Kinetics of waves and particles in plasmas

I. Introduction into kinetics of waves and particles

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1 Description of plasmas

Following *Langmuir, (1923)* we call *plasma* a fully ionized gas. However, nowadays this term became more broader and any media with charged particles can be called a plasma.

Generally a system of N charged particles can be easily described by the system of $2 \times N$ Hamiltonian equations

$$\frac{\partial \mathbf{p_i}}{\partial t} = -\frac{\partial H(\mathbf{p_1}, ..., \mathbf{p_N}; \mathbf{r_1}, ..., \mathbf{r_N})}{\partial \mathbf{r}_i}, \tag{1}$$

$$\frac{\partial \mathbf{r_i}}{\partial t} = \frac{\partial H(\mathbf{p_1}, ..., \mathbf{p_N}; \mathbf{r_1}, ..., \mathbf{r_N})}{\partial \mathbf{p}_i} \tag{2}$$

where $\mathbf{r_i}, \mathbf{p_i}$ are position and momentum of i-th particle, $H(\mathbf{p_1}, ..., \mathbf{p_N}; \mathbf{r_1}, ..., \mathbf{r_N})$ is a Hamiltonian of the system. However, as soon as $N \gg 1$ solution of the system becomes increasingly difficult to find and alternative methods of description should be involved.

There are different types of plasma description:

- $M < N$ particle description assuming slow change of macroscopic values
- Fluid (MHD) description of plasma when plasma is assumed to be a continues media at $L >> l$, where L scale of plasma processes, and l is the mean free path of a particle in a plasma.
- Kinetic theory introducing statistical tool to deal with plasma

1.1 Kinetic description

The statistical description of plasma is based on *the distribution function* $f(t, \mathbf{r}, \mathbf{p})$ in phase space ($\mathbf{r}$,$\mathbf{p}$). The value $f \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{p}$ is the average number of particle in the phase volume $\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{p}$, i.e. in the range $(\mathbf{r}, \mathbf{r} + \mathrm{d}\mathbf{r})$ and $(\mathbf{p}, \mathbf{p} + \mathrm{d}\mathbf{p})$. Note: the number of particles with $\mathbf{p} = \mathbf{p_0}$ and $\mathbf{r} = \mathbf{r_0}$ is equal to zero.

If we ignore collisions than each particle is a closed subsystem and the corresponding distribution function obeys *Liouville theorem* as a result of which we can write

$$\frac{\mathrm{d}f(t, \mathbf{r}, \mathbf{p})}{\mathrm{d}t} = 0 \tag{3}$$

where the derivative here means the time derivative along a trajectory in the phase space.

1.2 Kinetic equations

We immediately derive

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \mathbf{F}\frac{\partial f}{\partial \mathbf{p}} = 0. \tag{4}$$

The third term in this equations shows the influence of *the external field* on the the particles. When the interaction between particles cannot be neglected we have

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \mathbf{F}\frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t}\right)_c \tag{5}$$

where $\left(\frac{\partial f}{\partial t}\right)_c$ is integral of collisions and the equations of the type (5) are called *kinetic equations*. The term in the right hand side of equation (5) is a source or a sink of particles in the phase space volume $d\mathbf{r}d\mathbf{p}$.

1.3 Kinetic equations of plasma

The main force acting on a particle in a plasma is electromagnetic. When the mean free path of a particle $l \sim v/\nu$ is much larger than the distance of a field change $L \sim 1/k$

$$\nu \ll kv \tag{6}$$

where v is the velocity of particles, ν is the rate of collisions. At this condition the collisional term in the right hand side is a small value and the kinetic equations will take the form

$$\frac{\partial f_{e,i}}{\partial t} + \mathbf{v}\frac{\partial f_{e,i}}{\partial \mathbf{r}} + q_{e,i}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)\frac{\partial f_{e,i}}{\partial \mathbf{p}} = 0 \tag{7}$$

where $\mathbf{E}, \mathbf{B}$ are the average values of electric and magnetic field respectively. This equation should be completed with the system of Maxwell equations,

$$\triangledown \mathbf{B} = 0, \quad \triangledown \mathbf{E} = 4\pi\rho, \quad \triangledown \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \triangledown \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{8}$$

and with the sources

$$\rho = \sum_j q_j \int f_j \mathrm{d}\mathbf{p}, \quad \mathbf{j} = \sum_j q_j \int \mathbf{v} f_j \mathrm{d}\mathbf{p} \tag{9}$$

