Complex integration and real definite integrals

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0.1 Analytic function

A complex function is said to be *analytic* on a region D if it is complex differentiable at every point in D. Many mathematicians prefer the term *holomorphic function* (or "holomorphic map") to *analytic function*, while "analytic" appears to be in widespread use among physicists, engineers, and in some older texts.

Let's assume we have a complex function

$$f(x, y) = u(x, y) + iv(x, y) \tag{1}$$

of a complex variable

$$z = x + iy \tag{2}$$

If we want the function (1) to be differentiable i.e. to have a derivative $df(z)/dz$ then the functions $u(x, y)$ and $v(x, y)$ should follow *Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{3}$$

Note: The reverse statement is also correct.

0.2 Cauchy Integral Theorem

If the function of complex variable is holomorphic in the whole region D then the integral

$$\oint\limits_{\partial D} f(z)\mathrm{d}z = 0 \tag{4}$$

where ∂D is any closed contour completely contained in D.

Indeed,

$$\oint\limits_{\partial D} f(z)\mathrm{d}z = \oint\limits_{\partial D} (u(x,y) + iv(x,y))(dx + idy) = \tag{5}$$

where using Green's theorem we have

$$= -\int\int\limits_{D} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy + i\int\int\limits_{D} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \tag{6}$$

Applying Cauchy-Reimann equations we prove the theorem (4).

0.3 Residue Theorem

If there are *singular points* $z = z_j$ in region D, where complex function $f(z)$ is **not** analytic, then the integral

$$\oint_{\partial D} f(z)\mathrm{d}z = 2\pi i \sum_{j=1}^{N} \mathrm{res}(f(z), z_j) \tag{7}$$

where a residue $\mathrm{Res}(f(z), z_j)$ is the coefficient a_{-1} of the *Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_j)^n \tag{8}$$

For example

$$\mathrm{Res}\left(\frac{1}{z}, z = 0\right) = 1, \quad \mathrm{Res}\left(\frac{1}{z^3}, z = 0\right) = 0 \tag{9}$$

0.4 Application to definite integrals I

Let us consider integral

$$I = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} \tag{10}$$

We can consider integral (12) as a part of a complex integral

$$\oint \frac{\mathrm{d}z}{1+z^2} = I + \int\limits_{R\to\infty} \frac{\mathrm{d}z}{1+z^2} = I + 0 \tag{11}$$

where the second integral in (12) is along counterclockwise contour. On the other hand, the integral over closed contour in complex plain can be calculated using *residue theorem*

$$\oint \frac{\mathrm{d}z}{1+z^2} = 2\pi i \mathrm{Res}\left(\frac{1}{(z-i)(z+i)}, z=i\right) = \pi \tag{12}$$

That immediately gives us $I = \pi$.

0.5 Application to definite integrals II

The use of the residues of a complex function gives a way to evaluate many definite integrals, including what seem to be impossible to evaluate

$$I = \int\limits_{-\infty}^{\infty} \frac{\sin(x)}{x} \mathrm{d}x \tag{13}$$

Using that $sin(x) = \operatorname{Im} \exp(ix)$ we construct the integral in complex plain

$$\oint \frac{\exp(iz)}{z} \mathrm{d}z = \int\limits_{-\infty}^{\infty} \frac{\exp(iz)}{z} \mathrm{d}z + \int\limits_{R \to \infty} \frac{\exp(iz)}{z} \mathrm{d}z \tag{14}$$

The second integral is zero due to *Jordan's lemma*. On the other hand integral (14) is a half of a residue at $z = 0$

$$\oint \frac{\exp(iz)}{z} \mathrm{d}z = \frac{1}{2} 2\pi i \operatorname{Res} \left(\frac{\exp(iz)}{z}, z = 0 \right) = i\pi \tag{15}$$

Taking the imaginary part of (15) we have $I = \pi$.