Chapter 1

Resumé of Gravitation & Relativity

Part I

1.1 Foundations of General Relativity

General relativity (GR) explains gravitation as a consequence of the curvature of spacetime. In turn spacetime curvature is a consequence of the presence of matter. Spacetime curvature affects the movement of matter, which reciprocally determines the geometric properties and evolution of spacetime. We can sum this up neatly as follows:-

> *"Spacetime tells matter how to move,*
>
> *and matter tells spacetime how to curve"*

1.1.1 The weak principle of equivalence

This simply states that the inertial mass, m_I, and the gravitational mass, m_G, of a body are equal. GR incorporates this result by demanding that *test particles* have worldlines that are *geodesics* in curved spacetime. Hence the worldline is independent of the mass of the test particle and depends only on the geometry of spacetime. Translating back into Newtonian language, this means that all bodies accelerate in a gravitational field at the same rate, regardless of their mass.

1.1.2 The strong principle of equivalence

This goes further and states that locally, i.e. in a local inertial frame (or free-falling frame), all physical phenomena are in agreement with special relativity. There are two important and immediate consequences of this principle. The first is that the path of a light ray should be bent by gravitational fields, and secondly, there should be a gravitational redshift.

1.1.3 Geodesic deviation

The separation, ξ, between the two free test particles is called the **geodesic deviation**. In general ξ is a vector. (In fact, it is a **four vector**, if we consider time separations as well). It is the acceleration of this geodesic deviation that indicates the presence of a gravitational field, or equivalently, the curvature of spacetime. In the flat spacetime of Minkowski free test particles have worldlines that are 'straight'. Thus the acceleration of the geodesic deviation is zero for Minkowski spacetime.

1.1.4 Covariance

In SR all inertial frames are equally valid; accordingly physical laws should be expressible in a manner such that they are **covariant** under all Lorentz transformations. This simply means that the same physical laws apply in all Lorentz frames. In GR we go one step further: physical laws should remain valid under *all* coordinate transformations – we call this the **principle of general covariance**.

1.1.5 Summation convention

Instead of constantly writing

$$\vec{F} = \sum_i F^i \vec{e}_i \tag{1.1}$$

for the components of a vector, $\vec{F}$, with respect to a particular coordinate basis, we can simply write

$$F^i \vec{e}_i \tag{1.2}$$

Thus, where one encounters repeated indices (upper-lower or lower-upper), this implies summation.

1.2 Manifolds and functions on a manifold

A **manifold** is a continuous space which is locally flat. More generally we can regard a manifold as any set which can be continuously parametrised: the number of independent parameters is the *dimension* of the manifold, and the parameters themselves are the *coordinates* of the manifold. A **differentiable manifold** is one which is both continuous and differentiable. This means that we can define a scalar function (or *scalar field*) – ϕ, say – at each point of the manifold, and that ϕ is differentiable.

A **Riemannian manifold** is a manifold on which a distance function, or metric, is defined. The Minkowski spacetime of special relativity is an example of a Riemannian manifold. If one imagines a curved manifold embedded in a higher dimensional Euclidean space (e.g. a 2-D sphere embedded in $\mathbf{E^3}$) then there is a natural metric which we can adopt, which is just the Euclidean distance function of the higher-dimensional space in which the manifold is embedded.

1.2.1 Scalar functions on a manifold

A **scalar function** is a geometrical object the numerical value of which at each point of the manifold is the same real number, no matter which coordinate representation is used.

