20. Direct and Retrograde Motion
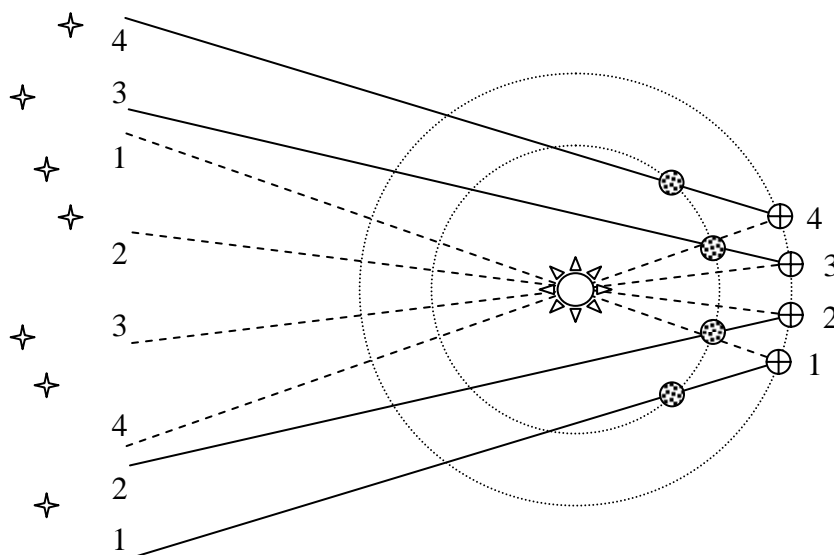
When the ecliptic longitude λ of an object increases with time, its apparent motion is said to be **direct**. When λ decreases with time, its apparent motion is said to be **retrograde**.

Since the simplified solar motion has the Sun move uniformly around the ecliptic once per year, the motion of the Sun is always **direct**. The planets (and other solar system objects) have apparent motion which is sometimes direct and at other times retrograde.
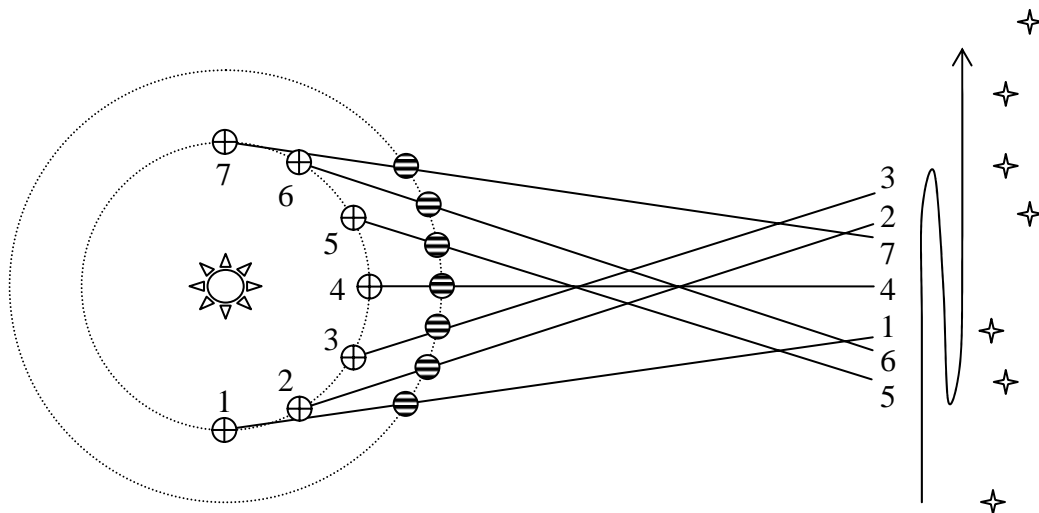
Inferior planets:

Consider a planet closer to the Sun than is the Earth. When the planet is on the opposite side of the Sun to the Earth (i.e. around its superior conjunction), both the orbital motion of the planet and of the Earth act in the same sense relative to the distant background stars. Therefore, the motion of the planet is *direct*.

