Astronomy A3/A4H

Statistical Astronomy I: Supplementary Handout

Proof of Poisson variable distribution function (Non-examinable)

Recall the three postulates which define a Poisson RV with pdf $p(r,t) \;=\; \frac{(\mu t)^r}{r!}\, e^{-\mu t}$:-

1. The probability of an event occuring in time interval, t, is independent of the past history of events prior to t
2. For small interval, δt, there is an intrinsic rate, μ (> 0) such that the probability of a single event in δt, $p(1, \delta t) = \mu\delta t + o(\delta t)$
3. The probability of two or more events happening at the same time is zero, i.e. $p(r, \delta t) = o(\delta t)$, for all $r \geq 2$.

(Here $o(\delta t)$ represents any function such that $o(\delta t)/\delta t \to 0$ as $\delta t \to 0$)

From postulate (1)

$$p(0, t + \delta t) = p(0, t)p(0, \delta t)$$

We can also write $p(0, \delta t)$ as

$$\begin{aligned} p(0, \delta t) &= 1 - \sum_{i=1}^{\infty} p(i, \delta t) \\ &= 1 - \mu\delta t - o(\delta t) \end{aligned}$$

(which follows from postulate (3) since $p(i, \delta t) = o(\delta t)$ for $i \geq 2$)

Hence, we may write

$$\begin{aligned} p(0, t + \delta t) &= p(0, t)\,[1 - \mu\delta t - o(\delta t)] \\ \Leftrightarrow \quad \frac{p(0, t + \delta t) - p(0, \delta t)}{\delta t} &= -\mu\, p(0, t) - \frac{o(\delta t)}{\delta t}\, p(0, t) \\ \Leftrightarrow \quad \frac{d\,p(0, t)}{dt} &= -\mu\, p(0, t) \end{aligned}$$

in the limit as $\delta t \to 0$. Solving this differential equation we obtain

$$p(0, t) = Ae^{-\mu t}$$

for some constant A. Since $p(0, 0) = 1$, it follows that $A = 1$.

Consider now $p(r, t + \delta t)$ where $r \geq 1$. Postulates (1) and (3) imply that (in the limit as $\delta t \to 0$)

$$\begin{aligned} p(r, t + \delta t) &= p(r, t)\, p(0, \delta t) + p(r-1, t)\, p(1, \delta t) + o(\delta t) \\ \Leftrightarrow \quad \frac{p(r, t + \delta t) - p(r, \delta t)}{\delta t} &= -\mu\, p(r, t) + \mu\, p(r-1, t) \\ \Leftrightarrow \quad \frac{d\,p(r, t)}{dt} &= -\mu\, p(r, t) + \mu\, p(r-1, t) \end{aligned}$$

We show by induction that $p(r,t) \;=\; \frac{(\mu t)^r}{r!}\, e^{-\mu t}$ is a solution to this equation, for all r. Consider $r = 1$.

$$\frac{dp(1,t)}{dt} = \mu e^{-\mu t} - \mu t e^{-\mu t}$$
$$= \mu p(0,t) - \mu p(1,t)$$

Hence, the assumed functional form of $p(r,t)$ is a solution to the above equation for $r = 1$. Suppose now that $p(s,t) = \frac{(\mu t)^s}{s!} e^{-\mu t}$ is a solution to the equation, for some $s \geq 1$. Consider now the case where $r = s + 1$.

$$\frac{dp(s+1,t)}{dt} = \frac{(s+1)\mu^{s+1}t^s e^{-\mu t}}{(s+1)!} - \frac{\mu^{s+2}t^{s+1}e^{-\mu t}}{(s+1)!}$$
$$= \mu p(s,t) - \mu p(s+1,t)$$

Hence $p(r,t)$ is also a solution for $r = s + 1$. $p(r,t)$ thus satisfies the Poisson postulates for all r, and the proof is complete.

Proof of approximate expression for var$[f(X)]$ (Non-examinable)

$$\text{var}[f(X)] = \text{var}(X) \left(\frac{\partial f}{\partial x}\right)^2_{x=\overline{x}}$$

Proof

We expand $f(x)$ about the mean value of X, $\overline{x}$. Thus, to second order

$$f(X) = f(\overline{x}) + (X - \overline{x}) \left(\frac{\partial f}{\partial x}\right)_{\overline{x}} + \frac{1}{2}(X - \overline{x})^2 \left(\frac{\partial^2 f}{\partial x^2}\right)_{\overline{x}}$$

Hence, $E[f(X)]$ is given to second order by

$$E[f(X)] = f(\overline{x}) + \frac{1}{2}\text{var}(X) \left(\frac{\partial^2 f}{\partial x^2}\right)_{\overline{x}}$$

(since $E(X) = \overline{x}$). Thus

$$f(X) - E[f(X)] = (X - \overline{x}) \left(\frac{\partial f}{\partial x}\right)_{\overline{x}} + \frac{1}{2}[(X - \overline{x})^2 - \sigma^2] \left(\frac{\partial^2 f}{\partial x^2}\right)_{\overline{x}}$$

(where $\sigma^2 = \text{var}(X)$). It then follows easily that

$$\text{var}[f(X)] = E[(X - \overline{x})^2] \left(\frac{\partial f}{\partial x}\right)^2_{\overline{x}}$$
$$= \text{var}(X) \left(\frac{\partial f}{\partial x}\right)^2_{\overline{x}}$$

as all other terms are of higher than second order.