1.2.2 Contravariant vectors on a manifold

A **contravariant vector** is a geometrical object the components of which, at each point of the manifold, in different coordinate systems transform **linearly** according to the equation

$$A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu} \tag{1.3}$$

1.2.3 One-Forms on a manifold

A **covariant vector**, **covector**, or (in more modern literature) a **one-form** is a geometrical object the components of which, at each point of the manifold, transform **linearly** according to the equation

$$B'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} B_{\nu} \tag{1.4}$$

1.2.4 General tensors on a manifold

A **tensor** of type (l,m), defined on an n dimensional manifold, is a linear operator which maps l one-forms and m (contravariant) vectors into a real number (i.e. scalar). Such a tensor has a total of n^{l+m} components. The transformation law for a general (l,m) tensor is

$$A'^{u_1\,u_2\,...\,u_l}_{r_1\,r_2\,...\,r_m} = \frac{\partial x'^{u_1}}{\partial x^{t_1}} \cdots \frac{\partial x'^{u_l}}{\partial x^{t_l}} \frac{\partial x^{q_1}}{\partial x'^{r_1}} \cdots \frac{\partial x^{q_m}}{\partial x'^{r_m}} A^{t_1\,t_2\,...\,t_l}_{q_1\,q_2\,...\,q_m} \tag{1.5}$$

A $(2,0)$ tensor, say T^{ij}, is called a **contravariant tensor** of rank 2 and transforms according to the transformation law

$$T'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}T^{kl} \tag{1.6}$$

A $(0,2)$ tensor, say B_{ij}, is called a **covariant tensor** of rank 2, and transforms according to the law

$$B'_{ij} = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}B_{kl} \tag{1.7}$$

A tensor which has both upper and lower indices, which means that it has both contravariant and covariant terms in its transformation law, is known as a **mixed tensor**. The simplest example (after the trivial case of a $(0,0)$ tensor) is a $(1,1)$ tensor, D^i_j, say. Its transformation law is

$$D'^i_j = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^l}{\partial x'^j}D^k_l \tag{1.8}$$

An important example of a $(1,1)$ tensor is the Kronecker delta, δ^i_j.

1.2.5 Contraction of tensors

We can form the **inner product**, or **contraction** of a vector and one-form; i.e. we form the quantity A^iB_i (where, as usual, the summation convention is implied). This quantity can easily be shown to be an invariant or scalar in the sense that

$$A'^{j} B'_{j} = A^{i} B_{i} \tag{1.9}$$

We can generalise the operation of contraction to the case of any two tensors, and over an arbitrary number of indices, provided that an equal number of upper and lower indices are selected. In general, contraction over k indices will produce from a tensor of type (l, m) a new tensor of type $(l-k, m-k)$. For example, the contraction of the two tensors G^{ijk}_{lm} and R^{s}_{tu} over the indices i and t, j and u and l and s will give the $(1,1)$ tensor $G^{ijk}_{lm} R^{l}_{ij}$, where now only the indices k and m are free indices.

1.3 Spacetime and the metric

Spacetime is a 4 dimensional manifold. The points of this manifold are called **events**. We can also define a **distance**, or **interval**, between neighbouring events – i.e. spacetime is a Riemannian manifold. If the interval between these events in **timelike**, we use the interval between them to define the **proper time** recorded by a particle on whose worldline the events lie.

Suppose a coordinate system has been set up in spacetime. Each event, P, is provided with 4 coordinate values, say $\{x^0, x^1, x^2, x^3\}$. These coordinates can be quite general, and are not necessarily assumed to represent time and spatial coordinates. A neighbouring event, Q, has coordinates $\{x^0 + dx^0, x^1 + dx^1, x^2 + dx^2, x^3 + dx^3\}$. We can write the (invariant) interval between these events as

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{1.10}$$

$g_{\mu\nu}$ is called the **metric tensor** (stricly speaking $\mathbf{g}$, in coordinate-free notation, is the metric tensor and $g_{\mu\nu}$ is just the metric tensor components, but this distinction is rarely made in practice).

To say that spacetime is **locally Minkowskian** (sometimes loosely stated as 'locally flat') means that there exists a coordinate transformation that reduces equation (1.10) to the form of the Minkowski metric in the neighbourhood of event P.

