Cosmology II - Introduction to Cosmology?

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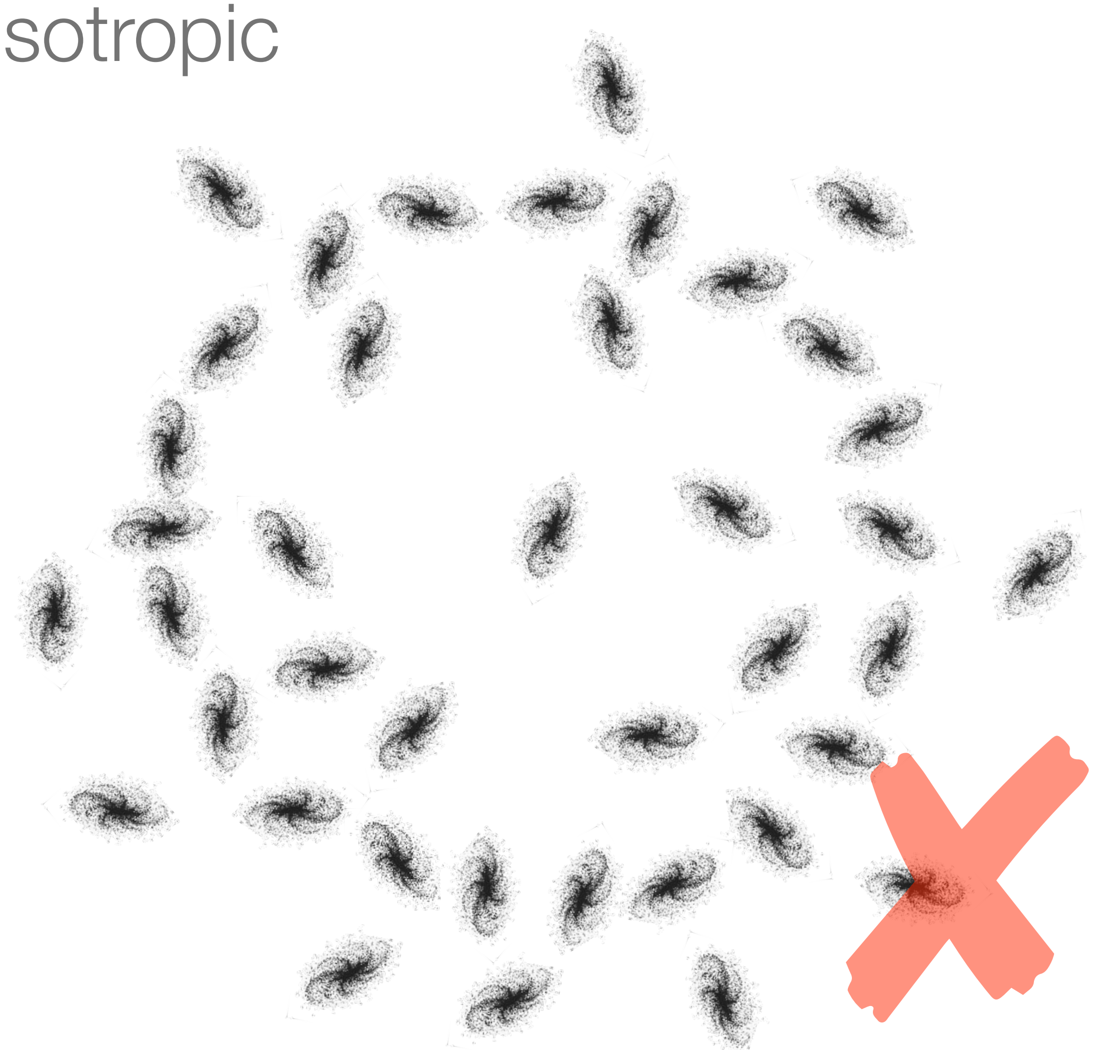
Lecture slides based on those of Leonard Susskind - Stanford University, USA

1. **Newtonian approach**
2. Adding radiation
3. Possible geometries

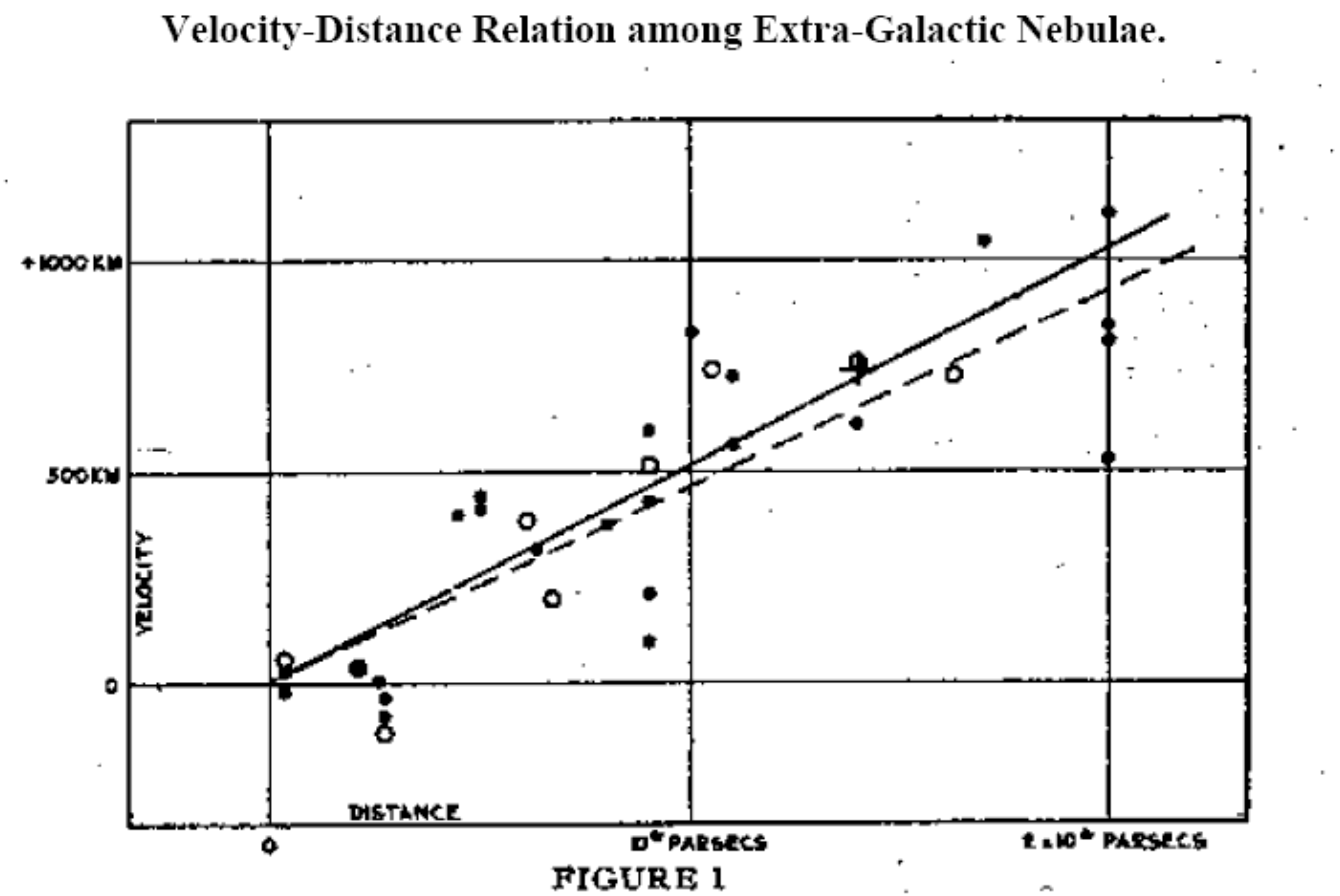
Assumptions

- Standard cosmology makes 2 fundamental assumptions:
  1. The properties of the Universe are isotropic
  2. Our position in the Universe is not preferred to any other (cosmological principle)