However, when the planet is on the same side of the Sun as the Earth (i.e. around its inferior conjunction), since the planet's orbital velocity is greater than the Earth's it 'overtakes' the Earth on the inside. The planet's orbital velocity 'cancels out' (and more) that of the Earth, making its apparent motion relative to the distant stars be *retrograde* in character.
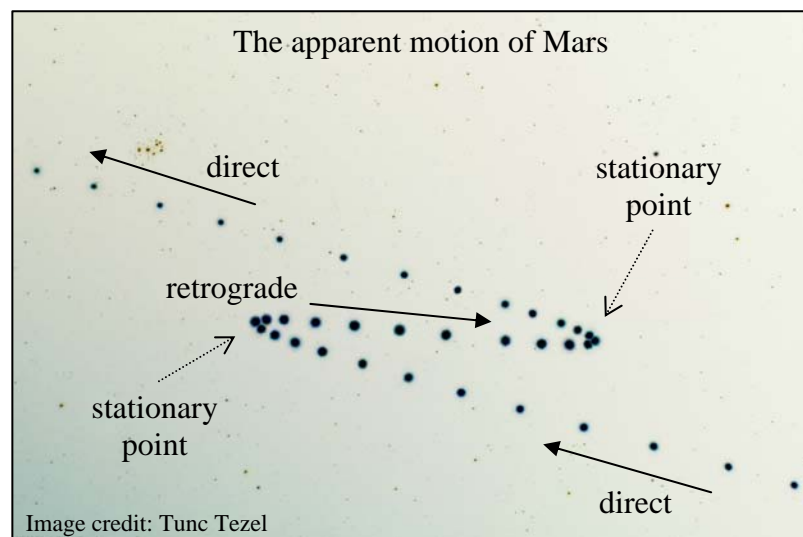
Superior planets:

Consider now a planet further away from the Sun than is the Earth. For most of its synodic period, this planet's apparent motion will be *direct*. However, near to its time of opposition, the faster-orbiting Earth will 'overtake' it on the inside, leading to a period of *retrograde* motion for the superior planet.
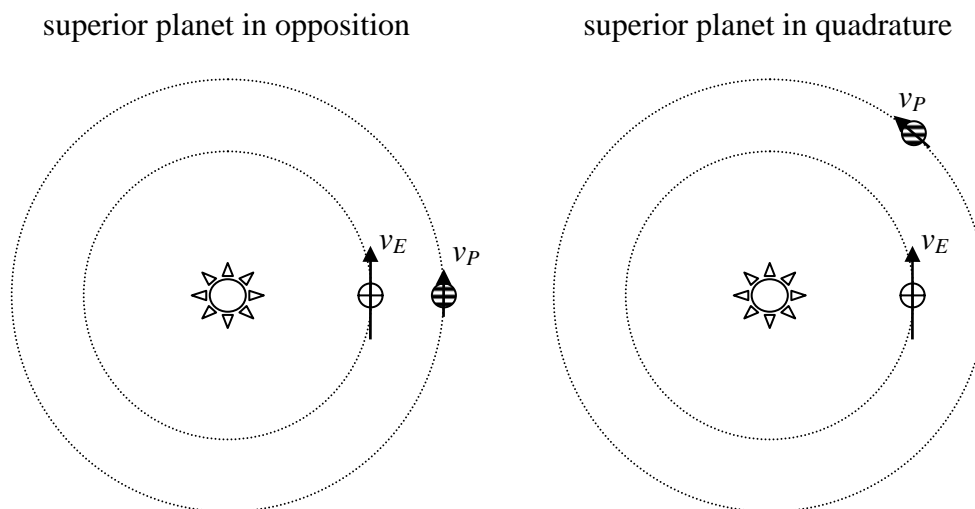

This is seen as a *retrograde loop* in the planet's apparent motion in the sky: the planet's motion is initially direct, then it slows to a halt at a **stationary point**, the planet's direction of motion reverses, it slows to another halt at its second stationary point, and then it proceeds onwards in direct motion again.