1.3.1 Transformation law for the metric

At any event, P, the value of $g_{\mu\nu}$ will depend on the coordinates used. It is easy to see that the metric must, in fact, transform as a $(0, 2)$ tensor, since ds^2 is invariant, and $dx^\mu dx^\nu$ transforms as a $(2, 0)$ tensor. Thus the metric has the transformation law

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \tag{1.11}$$

1.3.2 Contravariant components of the metric tensor

We can also define contravariant components of the metric tensor g^{jk} by requiring

$$g_{ik} g^{jk} = \delta_i^j \tag{1.12}$$

g^{jk} defined in this way must be unique, since g_{ij} is nonsingular. g^{jk} transforms as a $(2, 0)$ tensor, since g_{ik} transforms as a $(0, 2)$ tensor and δ_i^j as a $(1, 1)$ tensor.

1.3.3 Raising and lowering indices and contraction

Given any contravariant vector A^i it is possible to define, via the metric tensor, an associated one-form, which we denote as A_i and which is defined by

$$A_i = g_{ik} A^k \tag{1.13}$$

This operation is often called **lowering the index**.

Similarly by using g^{ij} we can raise the index of a covariant quantity B_i to obtain a contravariant quantity B^i, viz.

$$B^i = g^{ij} B_j \tag{1.14}$$

The process of raising or lowering indices can be carried out with tensors of any rank and type. For example

$$D^{ijk..}_{lm} = g_{lp} g_{mq} D^{ijkpq} \tag{1.15}$$

Some care must be taken in positioning the indices. The dots have been placed here to indicate the indices over which contraction has taken place, although in general we shall omit the dots and just write D^{ijk}_{lm}. Note that $D^{..ijk}_{lm}$ defined by

$$D^{..ijk}_{lm} = g_{lp} g_{mq} D^{pqijk} \tag{1.16}$$

is not the same as $D^{ijk..}_{lm}$ unless D^{ijkpq} possesses some symmetry.

The **magnitude** of a vector A^i is $g_{ij} A^i A^j$, which is of course invariant, since g_{ij} is a $(0, 2)$ tensor and A^i and A^j are both $(1, 0)$ tensors. Notice

$$g_{ij} A^i A^j = A_j A^j = g^{ij} A_i A_j \tag{1.17}$$

$g_{ij} A^i B^j$ may be regarded as the scalar product of two vectors.

1.4 Covariant differentiation

Any dynamical physical theory must deal in time varying quantities, and if this theory is also to be relativistic, spatially varying quantities too. Since GR is a covariant theory, we are confronted with the problem of constructing quantities that represent rates of change, but which can be defined in any coordinate system. In other words, we need to define a *derivative* which transforms covariantly (i.e. as a tensor) under a general coordinate transformation; we call this the **covariant derivative**, denoted by a semi-colon.

1.4.1 Covariant differentiation of a scalar

We define the covariant derivative of a scalar function, ϕ, simply to be equal to the partial derivative of ϕ, i.e.

$$\phi_{;k} = \phi_{,k} \tag{1.18}$$

1.4.2 Covariant differentiation of a one-form

For a $(0,1)$ tensor, or one-form, with components B_i,

$$B_{i;k} = B_{i,k} - \Gamma^j_{ik} B_j \tag{1.19}$$

1.4.3 Covariant differentiation of a vector

For a $(1,0)$ tensor, or contravariant vector, with components A^i,

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{jk} A^j \tag{1.20}$$

1.4.4 Covariant differentiation of tensor

It is straightforward (although tedious) to define covariant differentiation of a tensor of arbitrary rank. For example, the covariant derivative of a $(3,3)$ tensor is given by

$$T^{ijk}_{lmn;p} = T^{ijk}_{lmn,p} + \Gamma^i_{rp} T^{rjk}_{lmn} + \Gamma^j_{rp} T^{irk}_{lmn} + \Gamma^k_{rp} T^{ijr}_{lmn} - \Gamma^r_{lp} T^{ijk}_{rmn} - \Gamma^r_{mp} T^{ijk}_{lrn} - \Gamma^r_{np} T^{ijk}_{lmr} \tag{1.21}$$

1.4.5 Christoffel symbols

For a Riemannian manifold there is a very natural definition of the **Christoffel symbols**, or affine connections, in terms of g_{ij}

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l}) \tag{1.22}$$

1.4.6 Geodesic coordinates

In a Riemannian manifold we can always find a coordinate system in which at a given point not only does g_{ij} reduce to diagonal form (i.e. space is locally Minkowskian), but the first derivates of g_{ij} are also zero, i.e. $g_{ij,k} = 0$. In such a coordinate system the Christoffel symbols are all identically zero. We call such a coordinate system a **geodesic coordinate system**.