This set of equations is obviously nonlinear and the equation (7) was first derived by *Vlasov in 1937*

1.4 Dielectric tensor of isotropic, unmagnetized plasma

We first linearize the Vlasov equations by separating out zero and first order terms

$$f = f_0 + \delta f, \quad E = 0 + \delta E, \quad \text{etc.} \tag{10}$$

Avoiding the terms of second order, we have

$$\frac{\partial \delta f_{e,i}}{\partial t} + \mathbf{v}\frac{\partial \delta f_{e,i}}{\partial \mathbf{r}} + q_{e,i}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)\frac{\partial f_{0,e,i}}{\partial \mathbf{p}} = 0 \tag{11}$$

Interesting to note that since isotropic distribution function depends only on absolute value of $\mathbf{p}$, for such function $\partial f/\partial p$ is parallel to $\mathbf{p} = m\mathbf{v}$, and therefore last term in (13) is zero.

We also assuming perturbations to vary as

$$\delta f, \delta E, \delta B, \text{etc.} \sim exp(i\mathbf{kr} - i\omega t) \tag{12}$$

that gives us

$$-i\omega\delta f_{e,i} + i\mathbf{v}\mathbf{k}\delta f_{e,i} - e\mathbf{E}\frac{\partial f_{0,e,i}}{\partial \mathbf{p}} = 0 \tag{13}$$

The Maxwell equations produce

$$i\mathbf{k}\mathbf{B} = 0, \quad i\mathbf{k}\mathbf{E} = 4\pi\rho, \quad ik \times \mathbf{E} = \frac{i\omega}{c}\mathbf{B}, \quad i\mathbf{k} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} - \frac{i\omega}{c}\mathbf{B} \tag{14}$$

where in the neutral plasma charge density of electrons is compensated by ions

$$\rho = -e\sum_j \int \delta f_j \mathrm{d}\mathbf{p}, \quad \mathbf{j} = -e\sum_j \int \mathbf{v}\delta f_j \mathrm{d}\mathbf{p}. \tag{15}$$

Solving the system of equations we can find dielectric tensor

$$\varepsilon_l(\omega, \mathbf{k}) = 1 - \sum_j \frac{4\pi e^2}{k^2} \int \mathbf{k}\frac{\partial f}{\partial \mathbf{p}} \frac{d\mathbf{p}}{(\mathbf{k}\mathbf{v} - \omega)} \tag{16}$$

$$\varepsilon_t(\omega, \mathbf{k}) = 1 - \sum_j \frac{2\pi e^2}{\omega} \int \mathbf{v}_\perp \frac{\partial f}{\partial \mathbf{p}_\perp} \frac{d\mathbf{p}}{(\mathbf{k}\mathbf{v} - \omega)} \tag{17}$$

and here where problem begins.

Obviously integrals (16,17) have a pole $\omega = kv$, but the actual value depends on the path: above, below the pole or the average of two. We take the path above the pole i.e. add a small value $\omega \to \omega + i0$. This rule was suggested by Landau in 1946 and is named after him.

$$\int\limits_{-\infty}^{\infty} \frac{\partial f(z)dz}{(z - i0)} = \mathrm{PV} \int\limits_{-\infty}^{\infty} \frac{\partial f(z)dz}{z} + i\pi f(0) \tag{18}$$

This gives us

$$\sigma(\omega, \mathbf{k})_l = 1 - \sum_j \frac{4\pi e^2}{k^2} \int \mathbf{k} \frac{\partial f}{\partial \mathbf{p}} \frac{d\mathbf{p}}{(\mathbf{kv} - \omega - i0)} \tag{19}$$

$$\varepsilon_t(\omega, \mathbf{k}) = 1 - \sum_j \frac{2\pi e^2}{\omega} \int \mathbf{v}_\perp \frac{\partial f}{\partial \mathbf{p}_\perp} \frac{d\mathbf{p}}{(\mathbf{kv} - \omega - i0)} \tag{20}$$

1.5 Dielectric tensor of Maxwellian plasma

Let us apply Maxwellian distribution to the expressions of the previous section. We have

$$f(p) = \frac{n_{e,i}}{(2\pi m T_{e,i})^{1/2}} \exp\left(-\frac{p^2}{2mT_{e,i}}\right) \tag{21}$$