- Such a Universe is homogeneous and isotropic



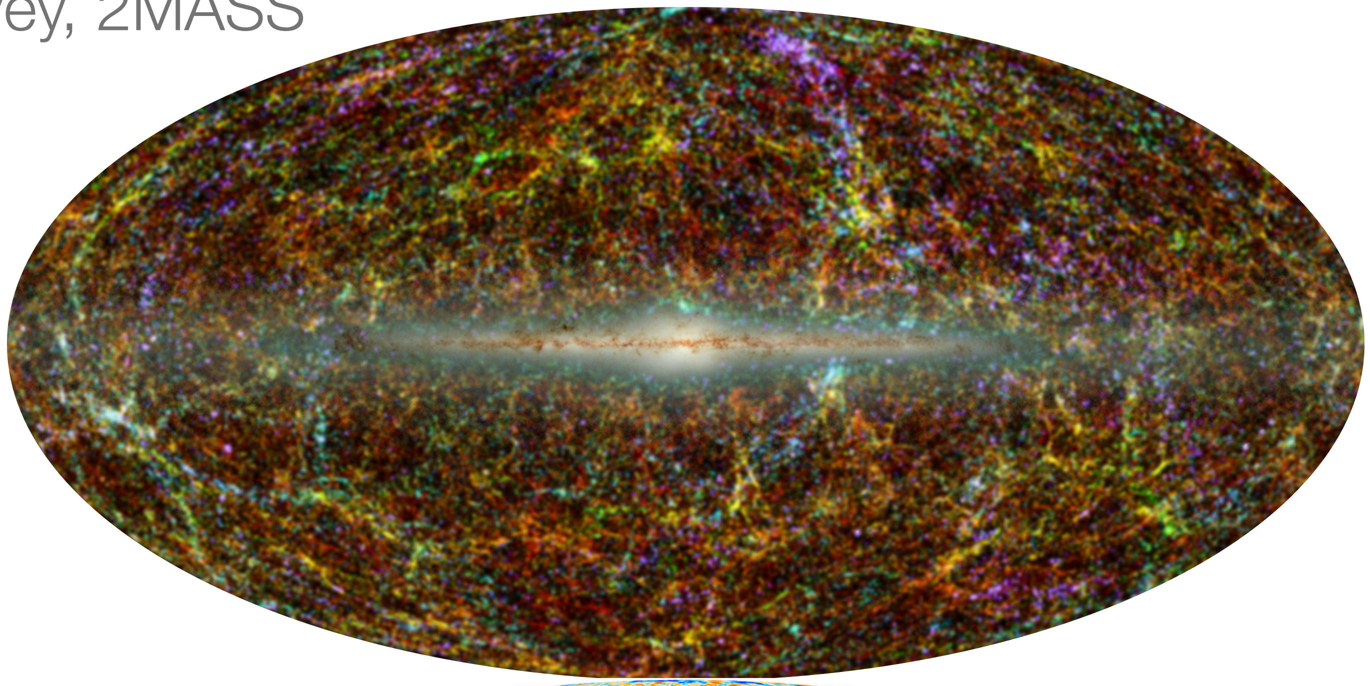
Motivation - Hubble

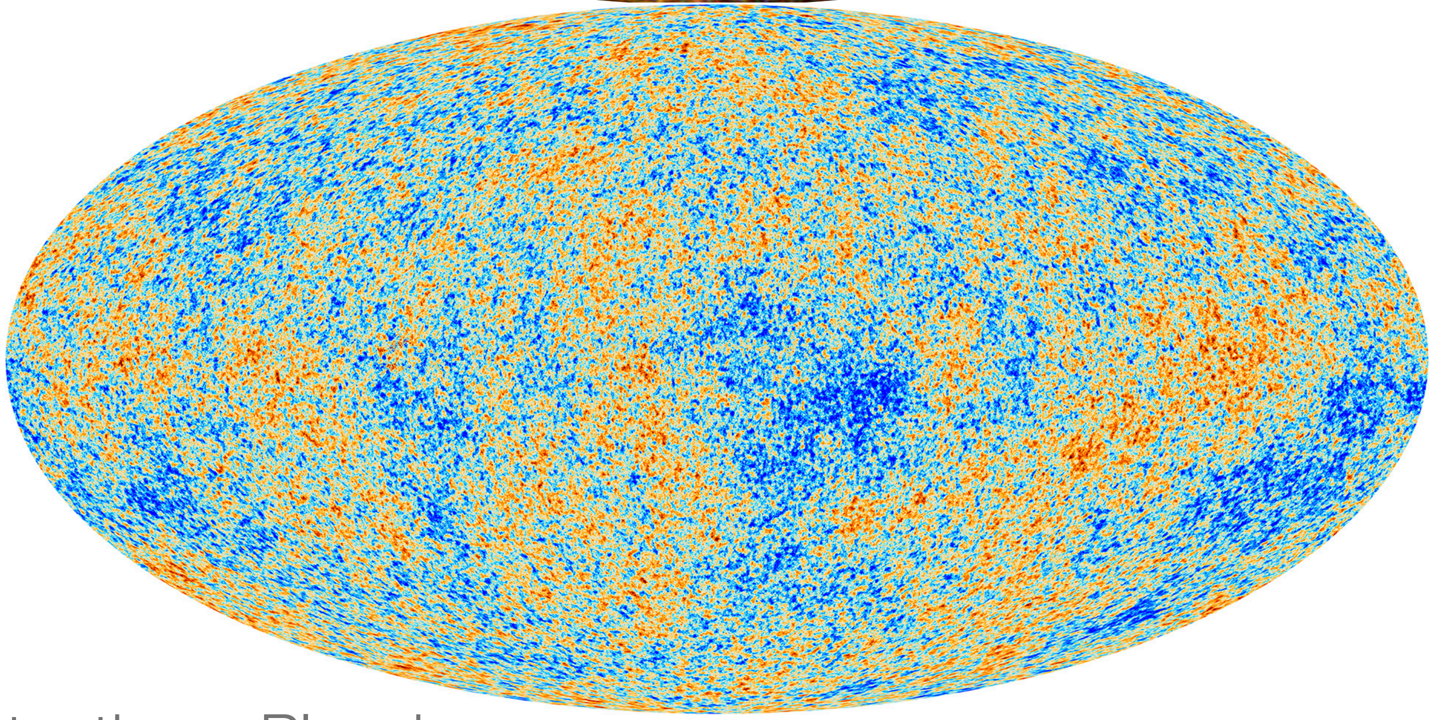


Hubble's original 1929 paper

Isotropic and Homogeneous

2-Micron All-Sky Survey, 2MASS

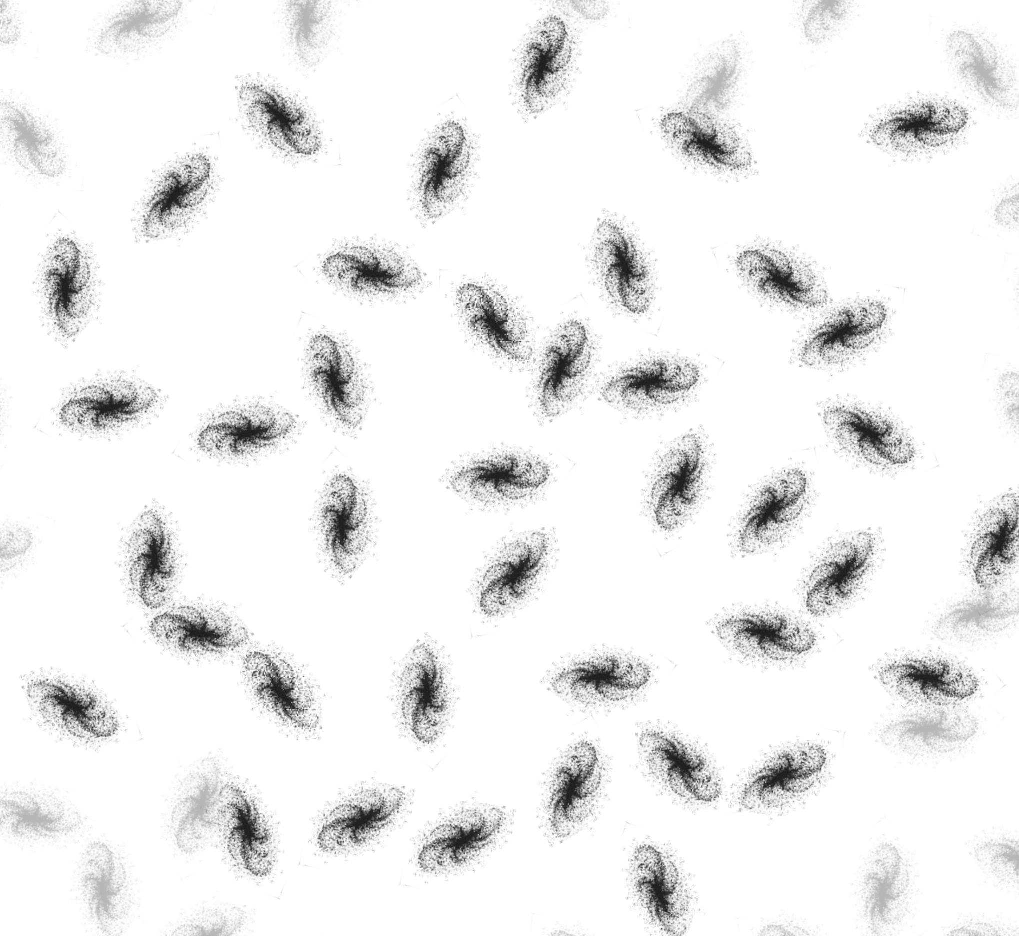




CMB temperature fluctuations, Planck

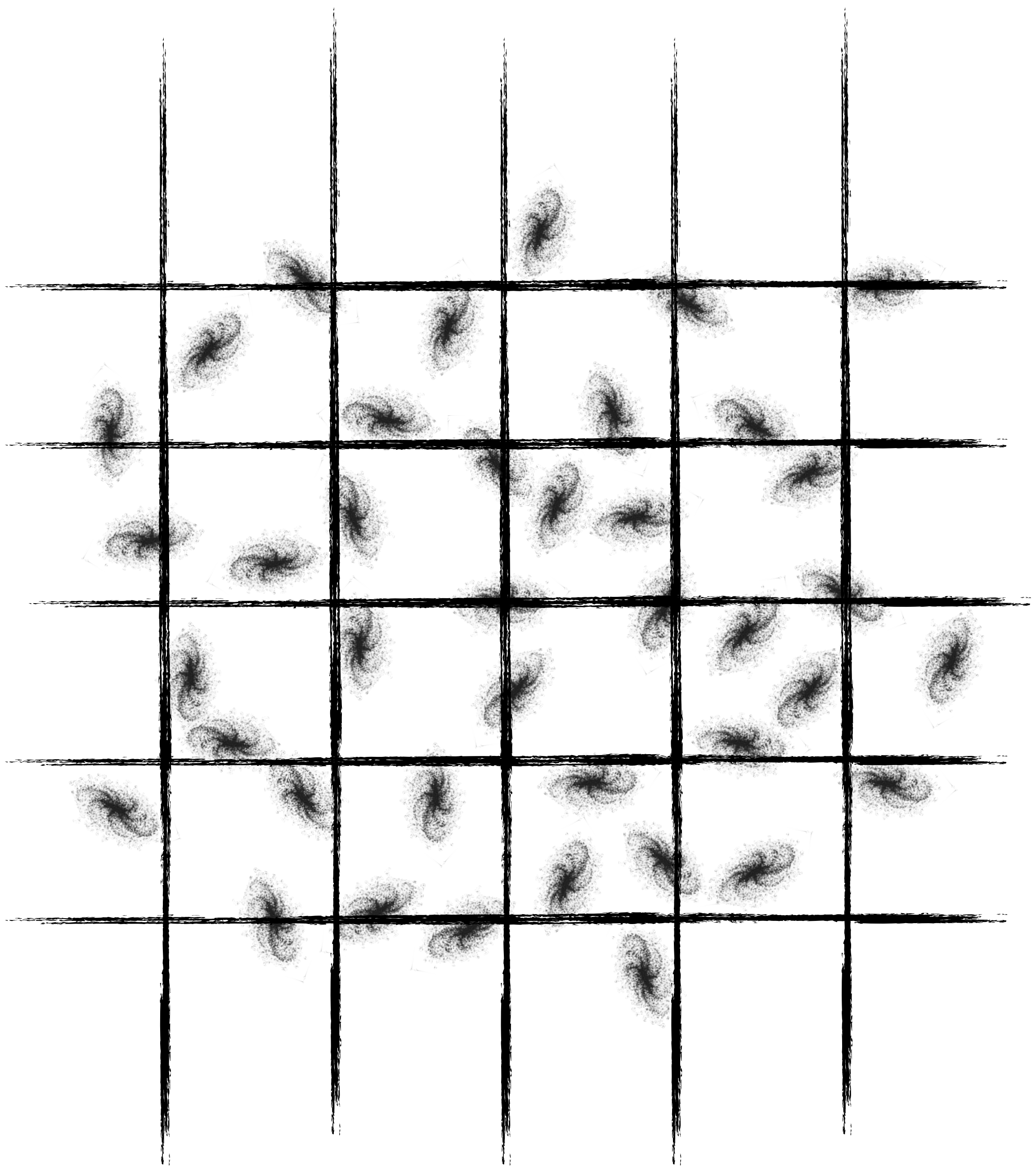
Uniform gas of particles

- Electrically neutral so gravity is the only important force
- You’d assume that everything just sits still (wrong).



Introduce co-moving coordinates

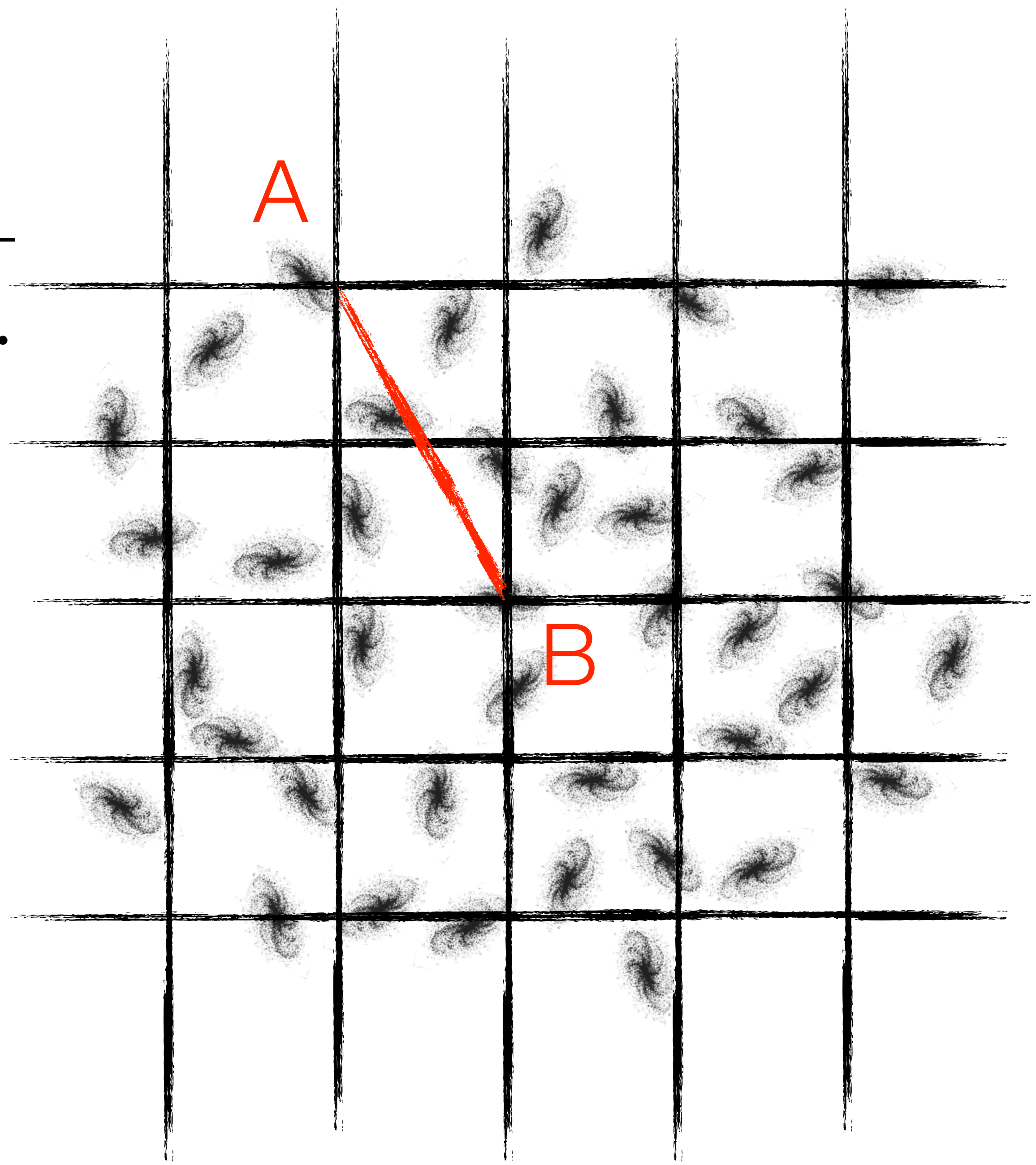
- Define the lattice of grid points to be co-moving with the galaxies
- Think of the galaxies as fixed to their location in the grid.
- Inherent assumption about behaviour of galaxies. Based on observation (Hubble).
- The galaxies themselves do not stretch!
- The x,y,z coords are ***not*** distances, they just label the grid lines.