1.5 Geodesics

Material particles not acted on by forces other than gravitational forces have worldlines that are **geodesics**. Similarly photons also follow geodesics. One can define a geodesic as an extremal path between two events, in the sense that the proper time along the path joining the two events is an extremum. Equivalently, one can define a **geodesic as a curve along which the tangent vector to the curve is parallel-transported**.

1.5.1 Geodesics of material particles

Consider the worldline of a material particle. In a given coordinate system the worldline may be written with the proper time, τ, along the worldline as the parameter, i.e.

$$x^\mu = x^\mu(\tau) \tag{1.23}$$

The four velocity of the particle is given by

$$v^\mu = \frac{dx^\mu}{d\tau} \tag{1.24}$$

and is the tangent vector to the worldline. Thus, we obtain the geodesic equation with τ as parameter, i.e.

$$\frac{dv^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} v^\alpha v^\beta = 0 \tag{1.25}$$

or alternatively

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{1.26}$$

One can show (see Example Sheet I-5) that, along a geodesic

$$\frac{d}{d\tau}(g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}) = 0 \tag{1.27}$$

and, in fact, if τ is the proper time then

$$g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = -1 \tag{1.28}$$

1.5.2 Geodesics of photons

For photons, the proper time τ cannot be used to parametrise the worldlines, since $d\tau$ is zero. If we use an arbitrary affine parameter λ the null geodesics will be described by

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{1.29}$$

and since it is a null geodesic, necessarily

$$g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0 \tag{1.30}$$

1.6 The energy momentum tensor

The energy momentum tensor (also known as the stress energy tensor) describes the presence and motion of gravitating matter. In GR-I and GR-II we discuss it for the particular case of a 'perfect fluid', which is a mathematical idealisation but one which is a good approximate description of the gravitating matter in many astrophysical situations.

1.6.1 Perfect fluids

The simplest type of relativistic fluid is known as 'dust'. To a physicist, a fluid element of dust means a collection of particles which are all at rest with respect to some Lorentz frame. Many textbooks (including Schutz) refer to this Lorentz frame as the **momentarily comoving rest frame** (MCRF) of the fluid element.

Generally the particles within a fluid element *will* have random motions, and these will give rise to **pressure** in the fluid (c.f. motions of the molecules in an ideal gas). A fluid element may also be able to exchange energy with its neighbours via **heat conduction**, and there may be **viscous forces** present between neighbouring fluid elements. When viscous forces exist they are directed parallel to the interface between neighbouring fluid elements, and result in a **shearing** of the fluid.

A relativistic fluid element is said to be a **perfect fluid** if, in its MCRF, the fluid element has no heat conduction or viscous forces. It follows from this definition that dust is the special case of a pressure-free perfect fluid.

1.6.2 Definition of the energy momentum tensor

We can define the energy momentum tensor, **T**, in terms of its components in some coordinate system, $\{x^1, x^2, ..., x^n\}$, for each fluid element. Thus we define $T^{\alpha\beta}$ for a fluid element to be equal to the **flux of the α component of four momentum of all gravitating matter across a surface of constant x^β.**

1.6.3 Symmetry of the energy momentum tensor

An important property of the energy momentum tensor which we will state here, but not prove, is that **T** is **symmetric** – i.e. in any coordinate system $T^{\alpha\beta} = T^{\beta\alpha}$, $(\alpha, \beta = 0, 1, 2, 3)$. (See Example Sheet I-3 for a proof that the symmetry of a tensor must hold under any coordinate transformation). To see a proof that the energy momentum tensor is symmetric, in the MCRF of a fluid element and hence in any frame, refer to e.g. Green Schutz, p.102 et. seq.