At the stationary points, $\mathrm{d}\lambda / \mathrm{d}t = 0$

The apparent motion of Mars

direct

stationary point

retrograde

stationary point

direct

Image credit: Tunc Tezel

21. Elongation of the Stationary Points

We have seen that when a superior planet is at opposition, with an elongation of $\eta = 180°$, its motion is retrograde: the planet's orbital velocity vector lies parallel to the Earth's orbital velocity vector, and since the Earth's orbital velocity v_E is greater than the superior planet's orbital velocity v_P, the Earth 'overtakes' the superior planet at *opposition*, giving it an apparent *retrograde* motion relative to the background stars.
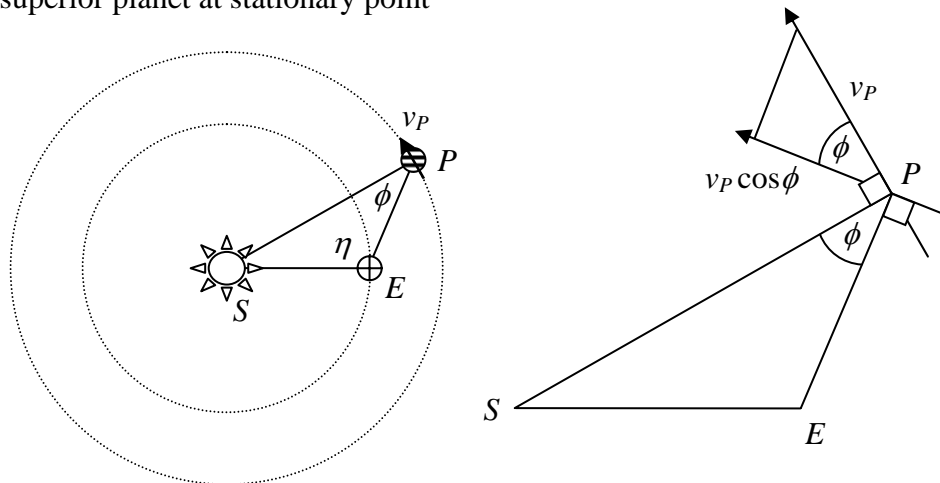
superior planet in opposition   superior planet in quadrature

When a superior planet is at quadrature, with an elongation of $\eta = 90°$, the Earth's velocity vector points straight at the planet. Thus, the Earth's orbital motion makes no contribution to the apparent movement of the superior planet against the background stars, as seen from Earth. Since we see only the planet's own orbital motion, at *quadrature* the superior planet's motion is *direct*.

At some elongation between 90° and 180°, the planet must switch over from having direct motion to having retrograde motion. This occurs at the stationary point, which therefore itself must have an elongation lying between 90° and 180°. To find the elongation of the stationary point, we must determine where the orbital velocity of the planet and the orbital velocity of the Earth combine to give zero apparent movement of the planet relative to the background stars.

Suppose a superior planet is at one of its stationary points. Suppose also that the Earth is perfectly motionless. We observe the planet along the Earth-planet line *EP*. We do not discern any motion along this line of sight, but motion perpendicular to this line of sight will be seen as an apparent shift of the position of the planet relative to the background stars.
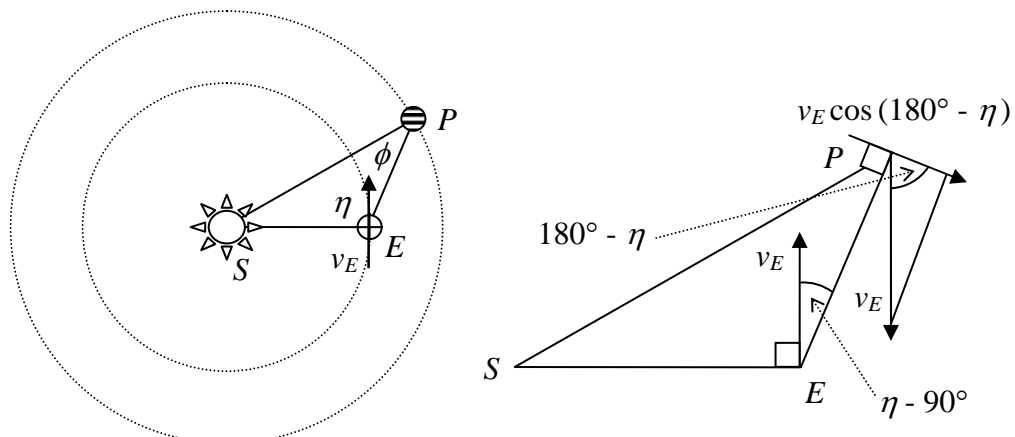