Scale parameter

- Let's define the scale parameter $a(t)$ such that

$$d_{AB} = a(t)\sqrt{\Delta x^2_{AB} + \Delta y^2_{AB} + \dots}$$
$$= a(t) R_{AB}$$



- The velocity is $v_{AB} = \dot{a}(t) R_{AB}$

- We then define the Hubble parameter/function

$$H(t) = \frac{\dot{a}}{a} = \frac{v_{AB}}{d_{AB}}$$

Mass in the model

- The mass within a coordinate volume is

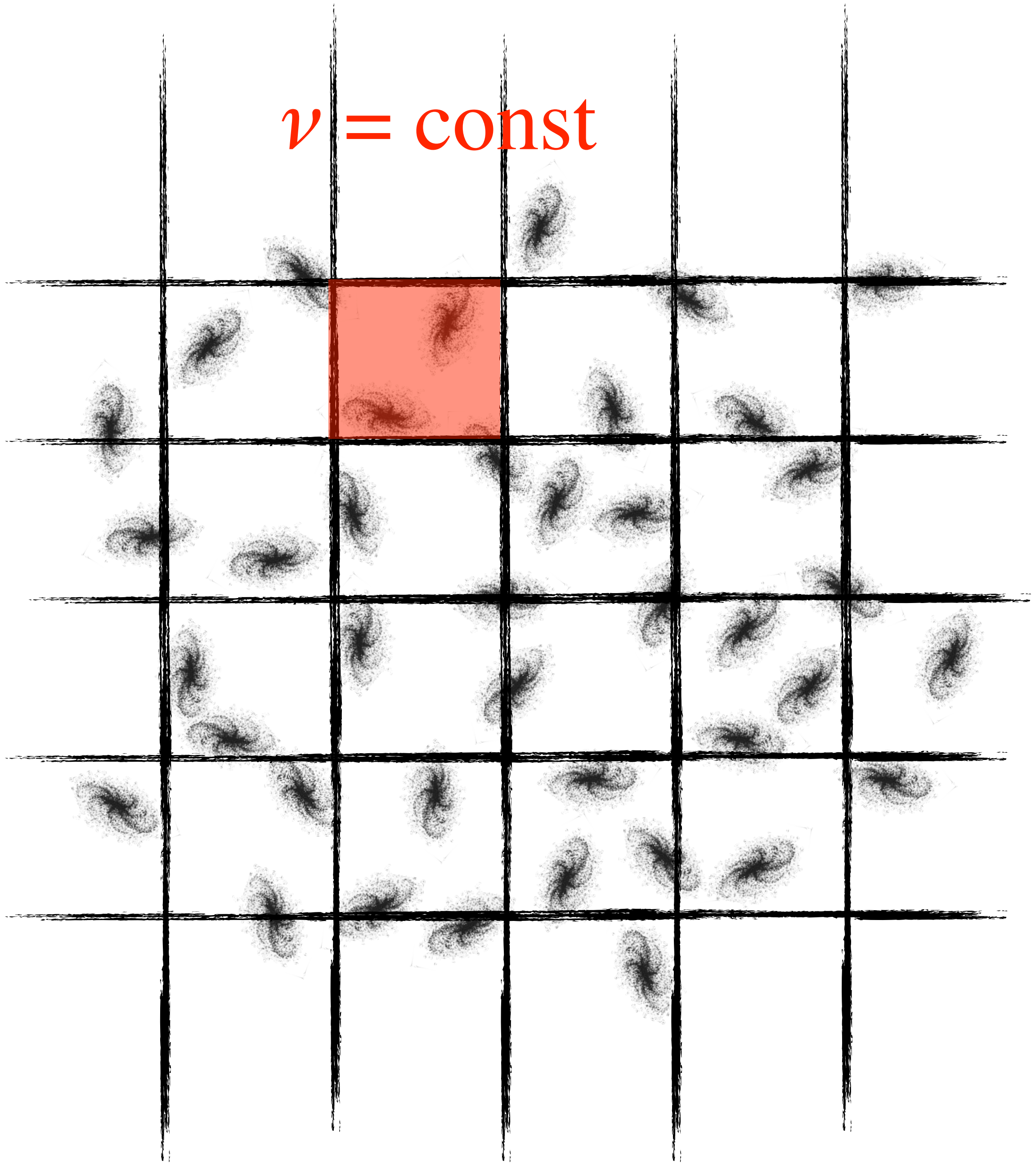
$$\Delta M = \nu \Delta x \Delta y \Delta z$$

- But the actual volume is

$$\Delta V = a^3(t) \Delta x \Delta y \Delta z$$

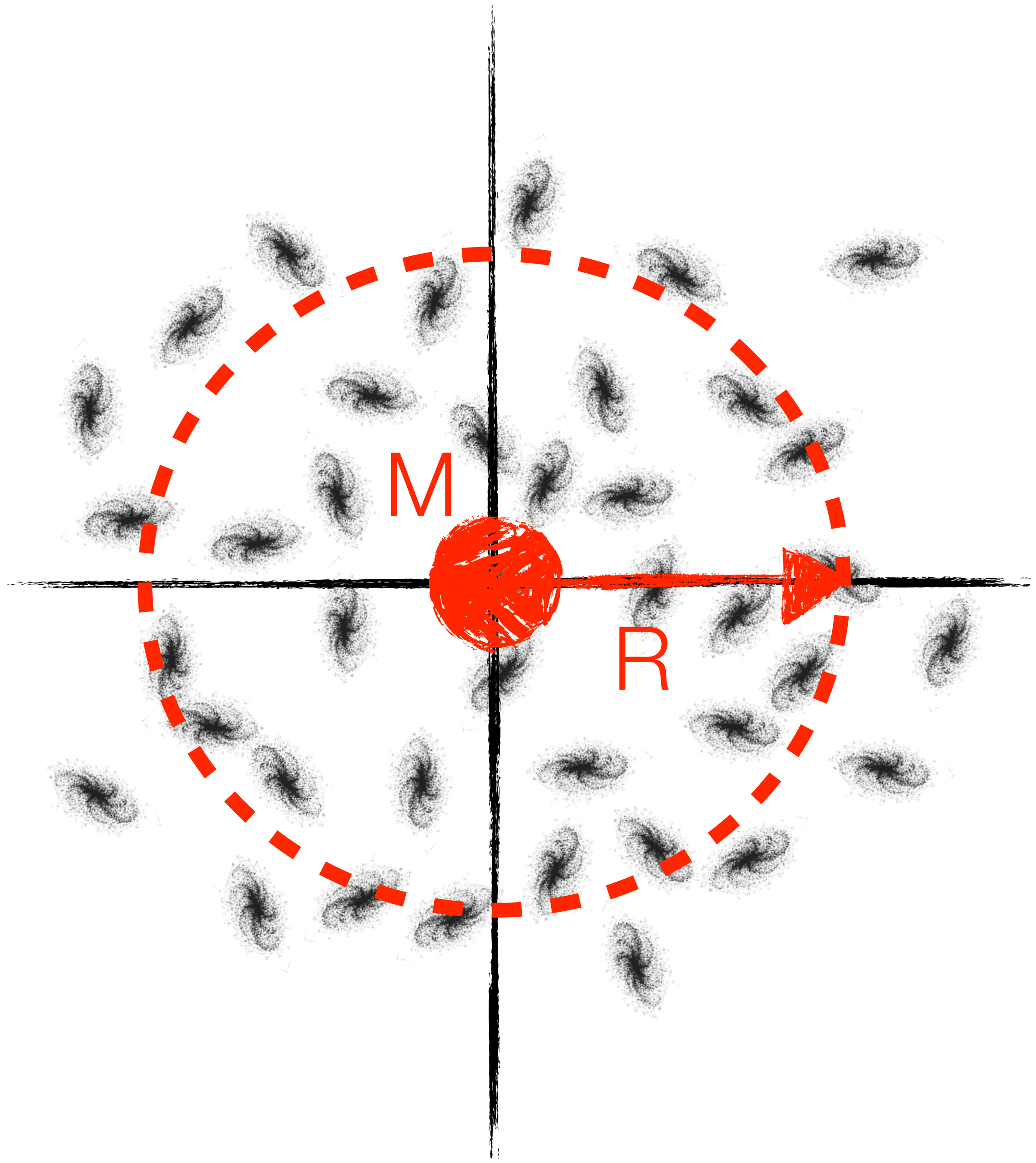
- So the density is $\rho = \dfrac{\nu}{a^3}$

- Mass per unit coordinate cell is constant but the density will change with $a(t)$



Newtonian Gravity

- Find the ***relative*** acceleration between us (at origin) and a distant galaxy.
- The acceleration due to the changing scale factor is
$$\ddot{d} = \ddot{a}R$$
- Use Newtons theorem to compute the gravitational acceleration on the galaxy
- Equate to Newtonian gravitational acceleration
$$\ddot{a}R = -\frac{GM}{d^2}$$



Equation of motion

- Since $d = a(t)R$

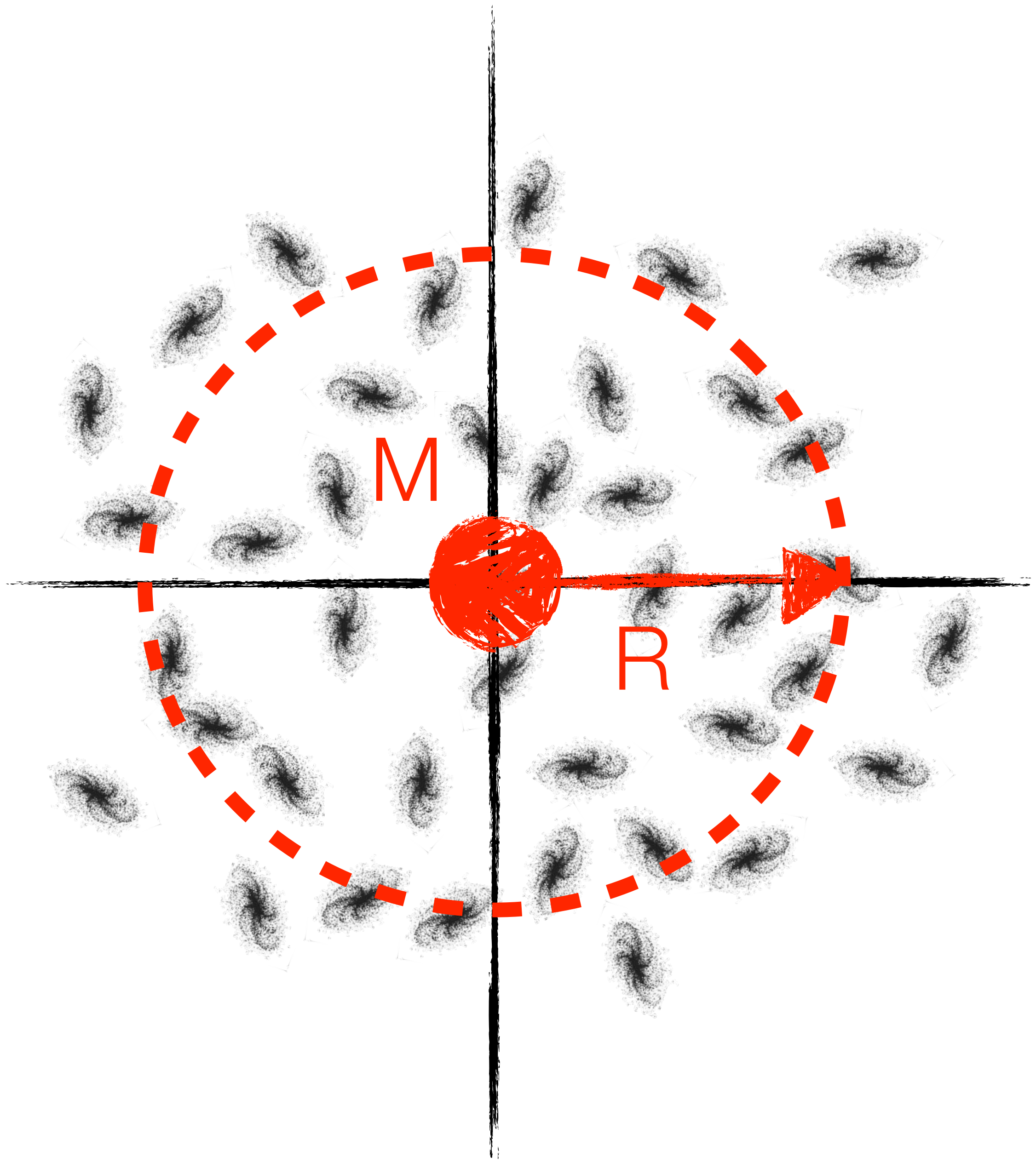
$$\ddot{a}R = -\frac{GM}{d^2} \Rightarrow \frac{\ddot{a}}{a} = -\frac{GM}{a^3R^3}$$

- The volume of the enclosing sphere is

$$V_{\rm sph} = \frac{4\pi}{3}(aR)^3$$

- and since $\varrho = M/V_{\rm sph} = \nu/a^3$ we get

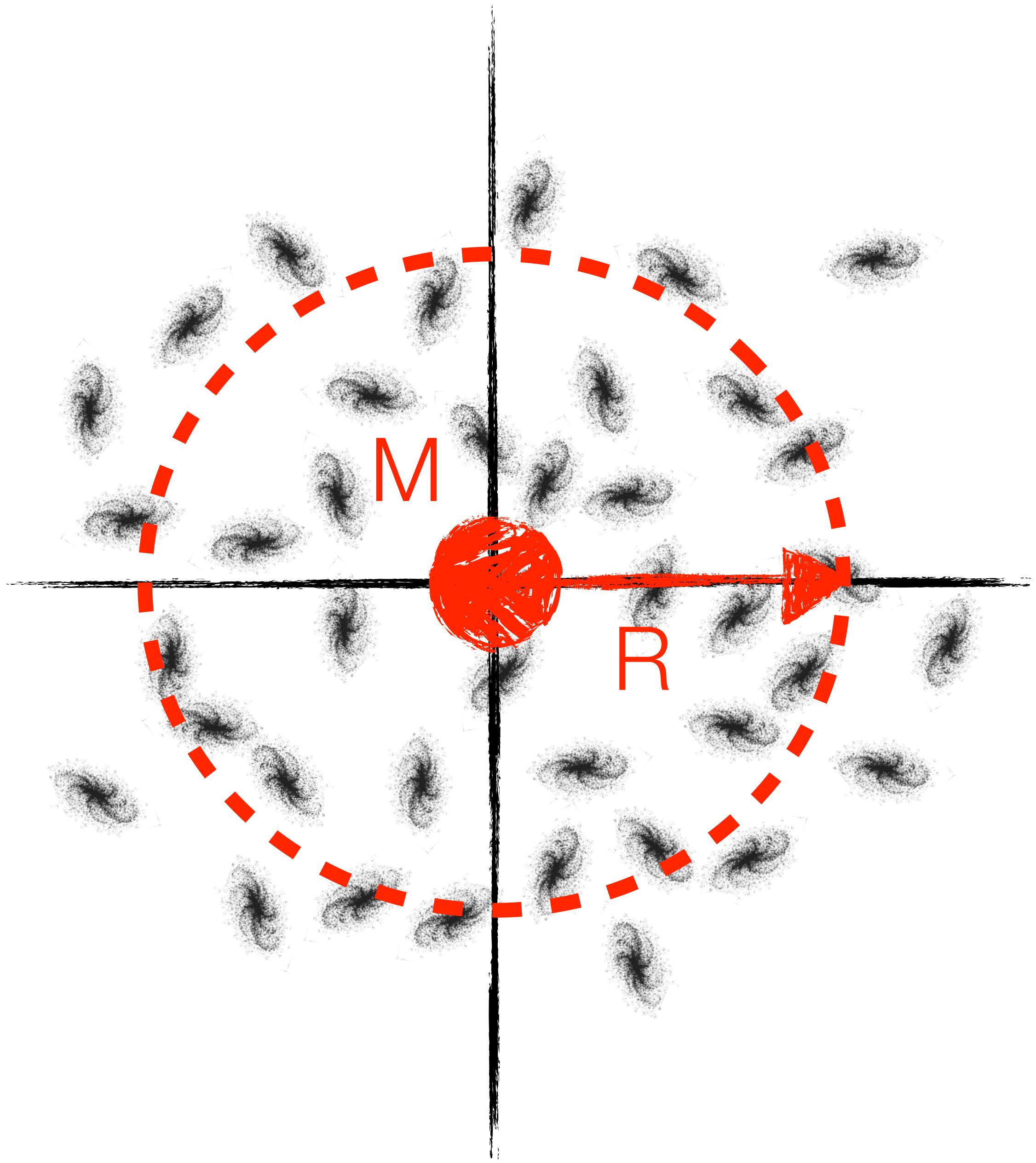
$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\rho$$



