Kinetics of waves and particles in plasmas

II. Wave-wave interactions in plasma

March 17, 2003

0.1 Waves as quasi-particles

When waves are not intense we can consider plasma turbulence as a gas *quasi-particles*. Similar to particles we introduce the number of waves of sort σ with a given $\mathbf{k}$.

The number of waves can be conveniently defined as

$$N(\mathbf{k}^{\sigma}) = \frac{W(\mathbf{k})^{\sigma}}{\hbar\omega^{\sigma}} \tag{1}$$

Let us start with three-waves interaction

$$\sigma' \leftrightarrow \sigma'' + \sigma \tag{2}$$

where σ', σ'', and σ are different sorts of waves. The corresponding conservation laws for momentum and energy require

$$\mathbf{k}^{'} = \mathbf{k}'' + \mathbf{k}, \quad \omega(\mathbf{k})_{\sigma'} = \omega(\mathbf{k})_{\sigma''} + \omega(\mathbf{k})_{\sigma} \tag{3}$$

where $\omega(\mathbf{k})$ is given by the dispersion relation for waves.

0.2 Kinetic equations for waves

Introducing the number of waves for each sort of waves, the kinetic equations can be constructed

$$\frac{\mathrm{d}N_{\mathbf{k}'}^{\sigma'}}{\mathrm{d}t} = \int w_{\sigma',\mathbf{k}'}^{\sigma'',\sigma} \left[N_{\mathbf{k}}^{\sigma} N_{\mathbf{k}'-\mathbf{k}}^{\sigma''} - N_{\mathbf{k}}^{\sigma} N_{\mathbf{k}'}^{\sigma'} - N_{\mathbf{k}'-\mathbf{k}}^{\sigma''} N_{\mathbf{k}'}^{\sigma'}\right] \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \tag{4}$$

$$\frac{\mathrm{d}N_{\mathbf{k}}^{\sigma}}{\mathrm{d}t} = \int w_{\sigma',\mathbf{k}'}^{\sigma'',\sigma} \left[N_{\mathbf{k}}^{\sigma} N_{\mathbf{k}'}^{\sigma'} + N_{\mathbf{k}'-\mathbf{k}}^{\sigma''} N_{\mathbf{k}'}^{\sigma'} - N_{\mathbf{k}}^{\sigma} N_{\mathbf{k}'-\mathbf{k}}^{\sigma''}\right] \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^3} \tag{5}$$

$$\frac{\mathrm{d}N_{\mathbf{k}''}^{\sigma''}}{\mathrm{d}t} = \int w_{\sigma',\mathbf{k}'}^{\sigma'',\sigma} \left[N_{\mathbf{k}}^{\sigma} N_{\mathbf{k}''+\mathbf{k}}^{\sigma'} + N_{\mathbf{k}''}^{\sigma''} N_{\mathbf{k}''+\mathbf{k}}^{\sigma'} - N_{\mathbf{k}}^{\sigma} N_{\mathbf{k}''}^{\sigma''}\right] \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \tag{6}$$

where $w_{\sigma',\mathbf{k}'}^{\sigma'',\sigma}$ is the probability of the process (2). The kinetic equations obviously conserves the total energy and momentum.

0.3 Decay of Langmuir waves into Langmuir and ion-sound waves

Let us start from the process $l' = l + s$, where l, l', and Langmuir waves, and s is ion-sound waves.

The probability of the process is given

$$w^{l,s}_{l',\mathbf{k}'} = \frac{\pi\omega_{pe}^2\omega_s(k)}{4n_eT_e}\frac{(\mathbf{k}'(\mathbf{k}'-\mathbf{k}))^2}{k'^2|\mathbf{k}'-\mathbf{k}|^2}\delta(\omega^l(\mathbf{k}') - \omega^{l'}(\mathbf{k}'-\mathbf{k}) - \omega^s(\mathbf{k})) \tag{7}$$