superior planet at stationary point

Thus, the planet's apparent motion due to its own orbital velocity is given by the transverse component of this velocity, $v_P \cos\phi$.

Suppose instead that the superior planet is perfectly motionless, so that all apparent motion of the planet is due only to the Earth's orbital motion. The planet's apparent motion due to Earth's orbital velocity is given by the transverse component of this velocity, $v_E \cos(180° - \eta) = -v_E \cos\eta$ (see Section 10).

superior planet at stationary point

The transverse components of the planet's orbital velocity and the Earth's orbital velocity act in opposing senses. Therefore, they will exactly cancel each other out (giving no apparent motion of the superior planet – a stationary point) when their magnitudes are equal:

$$v_P \cos\phi = v_E \cos(180° - \eta) = -v_E \cos\eta$$

The orbital velocity is given by $v = \omega r$ where ω is the planet's orbital angular velocity and r is its orbital radius. Therefore (by equation 17.3 and subsequent)

$$v_P = \omega_P a = \frac{2\pi}{T_P} a \quad \text{and} \quad v_E = \omega_E \times 1 = \frac{2\pi}{T_E} = 2\pi$$

where a is the superior planet's orbital radius in AU and T_P is its orbital period in years. Therefore

$$\frac{2\pi}{T_P} a \cos\phi = -2\pi\cos\eta$$

Since we are working in AU and years, by Kepler's 3rd Law (see Dynamical Astronomy), $a^3 = T_P^2$, so

$$2\pi \frac{a}{a^{3/2}} \cos\phi = -2\pi\cos\eta$$

$$\frac{\cos\phi}{a^{1/2}} = -\cos\eta$$

$$\cos\phi = -a^{1/2}\cos\eta$$

$$\cos^2\phi = a\cos^2\eta$$

$$1 - \sin^2\phi = a\cos^2\eta$$

where we use the identity $\cos^2 x + \sin^2 x = 1$.

We wish to obtain η by eliminating ϕ. From equation (17.1),

$$\sin\eta = a\sin\phi$$

Therefore

$$1 - \sin^2 \phi = 1 - \frac{\sin^2 \eta}{a^2} = a \cos^2 \eta$$

Dividing all terms by $\cos^2 \eta$ we have

$$\sec^2 \eta - \frac{\tan^2 \eta}{a^2} = a$$

where $\sec x = 1 / \cos x$ and $\tan x = \sin x / \cos x$.

Now, using the identity $\sec^2 x = 1 + \tan^2 x$, we isolate $\tan \eta$:

$$1 + \tan^2 \eta - \frac{\tan^2 \eta}{a^2} = a$$

$$\left(1 - \frac{1}{a^2} \right) \tan^2 \eta = \left(\frac{a^2 - 1}{a^2} \right) \tan^2 \eta = a - 1$$

$$\tan^2 \eta = (a - 1) \left(\frac{a^2}{a^2 - 1} \right) = \frac{(a - 1)a^2}{(a - 1)(a + 1)} = \frac{a^2}{a + 1}$$

We know that η must lie between 90° and 180°, i.e. in the 2nd quadrant (see page 36), so $\tan \eta$ is negative:

$$\tan \eta = \frac{-a}{\sqrt{a + 1}}$$

Also, since η lies in the 2nd quadrant, η will be 180° minus the principal angle (calculated using the modulus of $\tan \eta$), so

$$\eta = 180° - \arctan\left(\frac{a}{\sqrt{a + 1}} \right) \qquad [21.1]$$

Thus, the elongation of the stationary points of a superior planet depend only on the planet's orbital radius a.