Equation of motion

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G\rho$$

- Doesn't depend on R - true for any galaxy (hinges on ν being constant due to homogeneity).
- Also ***impossible*** to have a static universe (unless it's empty).
- Replace ρ by ν/a^3 $$\frac{\ddot{a}}{a} = -\frac{4\pi G\nu}{3a^3}$$
- Gives us an equation of motion of the scale factor - acceleration has to be negative



Energy Conservation

- Particle moving away from a large mass on x-axis

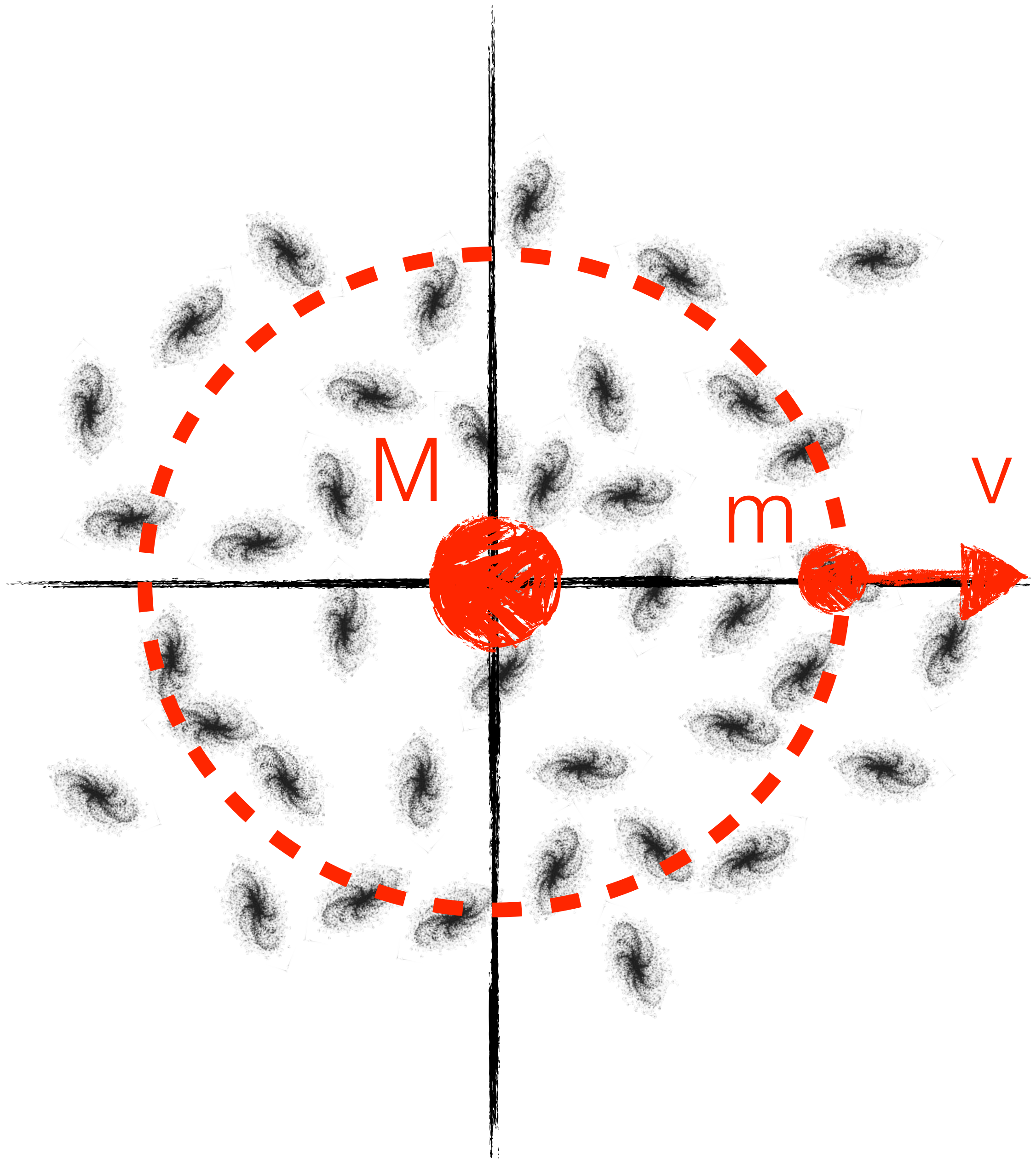
- Energy is conserved so

$$\frac{1}{2}m\dot{d}^2 - \frac{GmM}{d} = E$$

- If E is positive it ultimately escapes, if negative it falls back

- Escape velocity is therefore

$$\dot{d}^2 = \frac{2GM}{d}$$



Energy Conservation - special case

- Replacing and rearranging terms and setting $E = 0$ gives us

$$\frac{1}{2}m(\dot{a}R)^2 - \frac{GmM}{aR} = 0 \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{2GM}{(aR)^3}$$

- remembering that the volume of the sphere is

$$V_{\mathrm{sph}} = \frac{4\pi}{3}(aR)^3$$

- Gives us the ***Friedmann Equation*** (not general since we set $E = 0$)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho$$

- We already know that $\ddot{a} < 0$ so $\dot{a}$ is reducing, but RHS is +ve, so…

Solve for the scale factor

- Replacing ρ by ν/a^3 $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\nu}{3a^3} = H^2(t)$$
- The square of Hubble parameter never gets to zero (in this case)
- Can always choose ν to be whatever we want so only need to solve $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{a^3}$$
- Solution is $a \sim t^{2/3}$ (Newton could have done all of this but didn't have Hubble's observations to guide him)
- This particular Universe would have to be spatially flat, infinite, and matter dominated.

Energy - general case

- Looking at the general $E \neq 0$ case (@ $x = 1, d = a(t)$ and $\dot{d} = \dot{a}$)

$$\frac{1}{2}m\dot{d}^2 - \frac{GmM}{d} = E \Rightarrow \dot{a}^2 - \frac{2MG}{a} = C$$

- Rearrange to get nice ratios

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{2MG}{a^3} = \frac{C}{a^2}$$

- Since $x = 1$ the volume here is $V_{\rm sph} = (4/3)\pi a^3$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\nu}{3a^3} + \frac{C}{a^2}$$

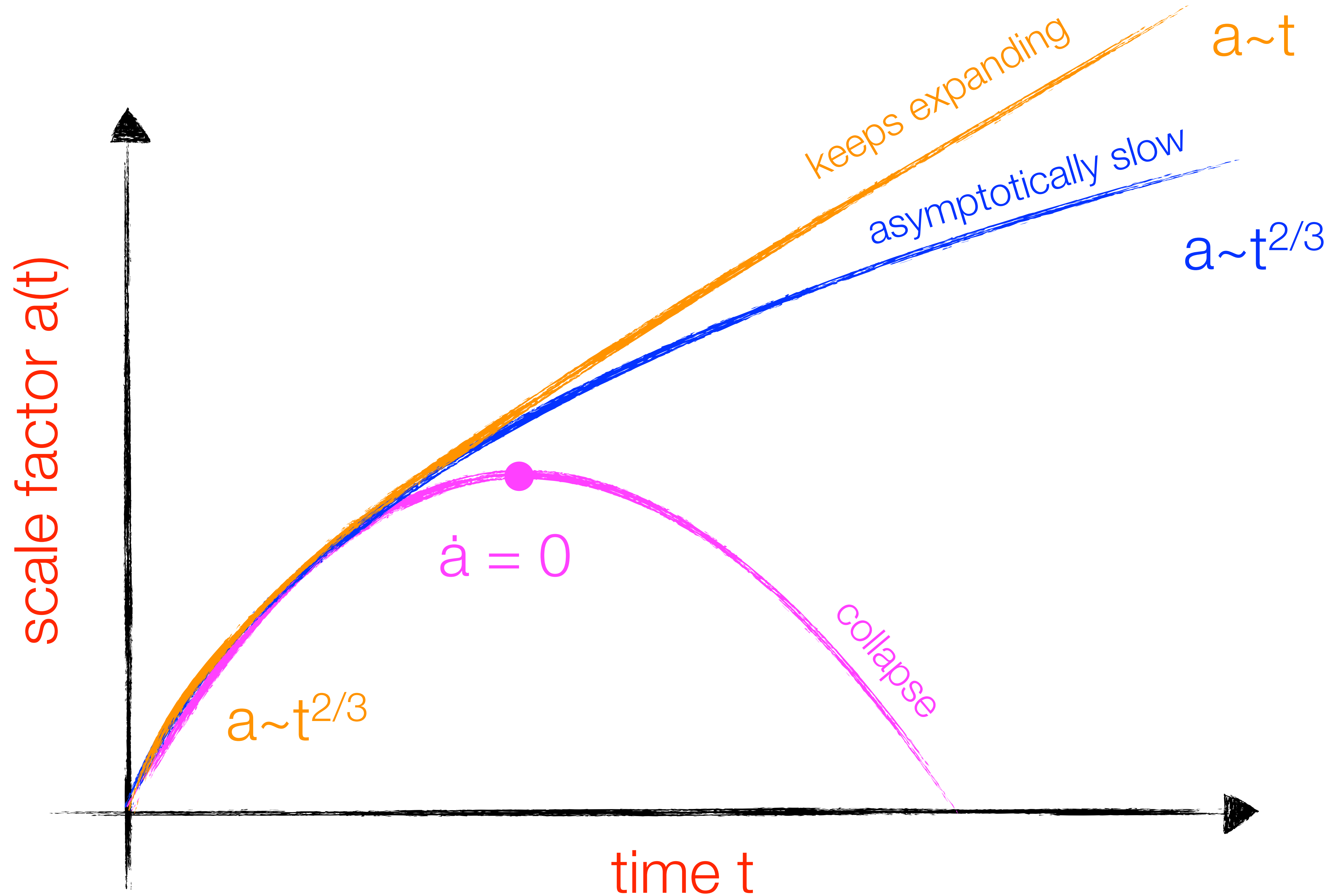
- If $E > 0$ then RHS is always +ve and $\dot{a} > 0$ (had to start being +ve)

Asymptotic behaviour

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\nu}{3a^3} + \frac{C}{a^2}$$

- For +ve energy ($C > 0$) at large a the new term dominates and we get $a \sim t$
- At small a we get the old result $a \sim t^{2/3}$
- For -ve energy ($C < 0$) the RHS can become zero so there is a turning point in the scale factor evolution ($\dot{a} = 0$)
- All of this is for the matter dominated universe (i.e., only considers matter)
- This is directly related to the geometry of the universe (C is related to the curvature of space)

Scale factor evolution



1. Newtonian approach

2. **Adding radiation**

3. Possible geometries

Connection to GR

- rearrange our equation to give

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{C}{a^3} = \frac{8\pi}{3} G\rho$$

- LHS has geometry, RHS is energy density, looks like Einsteins field equation(s)