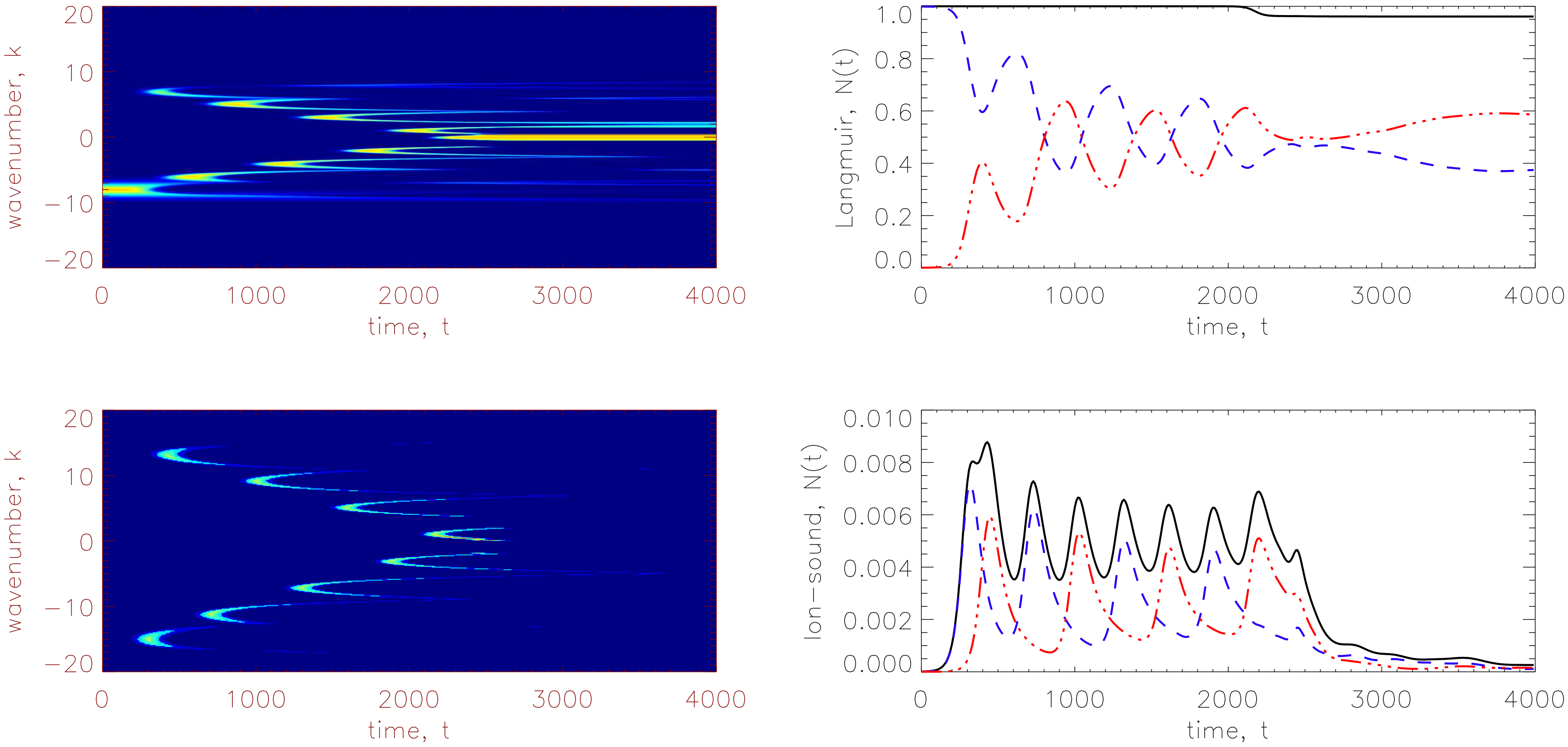
which can be written in 1D case as

$$w^{l,s}_{l',\mathbf{k}'} = \frac{\pi\omega_{pe}^2\omega_s(k)}{4n_eT_e}\frac{\delta(k'-k_1)}{(\omega^l(k') - \omega^{l'}(k'-k) - \omega^s(k))'_{k'}} \tag{8}$$

where k_1 is the solution of the energy conservation law

$$k_1 = k/2 + v_s\frac{\omega_{pe}}{3v_{Te}} \tag{9}$$

This process exists only in a plasma with $T_i \ll T_e$.



wavenumber, k
20
10
0
−10
−20
0
1000
2000
3000
4000
time, t
Langmuir, N(t)
1.0
0.8
0.6
0.4
0.2
0.0
Ion−sound, N(t)
0.010
0.008
0.006
0.004
0.002
0.000

