider black holes. For now, we need only note that – provided $r > 2M$ – the metric is perfectly well behaved. For most stars (like the Sun, for example!) the Schwarzschild radius is much smaller than the physical radius (i.e. the photosphere) of the star. When this is the case we can certainly apply the Schwarzschild metric in the form of equation (2.46) to describe the spacetime exterior to the surface of the star; we do exactly this in the next chapter.