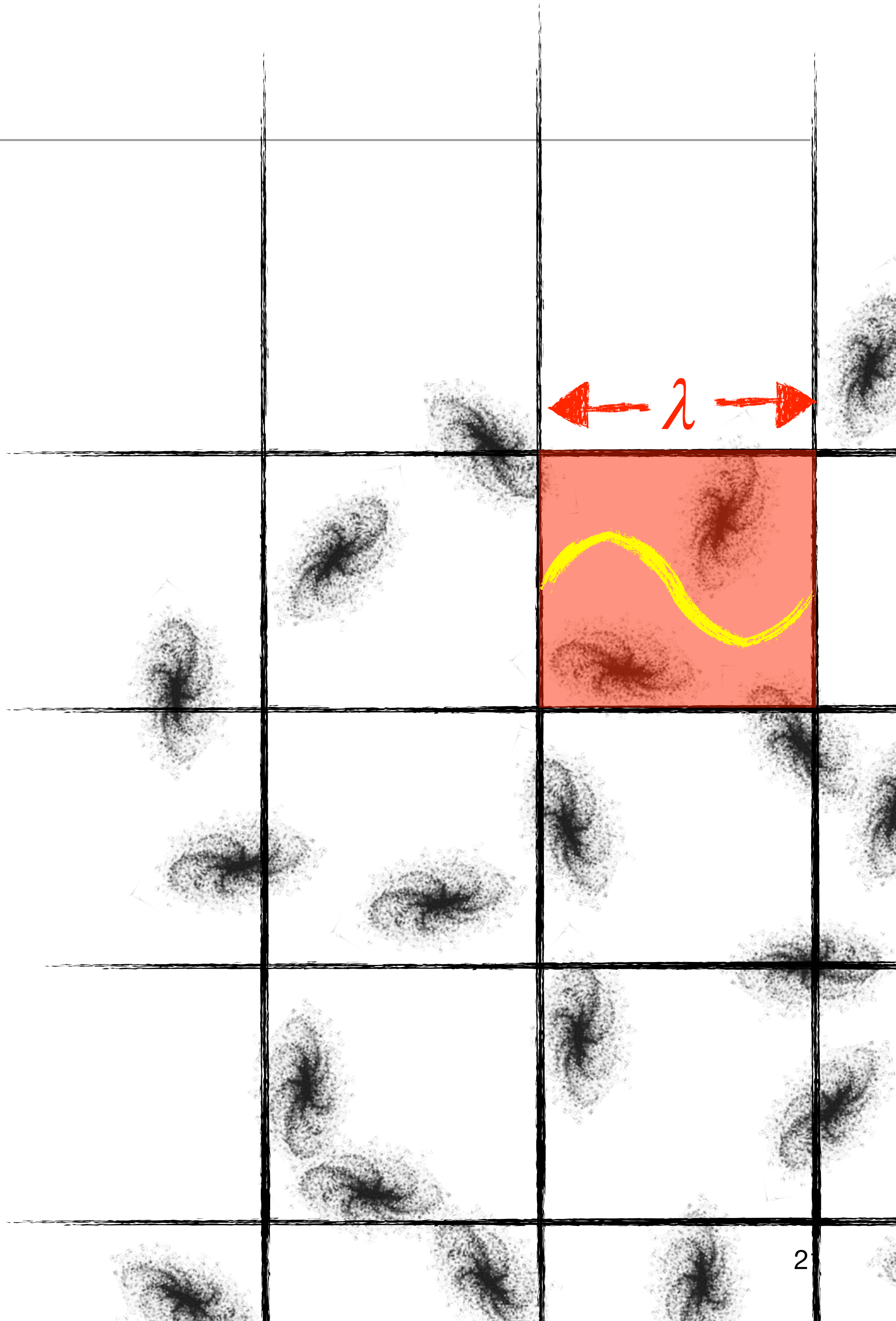
- So let’s now include radiation energy

$$\rho \Rightarrow M + \gamma$$

and see what happens

Add photons

- Consider a photon within our grid of coordinates
- The photons behave in such a way as to expand their wavelengths in proportion with the grid box they are in
- So since $$E = \frac{hc}{\lambda} \propto \frac{1}{a}$$
- The photon energy changes but the total number of photons remains constant.



Add photons

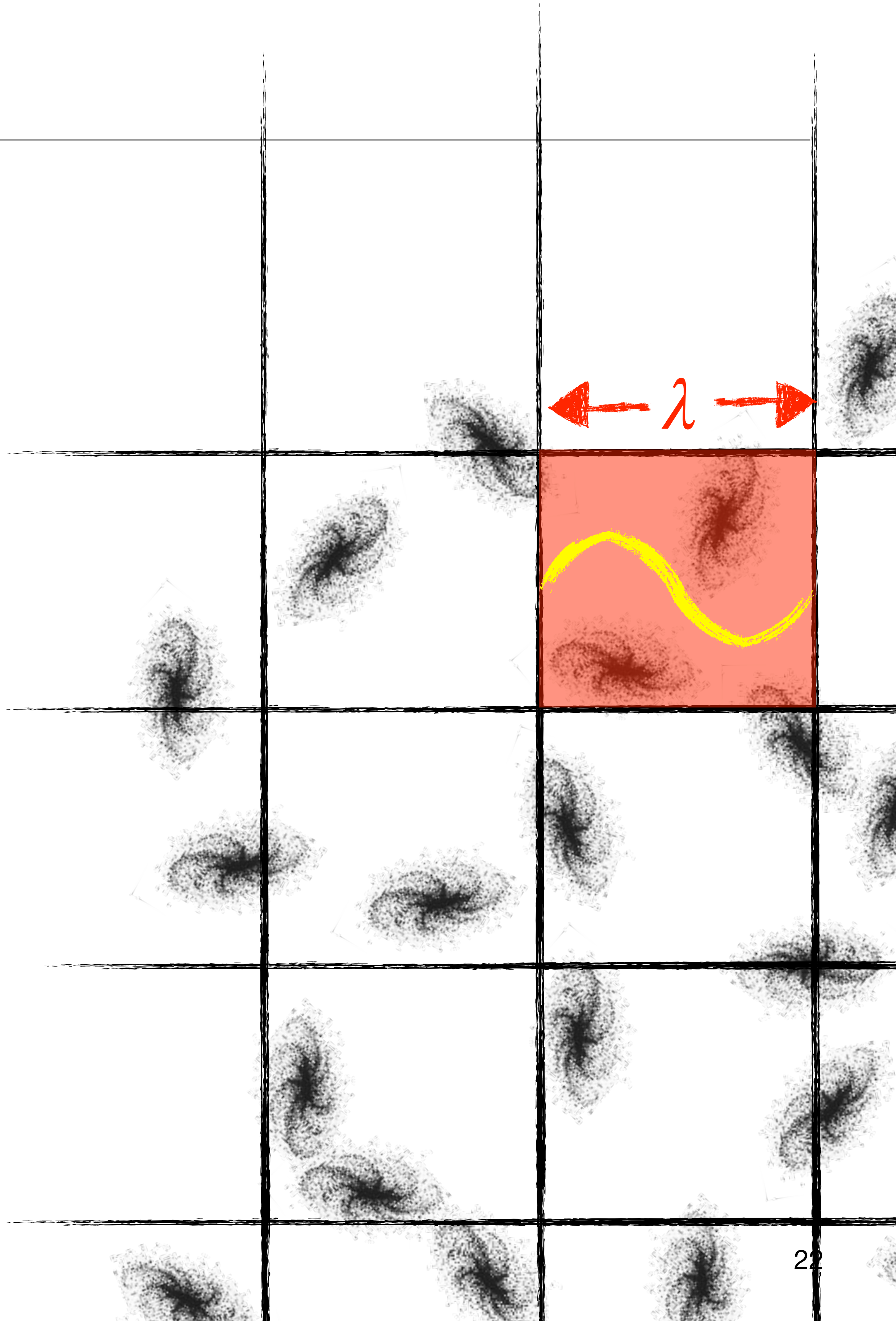
- So as $a(t)$ increases the photon energy in the box drops.

- Therefore compared to the mass case there is an extra factor of a in the denominator.

- So for radiation only and the zero energy case

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\nu}{3a^4}$$

- we get a solution $a \sim t^{1/2}$



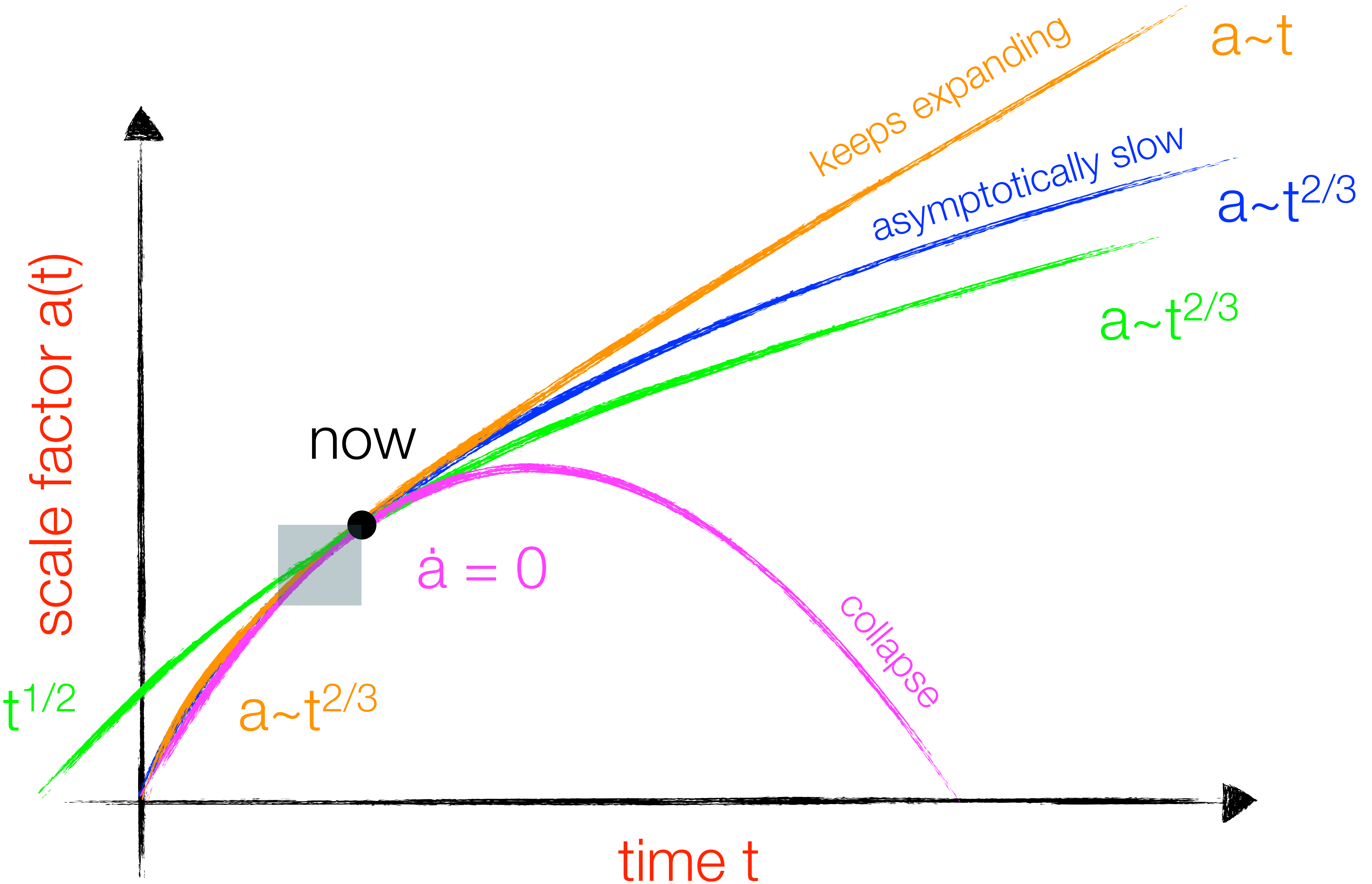
Mass plus radiation

- The general form of the equation will be

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{C_{\rm m}}{a^3} + \frac{C_\gamma}{a^4}$$

- So at large a matter dominates and at small a radiation dominates
- So the universe starts expanding at $t^{1/2}$ but later on switched to $t^{2/3}$
- The radiation part is made up of photons, neutrinos, gravitons (everything moving at ~speed of light).
- The matter component includes dark matter (all non-relativistic particles)

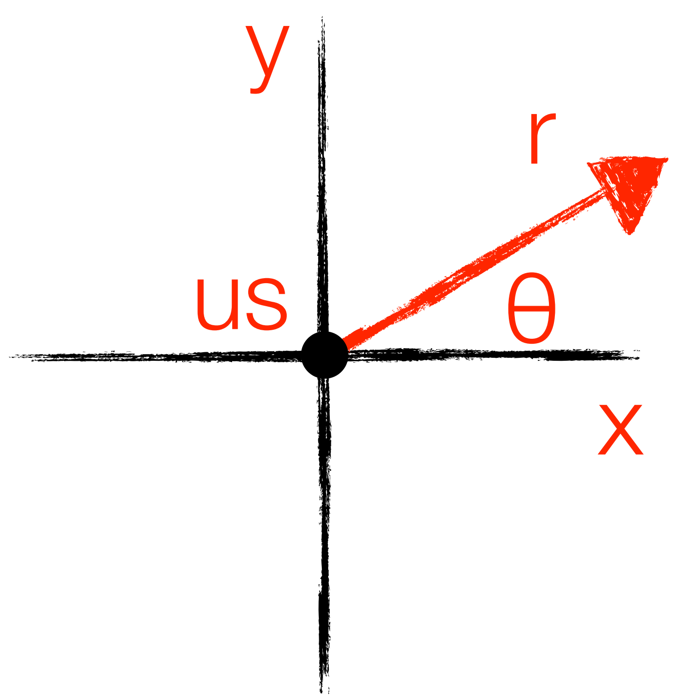
Scale factor evolution



1. Newtonian approach
2. Adding radiation
3. **Possible geometries**

General spaces

- Lets explore the possibility that space is not flat but still homogeneous - only a few options that are homogeneous.
    1. Flat space
    2. Spheres
    3. Hyperbolic space
    4. Toroidal + …