0.4 Scattering of Langmuir waves off ions

Let us consider the process $l + i = i' + l'$. The corresponding kinetic equation for waves has the form

$$\frac{\mathrm{d}N(\mathbf{k}')_l}{\mathrm{d}t} = \int\int w_i^{l,l'} \left[f_i(\mathbf{p})(N(\mathbf{k}') - N(\mathbf{k}') + N(\mathbf{k})N(\mathbf{k}')(\mathbf{k} - \mathbf{k}')\frac{\partial f_i(\mathbf{p})}{\partial \mathbf{p}} \right] \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\mathrm{d}\mathbf{p} \tag{10}$$

where the probability for Maxwellian distribution of ions in 1D case is given

$$w_{l',\mathbf{k}'}^{l,s} = \frac{\sqrt{\pi}\omega_{pe}^2}{4n_e v_{Ti}(k-k')(1+T_e/T_i)} \exp\left(-\frac{(\omega - \omega')^2}{2(k-k')^2 v_{Ti}^2} \right) \tag{11}$$

It can be seen that the scattering is most efficient for $k \sim k'$. Writing the energy conservation for waves we have

$$k' = -k + \frac{2v_{Ti}}{3v_{Te}}\omega_{pe} v_{Te} \tag{12}$$

0.5 Scattering of Langmuir waves off ions into electromagnetic waves

Let us consider the process $l + i = i' + t$. The corresponding kinetic equation for waves has the form

$$\frac{\mathrm{d}N(\mathbf{k}')_t}{\mathrm{d}t} = \int\int w_i^{l,l'} \left[f_i(\mathbf{p})(N^l(\mathbf{k}') - N^t(\mathbf{k}') + N_l(\mathbf{k})N^t(\mathbf{k}')(\mathbf{k} - \mathbf{k}')\frac{\partial f_i(\mathbf{p})}{\partial \mathbf{p}}\right] \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\mathrm{d}\mathbf{p} \tag{13}$$

The scattered transverse waves should have frequency $\omega^t \approx \omega_{pe}$, that leads to

$$k_t \approx \sqrt{3} k_l v_{Te}/c \tag{14}$$

This normally requires that $k_l >> k_t$. Therefore, the bandwidth of scattered electromagnetic radiation follows

$$\Delta\omega \approx k_l v_{Ti} \tag{15}$$

0.6 Coalescence of two Langmuir waves into an electromagnetic wave

There is another possible process that leads to emission of electromagnetic waves $l + l' = t$. The kinetic equation for this process follows:

$$\frac{\partial N_t(\mathbf{k})}{\partial t} = \int w_t^{ll}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \left[N(\mathbf{k}_1)N(\mathbf{k}_2) - N_t(\mathbf{k})N(\mathbf{k}_1) - N_t(\mathbf{k})N(\mathbf{k}_2)\right] \frac{\mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2}{(2\pi)^3}, \tag{16}$$

where right hand side is a collision integral describing the process of coalescence of two Langmuir waves into a transverse wave ($l + l = t$) and the reverse process ($t = l + l$).

$$w_t^{ll}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = \frac{e^2 (2\pi)^6}{32\pi m^2} \frac{(k_1^2 - k_2^2)^2}{k^2 \omega_{pe}} \frac{[\mathbf{k}_1 \mathbf{k}_2]}{k_1^2 k_2^2} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\delta(\omega - \omega_1 - \omega_2), \tag{17}$$

From the conservation of energy and momentum it follows that the wave vector of transverse waves should be $k = \sqrt{3}\omega_{pe}/c$. Langmuir waves should be concentrated in the region of small k, $k_1 = \sqrt{3}\omega_{pe}/2c$, although that leads to superluminal phase velocity of Langmuir waves. Therefore one should assume that waves interact with either thermal or scattered Langmuir waves.

However, the interacting plasma waves are thermal (16,17) then saturation is reached at the thermal level

$$N_t(\mathbf{k}) \sim 2\frac{N(\mathbf{k}_1)N(\mathbf{k}_2)}{N(\mathbf{k}_1 + N(\mathbf{k}_2)} \tag{18}$$

and one needs to include scattered waves.

0.7 Conclusions

- Weak turbulence theory presents self-consistent description of wave-wave and wave particle interaction

- Generally, the condition of applicability of WTT is smallness of wave amplitudes, i.e. the waves does affect the despersion properties of the plasma where they propagate

- Quasi-particle consideration of waves requires that the characteristic growth/damping time of waves is much more than the periods of oscillations

- Statistical description of waves is applicable when the number of waves is large with uniformly random distribution of phases

- What next ?