1.6.4 Energy momentum tensor for a perfect fluid

Components of T in the MCRF for dust

In this case the energy momentum tensor takes a very simple form. The only non-zero component is $T^{00} = \rho$, the energy density of the fluid element.

Components of T in the MCRF for a general perfect fluid

This case is only slightly less straightforward than that of dust. We can write **T** as

$$\mathbf{T} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{1.31}$$

where P is the pressure of the fluid element.

Components of T in a general Lorentz frame

If $\vec{u} = \{u^\alpha\}$ is the *four* velocity of the fluid element, then equation (1.31) further generalises to

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + P\eta^{\alpha\beta} \tag{1.32}$$

Here $\eta^{\alpha\beta}$ are the contravariant components of the metric tensor for Minkowski spacetime.

1.6.5 Conservation of Energy and Momentum

Conservation of momentum and energy requires that

$$T^{\mu\nu}_{\ ,\nu} = 0 \tag{1.33}$$

Thus, the **divergence** of the energy momentum tensor is equal to zero.

1.6.6 Extending to GR: 'Comma goes to semi-colon' rule

In the light of the properties of tensors, an immediate consequence of the strong principle of equivalence is as follows

Any physical law which can be expressed as a tensor equation in SR has exactly the same form in a local inertial frame of a curved spacetime

This statement holds since, in the LIF, physics – and hence the form of physical laws – is indistinguishable from the physics of SR. This is a very important result because it allows us to generalise the form of physical laws which are valid in SR to the case of GR, with semi-colons (denoting covariant derivatives) replacing commas (denoting partial derivatives) where appropriate.

Hence, the energy momentum tensor for a perfect fluid in GR takes the form

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu} \tag{1.34}$$

where $g^{\mu\nu}$ denotes the contravariant metric tensor for a general curved spacetime (which of course reduces locally to $\eta^{\mu\nu}$).

We can extend to GR in this way the result of equation (1.33), on the conservation of energy and momentum. Thus, for a fluid element in a general curved spacetime

$$T^{\mu\nu}_{\ ;\nu} = 0 \tag{1.35}$$

If this were *not* the case – i.e. if there existed some point, P, at which $T^{\mu\nu}_{\ ;\nu} \neq 0$ – then we could construct a LIF at P (e.g. by changing to geodesic coordinates) in which all Christoffel symbols are zero. In this new frame covariant derivatives reduce to partial derivatives, implying that $T^{\mu\nu}_{\ ,\nu} \neq 0$, which contradicts equation (1.33).

The general technique of using the principles of covariance and equivalence to extend the validity of tensor equations from SR to GR, usually by evaluating their components in the LIF where Christoffel symbols vanish, is a very powerful one and is commonly met in the relativity literature. It is sometimes referred to informally as the '**comma goes to semi colon rule**'.

1.7 The Riemann Christoffel tensor

The curvature of a metric space is described by the **Riemann Christoffel tensor** (often also referred to simply as the **Riemann tensor**), defined by

$$R^{\mu}{}_{\alpha\beta\gamma} = \Gamma^{\sigma}_{\alpha\gamma}\Gamma^{\mu}_{\sigma\beta} - \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\mu}_{\sigma\gamma} + \Gamma^{\mu}_{\alpha\gamma,\beta} - \Gamma^{\mu}_{\alpha\beta,\gamma} \tag{1.36}$$

Notice that if the spacetime is flat then

$$R^{\mu}{}_{\alpha\beta\gamma} = 0 \tag{1.37}$$

i.e. all components of the Riemann Christoffel tensor are identically zero.

1.7.1 Fully covariant form of the Riemann Christoffel tensor

We may obtain the components of the $(0, 4)$ tensor corresponding to $R^{\mu}{}_{\alpha\beta\gamma}$ by using the covariant components of the metric tensor to lower the index, viz.

