Astronomy A3/A4H

Statistical Astronomy I: Supplementary Handout

Example of Bivariate Normal Distribution

We consider here a simple example of a bivariate normal pdf, with only one free parameter, ρ, i.e. we assume $\mu_{\rm x} = \mu_{\rm y} = 0$ and $\sigma_{\rm x}^2 = \sigma_{\rm y}^2 = 1$. Thus, $p(x, y)$ is given by

$$p(x,y) \quad = \quad \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \left(x^2 - 2\rho xy + y^2 \right) \right]$$

Contours of constant $p(x, y)$, known as **isoprobability contours**, are thus seen to be **ELLIPSES**. The correlation coefficient, ρ, determines the **eccentricity** of these ellipses. When $\rho = 0$, contours are **circular** – x and y are independent. (In this case $p(x, y) = p(x)p(y)$). In general, however, for $\rho \neq 0$ the distribution of Y depends on the observed value of x, and vice versa. As $|\rho| \to 1$, the isoprobability contours become longer and thinner. This means that the **conditional pdf**, $p(y|x)$, has a smaller variance. In fact, it can be shown that

$$\sigma_{y|x}^2 = \sigma_y^2 (1 - \rho^2)$$

Thus, as $|\rho| \to 1$, $\sigma_{y|x}^2 \to 0$, i.e. the value of y is increasingly tightly constrained by the observed value of x – i.e. x and y are increasingly **correlated**. Figure 1 shows isoprobability contour plots of $p(x, y)$ for different values of ρ. Also shown is the **regression line** of Y on X for each plot; this line shows the **conditional expectation** value of Y, given the observed value of x.

Figure 1: Isoprobability Contours and Regression Lines for Example Bivariate Normal Distribution Functions