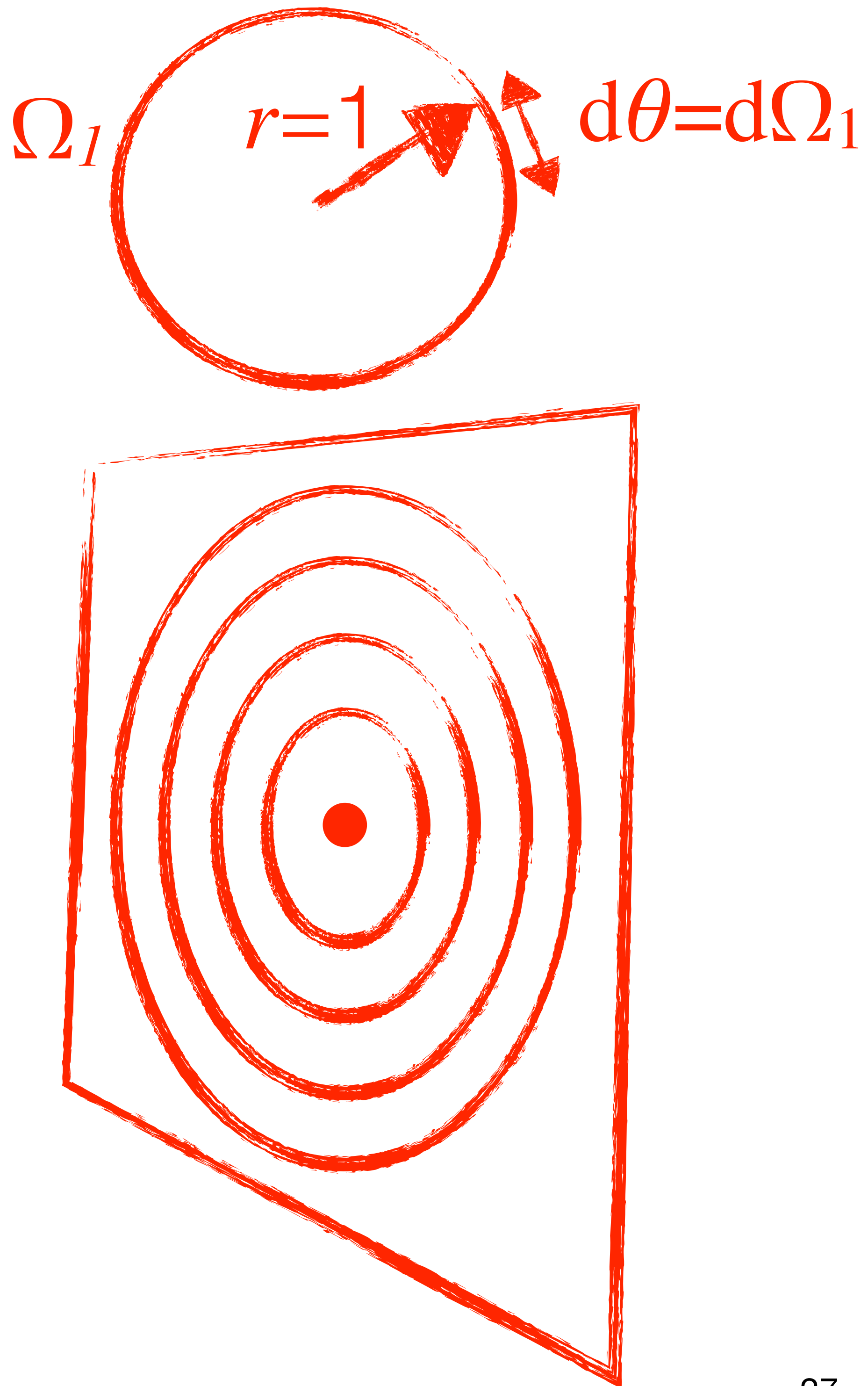
- In 2-D things like paraboloids, ellipsoids, bumpy spaces, are curved but not homogeneous
- 2-D flat space (plane) defined by metric $\mathrm{ds}^2 = \mathrm{dx}^2 + \mathrm{dy}^2$
- Useful to thing in polar coordinates (still flat) $\mathrm{ds}^2 = \mathrm{dr}^2 + \mathrm{r}^2 \mathrm{d}\theta^2$

1-spheres

- $d\theta^2$ is a metric for a particular space (a unit circle) - measures the squared distance along a circle (total distance is 2π)
- ***a unit circle is a 1-sphere***, a 1-D sphere - also called $\mathbf{\Omega_1}$
- Sometimes refer to $d\theta^2$ as $d\Omega_1{}^2$

$$ds^2 = dr^2 + r^2 d\theta^2 \equiv dr^2 + r^2 d\Omega_1^2$$

- Think of a flat 2-D space composed of a nested set of 1-spheres - space is flat

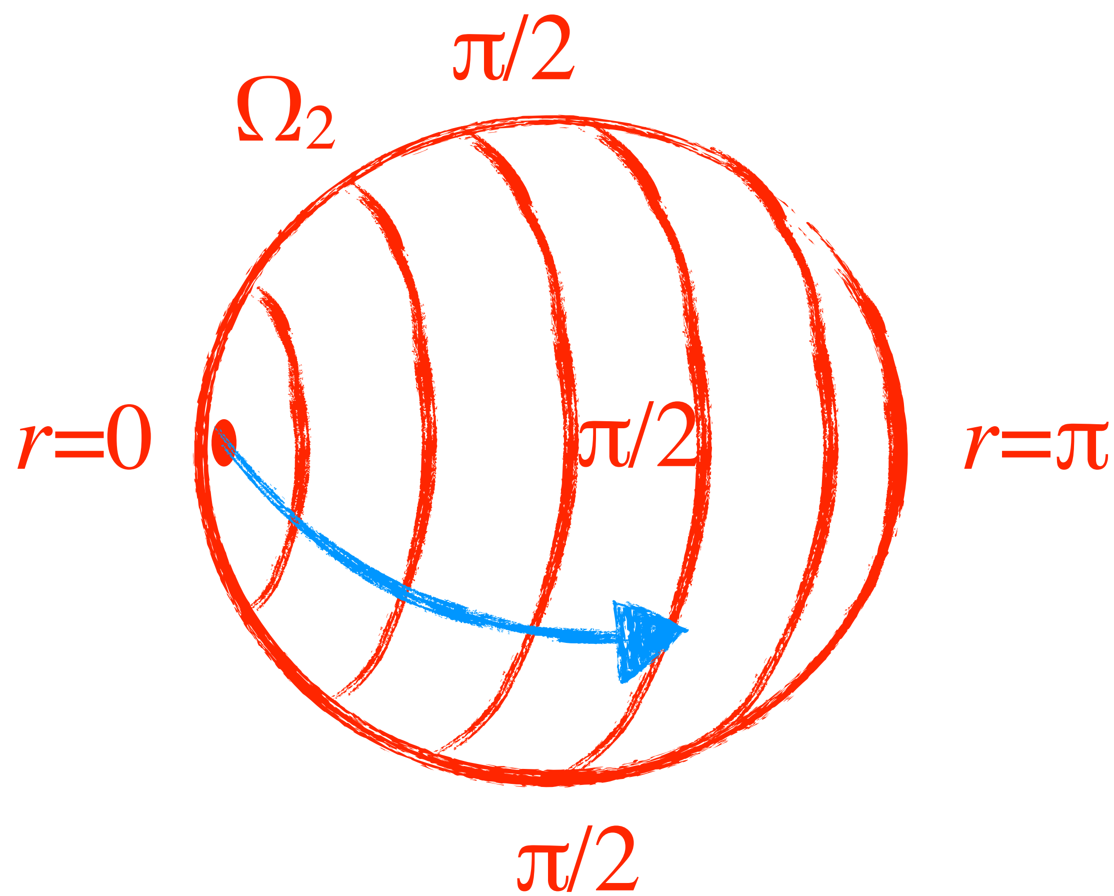


2-spheres

- A 2-D astronomer looks away from our own position on the 2-sphere and is surrounded by 1-spheres.
- The 1-spheres grow more slowly with distance and then get smaller.
- Use r to measure an angle,

  furthest we can see is $r = \pi$
- The metric in our notation is

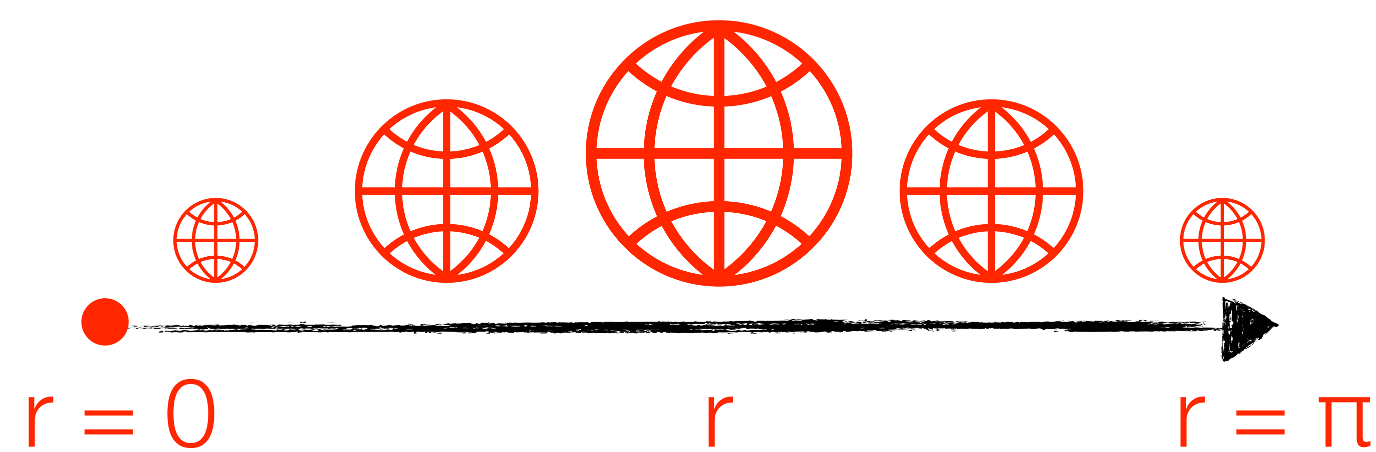
$$\mathrm{ds}^2 = \mathrm{dr}^2 + \sin^2 \mathrm{r}\, \mathrm{d}\Omega_1^2 = \mathrm{d}\Omega_2^2$$



3-spheres

- A 3-sphere is a 3-d space which is homogeneous and if you go in any direction you come back to yourself - ***hard to think about***
- Look into space at different distances and you see 2-spheres around you. Look further and further and the 2 spheres start to get smaller - ***hard to visualise***.
- They are series of nested 2-spheres that grow and collapse as r increases.
- The metric is

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\mathrm{r}^2 + \sin^2 \mathrm{r}\, \mathrm{d}\Omega_2^2$$

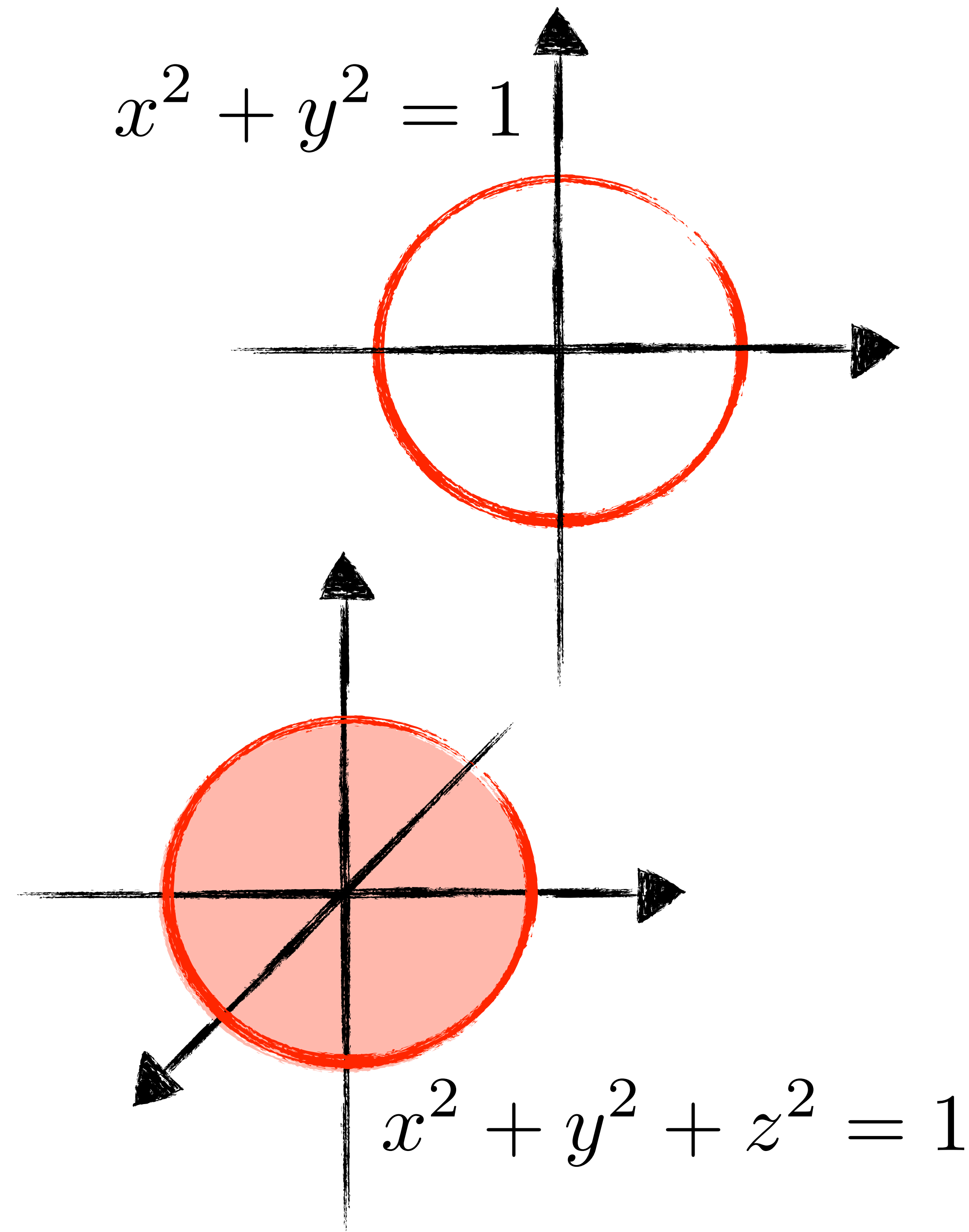


Another way to view spheres - embedding

- We can embed a 1-sphere (circle) in 2-D, and a 2-sphere in 3-D.
- The extra dimension is just a trick - the previous metrics still apply
- To construct a 3-sphere in this way you have to embed it in a 4-D space.

$$x^2 + y^2 + z^2 + w^2 = 1$$

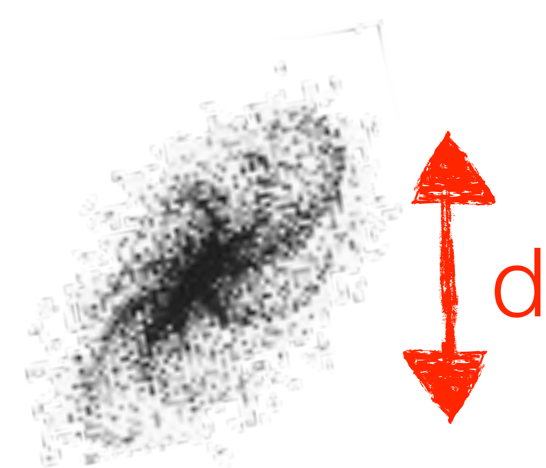
- Clearly not sensible coordinates for traversing the space.