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\sigma}R^{\sigma}{}_{\beta\gamma\delta} \tag{1.38}$$

$R_{\alpha\beta\gamma\delta}$ has the following important symmetries, which reduce the number of independent components in $4d$ to 20.

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \tag{1.39}$$

$$R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\gamma\delta} \tag{1.40}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0 \tag{1.41}$$

1.7.2 The Bianchi identities

The Bianchi identities, for the derivatives of the Riemann Christoffel tensor, state that

$$R_{\alpha\beta\gamma\delta;\lambda} + R_{\alpha\beta\lambda\gamma;\delta} + R_{\alpha\beta\delta\lambda;\gamma} = 0 \tag{1.42}$$

Note that the three terms cyclically permute the final three indices.

1.8 Einstein's equations

The Riemann Christoffel tensor, $R^{\mu}{}_{\alpha\beta\gamma}$, describes the curvature of spacetime. Einstein's equations relate this curvature to the matter and energy content of the Universe, as described by the energy momentum tensor, $\mathbf{T}$. But $\mathbf{T}$ is a tensor of rank $(2, 0)$. Thus, Einstein's equations involve various *contractions* of the Riemann Christoffel tensor.

1.8.1 Ricci tensor

We can contract the Riemann Christoffel tensor to form a $(0, 2)$ tensor, which we call the **Ricci tensor** defined by

$$R_{\alpha\gamma} = R^{\mu}_{\alpha\mu\gamma} \tag{1.43}$$

i.e. contracting on the *second* of the lower indices. (N.B. some authors choose to define $R_{\alpha\gamma}$ as minus this value). From equation (1.38) it follows that we can also write the components of the Ricci tensor as

$$R_{\alpha\gamma} = g^{\sigma\delta} R_{\sigma\alpha\delta\gamma} \tag{1.44}$$

It is easy to show (see Example Sheet I-6) that $R_{\alpha\beta} = R_{\beta\alpha}$, i.e. the Ricci tensor is symmetric.

1.8.2 Curvature Scalar, R, and contravariant Ricci tensor

By further contracting the Ricci tensor with the contravariant components of the metric, one obtains the **curvature scalar**, viz:

$$R = g^{\alpha\beta} R_{\alpha\beta} \tag{1.45}$$

One may also use the metric to raise the indices of the Ricci tensor, and thus express it in contravariant form, viz:

$$R^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta} \tag{1.46}$$

$R^{\mu\nu}$ is also symmetric.

1.8.3 The Einstein tensor

Using the contravariant form of the Ricci tensor, we define the **Einstein tensor**, **G**, viz:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{1.47}$$

where R is the curvature scalar. Note that since $R^{\mu\nu}$ is symmetric, so too is $G^{\mu\nu}$.

The Einstein tensor is of crucial physical significance in general relativity, since it can be shown from the Bianchi identities that

$$G^{\mu\nu}{}_{;\nu} = 0 \tag{1.48}$$

1.8.4 Einstein's equations

Conservation of energy and momentum implies that

$$T^{\mu\nu}_{;\nu} = 0 \tag{1.49}$$

From the Bianchi identities

$$G^{\mu\nu}_{;\nu} = 0 \tag{1.50}$$

Thus we have automatically that

$$T^{\mu\nu}_{;\nu} = G^{\mu\nu}_{;\nu} \tag{1.51}$$

These are Einstein's equations, and Einstein took as their solution

$$G^{\mu\nu} = kT^{\mu\nu} \tag{1.52}$$

i.e. the Einstein tensor is equal to the energy momentum tensor, multiplied by a constant, k.

Thus, the geometry of spacetime (described by the Einstein tensor) and the mass-energy content of spacetime (described by the energy momentum tensor) are fundamentally interlinked.

Note that there are 10 independent equations incorporated in equation (1.52), since $G^{\mu\nu}$ and $T^{\mu\nu}$ are symmetric.

We determine the constant k by requiring that we should recover the laws of Newtonian gravity and dynamics in the limit of a weak gravitational field and non-relativistic motion.