Substituting Maxwell distribution we immediately obtain

$$\varepsilon_l(\omega, k) = 1 + \sum_j \frac{1}{k^2 \lambda_{Dj}^2} \left[1 + F\left(\frac{\omega}{\sqrt{2} k v_{Tj}}\right)\right] \tag{22}$$

$$\varepsilon_t(\omega, k) = 1 + \sum_j \frac{\omega_{pj}^2}{\omega^2} F\left(\frac{\omega}{\sqrt{2} k v_{Tj}}\right) \tag{23}$$

where

$$\lambda_{Dj} = \sqrt{\frac{T_j}{4\pi e^2 n_j}}, \quad v_{Tj} = \sqrt{\frac{T_j}{m_j}}, \quad \omega_{pj} = \sqrt{\frac{4\pi e^2 n_j}{m_j}} \tag{24}$$

are electron/ion Debye length, and electron/ion thermal velocity. The function $F(x)$ is well-tabulated *Plasma Dispersion Function*

$$F(x) = \frac{x}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{\exp(-z^2)dz}{z - x - i0} \tag{25}$$

1.6 Landau damping

The imaginary part of ε_l follows

$$\mathrm{Im}\sigma(\omega, \mathbf{k})_l = -\sum_j \frac{4\pi^2 e^2}{k^2} \int \mathbf{k} \frac{\partial f}{\partial \mathbf{p}} \delta(\omega - \mathbf{k}\mathbf{v}) d\mathbf{p} \tag{26}$$

It says that the waves in plasma are dumped even in the collisionless plasma. For Maxwellian distribution we have

$$\gamma_k \approx -\sqrt{\frac{\pi}{8}} \sum_j \frac{\omega_{pj}}{(k\lambda_{Dj})^3} \exp\left(-\frac{1}{2(k\lambda_{Dj})^2}\right) \tag{27}$$

The damping is not randomizing collisions, but by a transfer of waves energy into resonant oscillation of particles. Note that $k\lambda_{De} > 1$ the damping rate would exceed frequency of oscillations.

1.7 Langmuir waves

The dispersion relation for longitudinal electrostatic oscillations is simply

$$\varepsilon(\omega, k)_l = 0 \tag{28}$$

The solutions of this equations as we see are complex. In the limit $\omega >> kv_{Te} >> kv_{Ti}$ we find for the real part

$$\omega(k)_l = \omega_{pe}(1 + \frac{3k^2 v_{Te}^2}{2}) \tag{29}$$

which is the dispersion relation for *Langmuir waves*. The imaginary part of the frequency is

$$\gamma(k) = -0.5\omega_{pe}\mathrm{Im}\varepsilon(\omega, k) \tag{30}$$

1.8 Transverse waves

For transverse waves the dispersion relation is given by

$$\omega^2 = c^2 k^2 / \varepsilon_t \tag{31}$$

High frequency waves $\omega \gg k v_{Te}$ corresponds to ordinary electromagnetic waves. With ε_t derived we find

$$\omega(k)_t^2 = \omega_{pe}^2 + k^2 c^2 \tag{32}$$

This relation is correct for all values of k. Note Landau damping is does not exist since phase velocities are greater than speed of light.

For low frequency waves $\omega >> k v_{Te}$ ions again are not important and the solution has only imaginary part

$$\omega(k)_t = -i \sqrt{\frac{2}{\pi}} \frac{k^3 c^2 v_{Te}}{\omega_{pe}^2} \tag{33}$$

and no waves can propagate.

1.9 Ion-sound waves

Let us consider the range $kv_{Ti} << \omega << kv_{Te}$ then we have the solution of dispersion equation $\varepsilon(\omega, k)_l = 0$ in the form

$$\omega(k)_s^2 = \omega_{pi}^2 \frac{k^2 \lambda_{De}^2}{1 + k^2 \lambda_{De}^2} \tag{34}$$