Testing the geometry

- How could you tell if you live on a sphere rather than an infinite flat plane?
- Use telescopes to determine the distances to objects (using tricks like the Hubble law).

- Look at how bright a galaxy is (assume all galaxies are the same intrinsic brightness)
- Look at how much angle they subtend on the sky? (assume all galaxies are the same intrinsic size)



Angular diameter - flat space

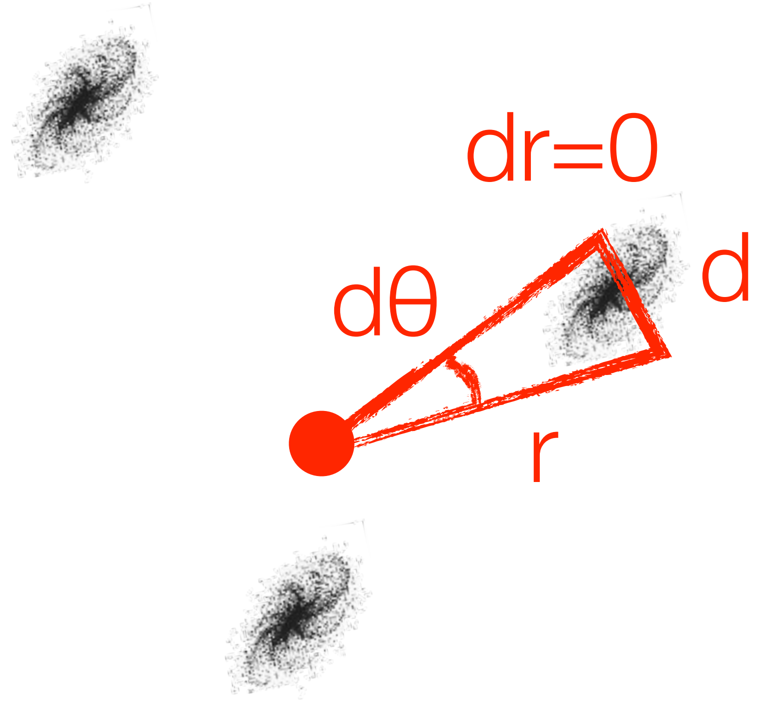
- In flat 2-D space we use the metric to give us

$$\mathrm{ds}^2 = \mathrm{r}^2\mathrm{d}\theta^2 = \mathrm{d}^2$$

- and therefore

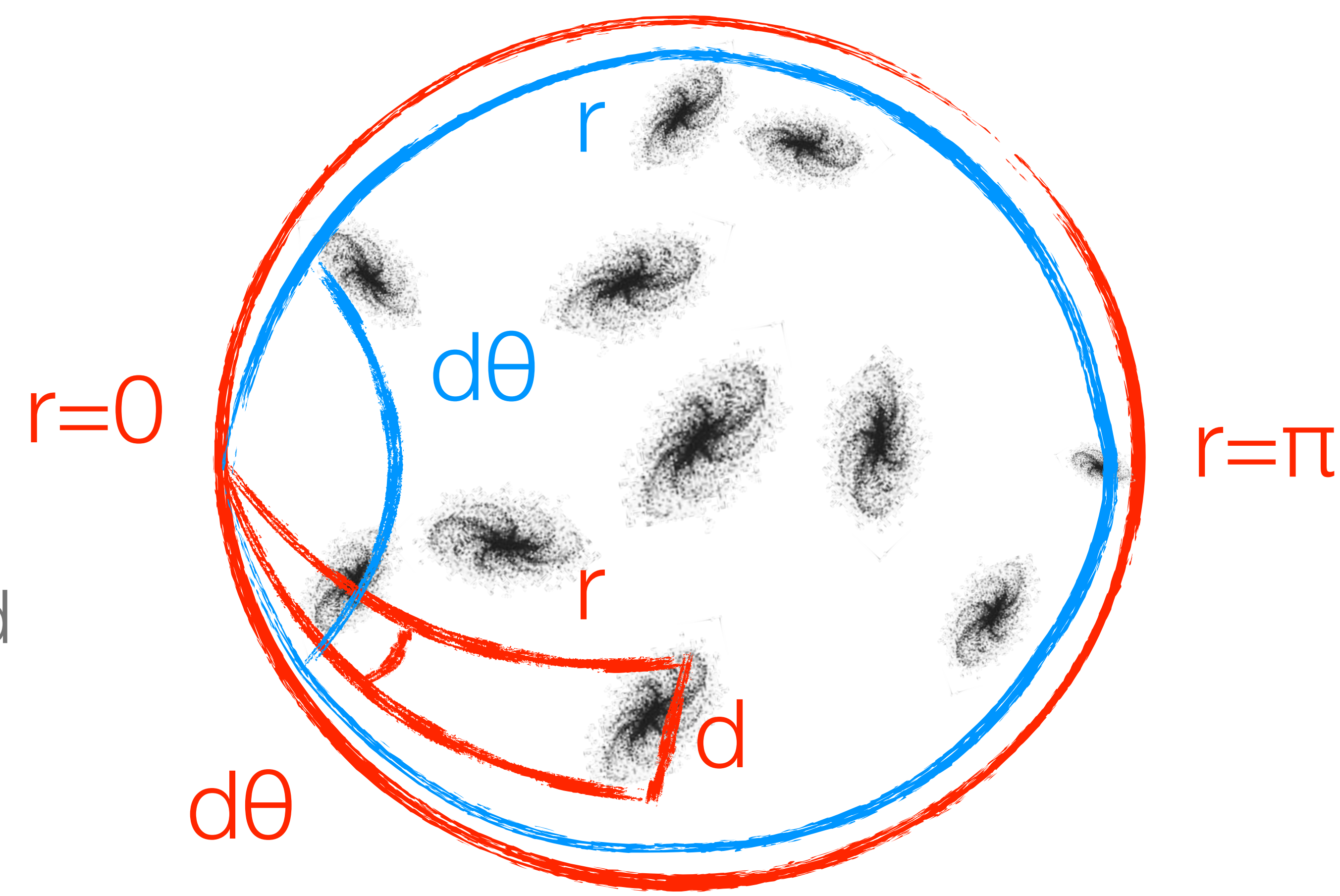
$$\mathrm{d}\theta = \frac{\mathrm{d}}{\mathrm{r}}$$

- As you might expect



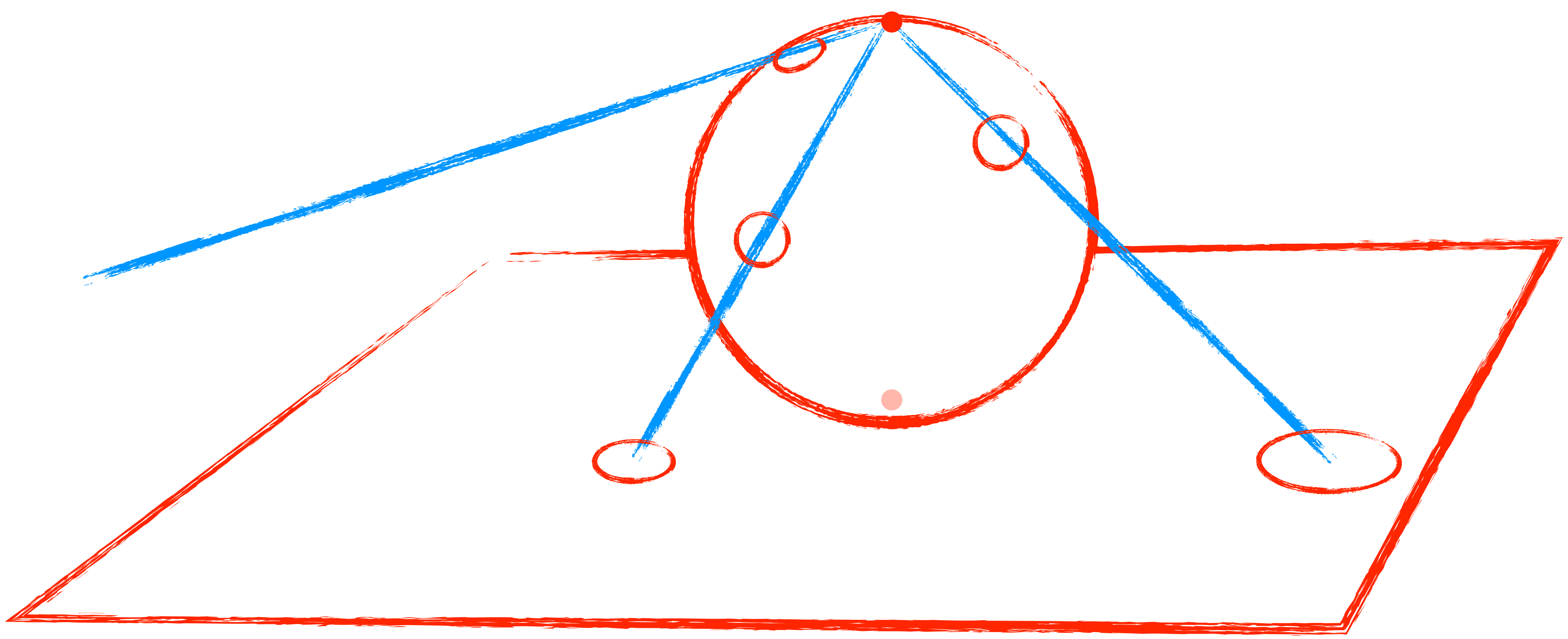
Angular diameter - spherical geometry

- In curved 2-D space we have $\mathrm{ds}^2 = \sin^2 \mathrm{r} \mathrm{d}\theta^2 = \mathrm{d}^2$
- So $\mathrm{d}\theta = \dfrac{d}{\sin r}$
- Since $\sin r \leq r$, the angle is larger than in the flat case.
- More distant galaxies will of course be dimmer (further away) but would ***appear larger than closer ones***.
- Could also count the number of galaxies as a function of r



Stereographic projection

- Every point on the sphere can be mapped to a point on the plane
- Just a way of representing it so that you can draw it on a plane (it distorts it)
- You can do the same thing for a 3-sphere (I can't draw it).

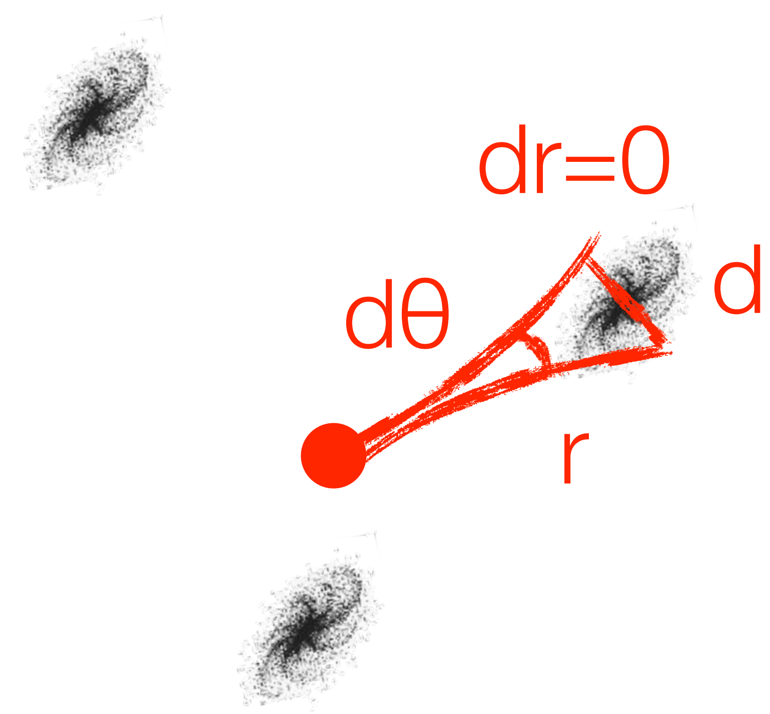


Hyperbolic space

- Harder to imagine that the sphere
- Instead of $\mathrm{d}\Omega$ we use $\mathrm{d}\mathscr{H}$ - so for 2-D space $\mathrm{d}\mathscr{H}_2{}^2 = \mathrm{d}r^2 + \sinh^2 r\,\mathrm{d}\Omega$
- Compare $$\sinh r = \frac{e^{ir} - e^{-ir}}{2i}, \quad \sin r = \frac{e^{r} - e^{-r}}{2}$$
- For very large r $\sinh r$ is dominated by e^r and grows exponentially but for $\sin r$ large r goes back to zero
- In hyperbolic space you still see circles around you in 2-D but the circles grow exponentially
- In 3-D you get $$\mathrm{d}\mathcal{H}_3^2 = \mathrm{d}r^2 + \sinh^2 r\,\mathrm{d}\Omega_2^2$$

