

Astronomy A3/A4M, Physics P4M

Gravitation and Relativity II: Example Sheet 2

1. Assuming the equation of motion of a test mass orbiting a star in the Schwarzschild form given by equation (3.11) of your notes:-

$$\left(\frac{dr}{d\tau}\right)^2 = k^2 - 1 - \frac{h^2}{r^2} + \frac{2M}{r} \left(1 + \frac{h^2}{r^2}\right)$$

show that the values of h and k for a circular orbit of coordinate radius a are given by

$$h^2 = \frac{Ma}{\left(1 - \frac{3M}{a}\right)}$$

and

$$k^2 = \frac{\left(1 - \frac{2M}{a}\right)^2}{\left(1 - \frac{3M}{a}\right)}$$

Why are there no circular orbits for $a < 3M$?

2. Using equation (3.52), show that a photon could be moving in a circular orbit about a Schwarzschild black hole (i.e. a static black hole with exterior metric given by the Schwarzschild metric). Find the coordinate radius of the orbit, and the corresponding value of the angular momentum, h .
3. Verify that equation (4.26) of your notes may be re-written in the form of equation (4.29), i.e.

$$\frac{d}{dr} [r(1 - e^{-\lambda})] = 8\pi\rho r^2$$

4. Using equation (4.44) to eliminate ν' from equation (4.35), derive the Oppenheimer-Volkoff equation, i.e.

$$\frac{dP}{dr} = -\frac{(\rho + P)(4\pi Pr^3 + m)}{r(r - 2m)}$$

and show that this equation reduces to the classical Newtonian equation of hydrostatic equilibrium in the weak-field limit.

5. Verify, using the method of partial fractions or otherwise, that equation (4.51)

$$\frac{dP}{(\rho_0 + P)(\rho_0 + 3P)} = -\frac{4\pi r dr}{3\left(1 - \frac{8\pi\rho_0 r^2}{3}\right)}$$

reduces to equation (4.52)

$$\frac{1}{2\rho_0} \left[\frac{3dP}{(\rho_0 + 3P)} - \frac{dP}{(\rho_0 + P)} \right] = -\frac{4\pi}{3} \frac{r dr}{\left(1 - \frac{8\pi\rho_0 r^2}{3}\right)}$$

Integrating both sides, and applying the boundary conditions that $P = P_0$ for $r = 0$, and $P = 0$ for $r = R$, show that the central pressure for a spherically symmetric star of constant density is given by

$$P_0 = \frac{\rho_0 \left[1 - \left(1 - \frac{2M}{R}\right)^{1/2}\right]}{3\left(1 - \frac{2M}{R}\right)^{1/2} - 1}$$

Show also that

$$P_0 \rightarrow \infty \quad \text{when} \quad \frac{M}{R} \rightarrow \frac{4}{9}$$

6. Prove that, if we are considering *Background Lorentz Transformations* – i.e. transformations for which the weak-field metric perturbations, $h_{\mu\nu}$, transform as if they are a $(0, 2)$ tensor on a Background Minkowski spacetime, then the linearised Riemann-Christoffel tensor reduces to the form of (equation 5.22)

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2} (h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma})$$

7. Show, further, that if we carry out a gauge transformation of the weak-field metric perturbations

$$h'_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$$

where

$$|\xi^\alpha{}_{,\beta}| \ll 1 \quad \text{for all } \alpha, \beta$$

then to first order the linearised Riemann-Christoffel tensor in this new coordinate system is unchanged, i.e.

$$R_{\alpha\beta\gamma\delta} = R'_{\alpha\beta\gamma\delta} = \frac{1}{2} (h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma})$$

8. Consider the hypersurface

$$t^2 - x^2 - y^2 - z^2 = T^2$$

in Minkowski spacetime, where T is a constant. Show that the interval between two neighbouring points on this hypersurface is spacelike. Show further that all inertial observers that were present at the spatial origin at time $t = 0$ will cross this hypersurface a proper time, T , later – regardless of their velocity.

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