Bivariate Normal Distribution : Examples

Consider $p(x,y)$ with $\mu_x = \mu_y = 0$, $\sigma_x = \sigma_y = 1$

$$p(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)} \left(x^2 - 2\rho xy + y^2 \right) \right]$$

Contours of constant $p(x,y)$ (isoprobability contours) are ELLIPSES

Correlation coefficient determines the ECCENTRICITY of these ellipses

$\rho = 0 \Rightarrow$ contours are CIRCULAR (x, y independent)

(In this case $p(x,y) = p(x)\,p(y)$)

In general, however, for $\rho \neq 0$ distribution of y depends on observed value of x; and vice versa

$E(y|x)$ is a STRAIGHT LINE of slope ρ

Generally the equation of the straight line is $y = \mu_y + \frac{\sigma_y}{\sigma_x}\rho(x - \mu_x)$

As $|\rho| \to 1$, the isoprobability contours become longer and thinner. This means that the CONDITIONAL pdf $p(y|x)$ has a smaller variance. In fact $\sigma^2_{y|x} = \sigma^2_y (1-\rho^2)$

Thus, as $|\rho| \to 1$, the value of y is increasingly tightly constrained by the value of x. i.e. y and x are increasingly CORRELATED.

We call the line $y = \mu_y + \frac{\sigma_y}{\sigma_x}\rho(x - \mu_x)$ the regression line of Y on X. We expect real data drawn from a bivariate normal pdf to be scattered around this line.

(Analogously, we could consider the conditional pdf $p(x|y)$ and the regression line of X on Y)