These waves are called *ion-sound waves*. For $k\lambda_{De} << 1$ we have

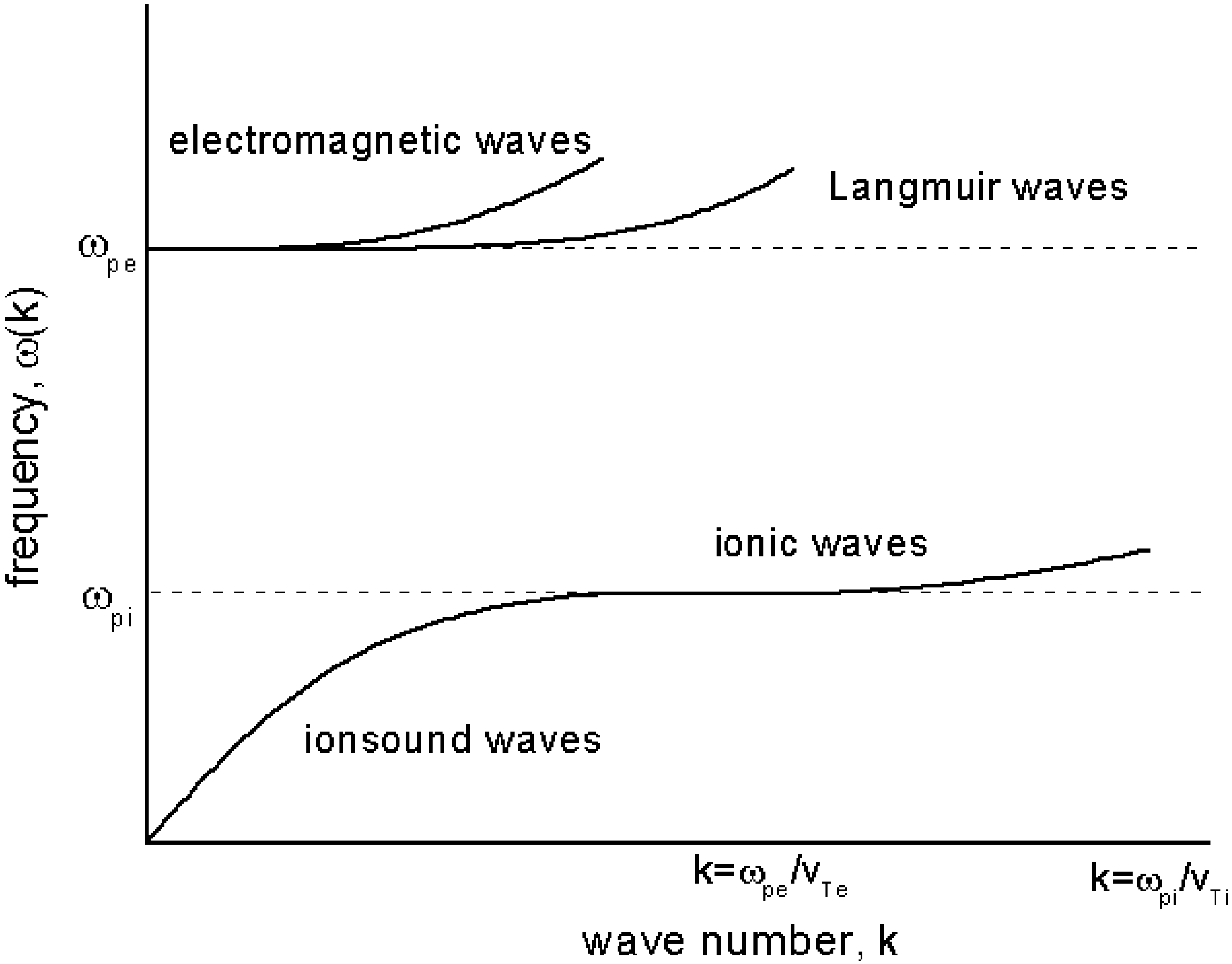
$$\omega(k)_s = k\sqrt{\frac{T_e}{M}} \tag{35}$$

with the dispersion as for ordinary sound waves.

This waves are subject to damping. The rate accounting only electrons is

$$\gamma(k)_s = \omega\sqrt{\frac{\pi m}{8M}} \tag{36}$$

Note that these waves require $T_e >> T_i$.


electromagnetic waves
Langmuir waves
ω_{pe}
frequency, $\omega(k)$
ionic waves
ω_{pi}
ionsound waves
$k=\omega_{pe}/v_{Te}$
$k=\omega_{pi}/v_{Ti}$
wave number, k