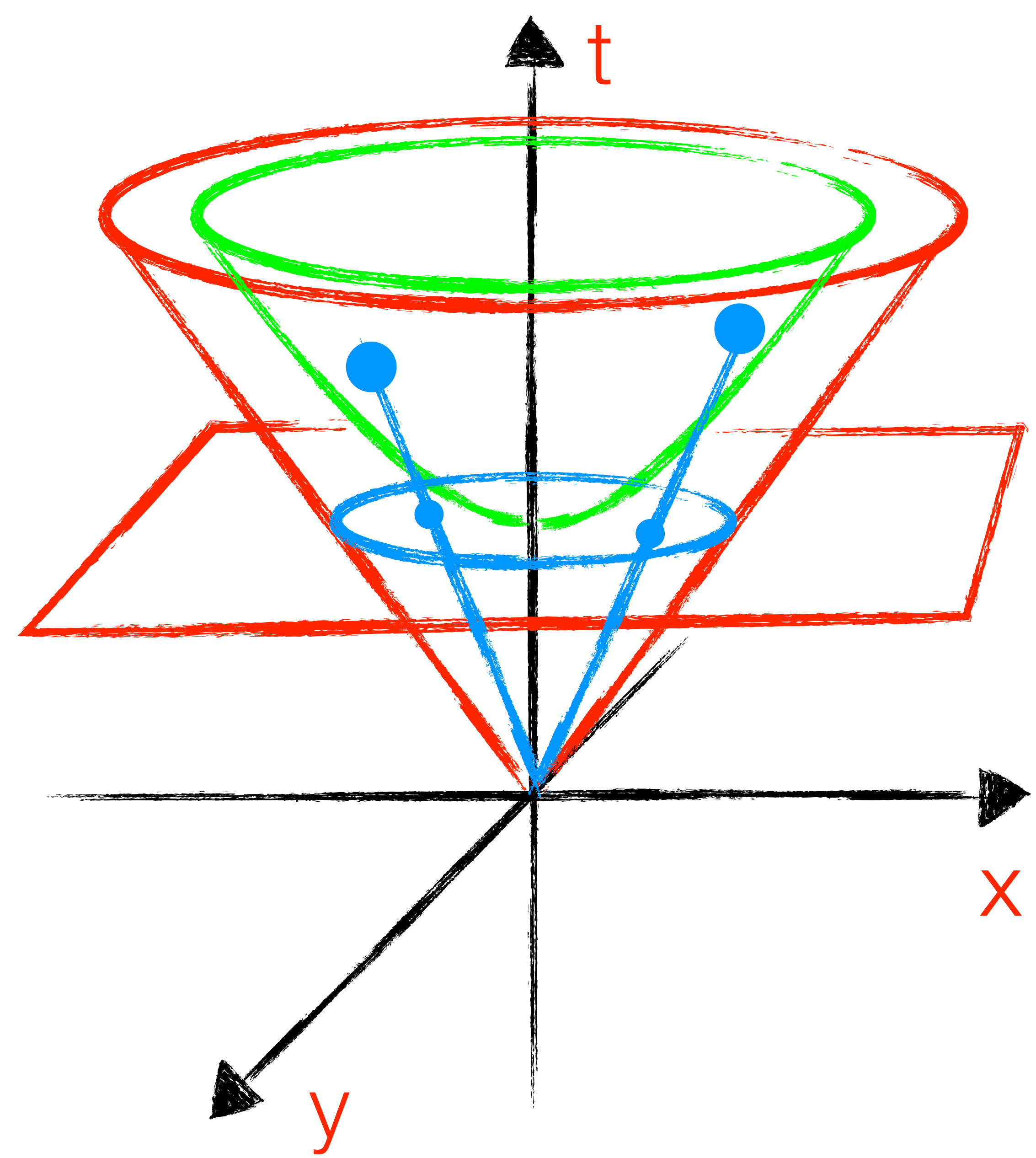
Hyperbolic space

- In hyperbolic 2-D space we have $\mathrm{d}s^2 = \sinh^2 r \, \mathrm{d}\theta^2 = d^2$
- and therefore $\theta = \dfrac{d}{\sinh r}$
- For large r we find that $\theta \approx \dfrac{2d}{e^r}$
- So the angle shrinks fast and the number of galaxies grows fast
- You would notice that distant galaxies look too small and there would be very many of them.

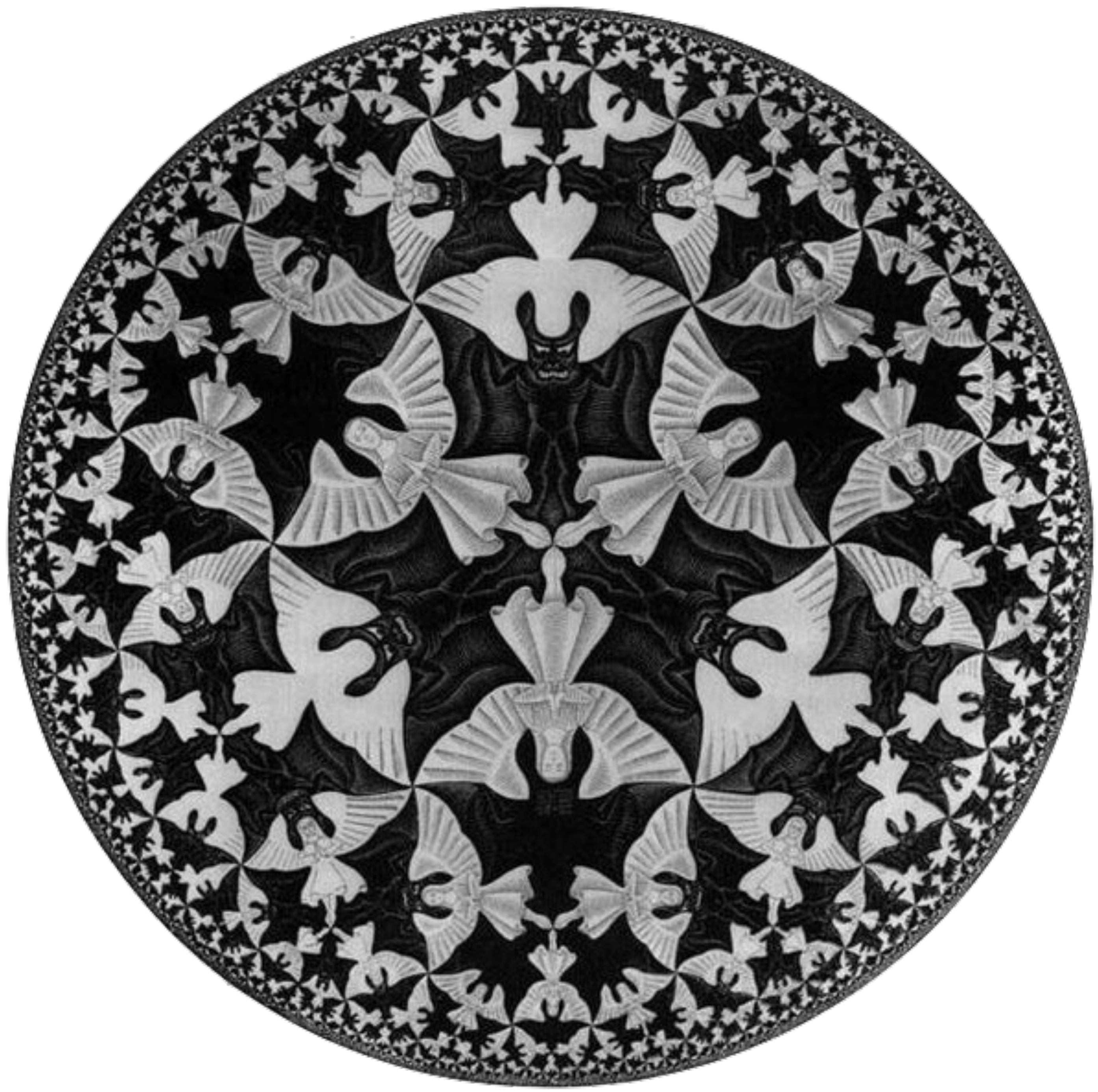


Hyperbolic projection

- An embedded 2-sphere can be written as
$$x^2 + y^2 + z^2 = 1$$
- For a 2-D hyperbolic space the equivalent is (t is a dummy coordinate - defines a hyperboloid
$$t^2 - x^2 - y^2 = 1$$
- Doesn't look like every point is equivalent on the hyperboloid but it is
- Lots of distortion in this projection
- Just as a sphere has a radius so does the hyperboloid



Hyperbolic projection



Metrics with the scale factor

- The metric of an ordinary sphere of radius a is

$$\mathrm{d}s^2 = a^2 \left(\mathrm{d}r^2 + \sin^2 r \, \mathrm{d}\Omega_1^2\right)$$

- Same thing for the hyperbolic geometry

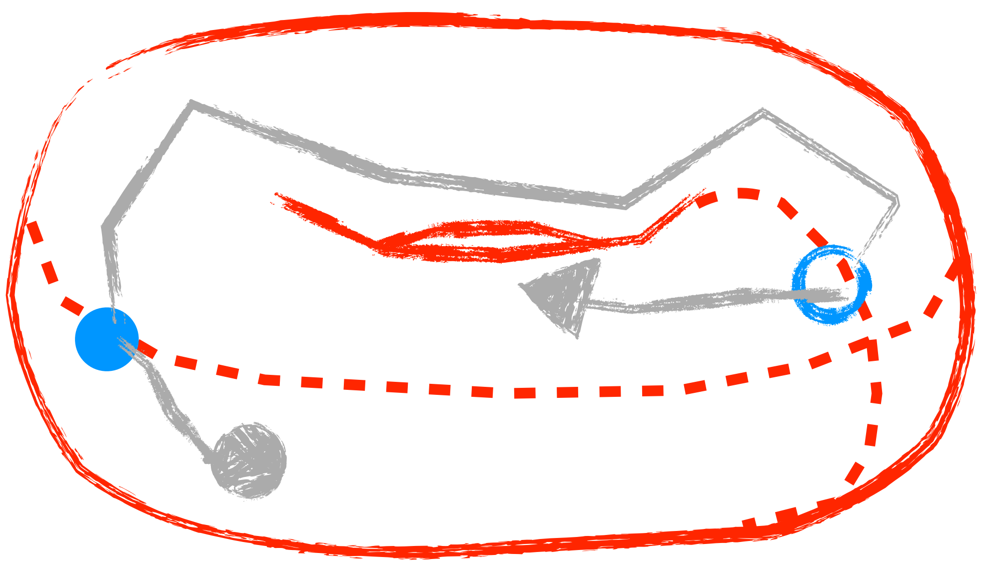
$$\mathrm{d}s^2 = a^2 \left(\mathrm{d}r^2 + \sinh^2 r \, \mathrm{d}\mathcal{H}_1^2\right)$$

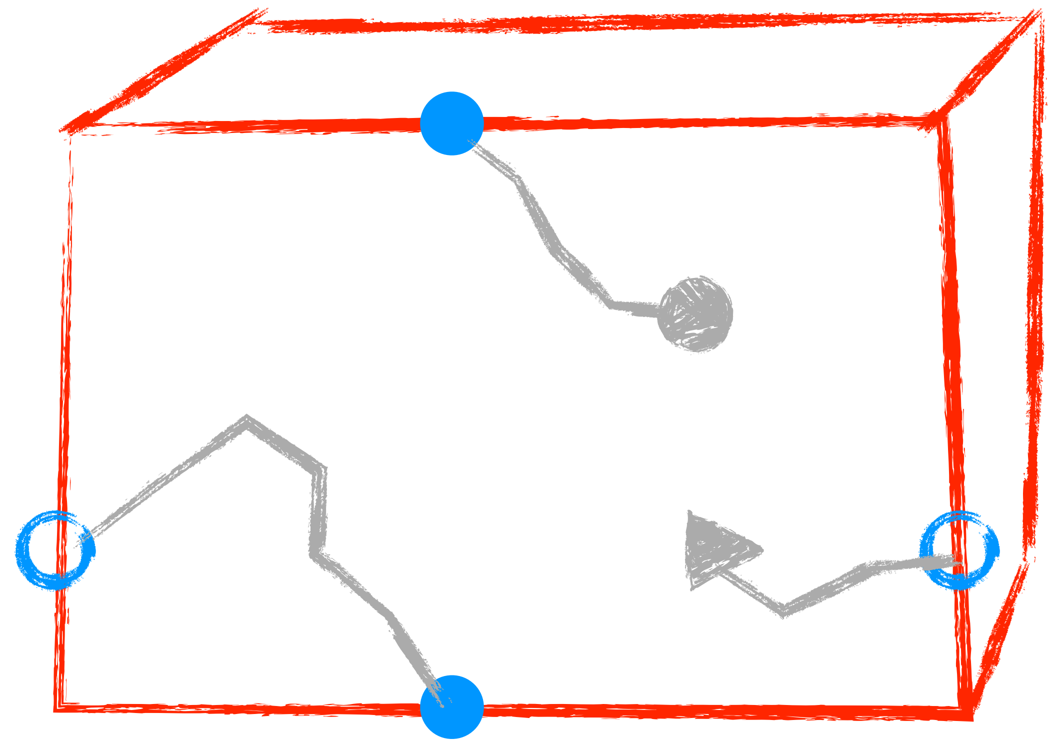
- It is assumed that we live in either the flat, +ve, or -ve curved universe.

- It currently looks flat out to distances that we can detect - the nearby distances are small compared to the radius of curvature.

Toroidal geometry

- Flat but periodic and homogeneous
- You would be able to see yourself in different directions
- As long as it was big enough then we might mistake it for infinite flat space
- Can do this in 3-D (maybe like portal?)
- A 1-D torus is a circle (1-sphere, Ω_1)





Space-time

- In ordinary flat Minkowski space (special relativity)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

- A photon travels on a null geodesic so $\mathrm{d}s^2 = 0$ and $\mathrm{d}x = \pm\,\mathrm{d}t$ along the x-axis

- We'll keep time as it is and substitute one of our geometries into the spatial part - also include the scale factor $a(t)$

- For the 2-sphere $$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\Omega_2^2$$

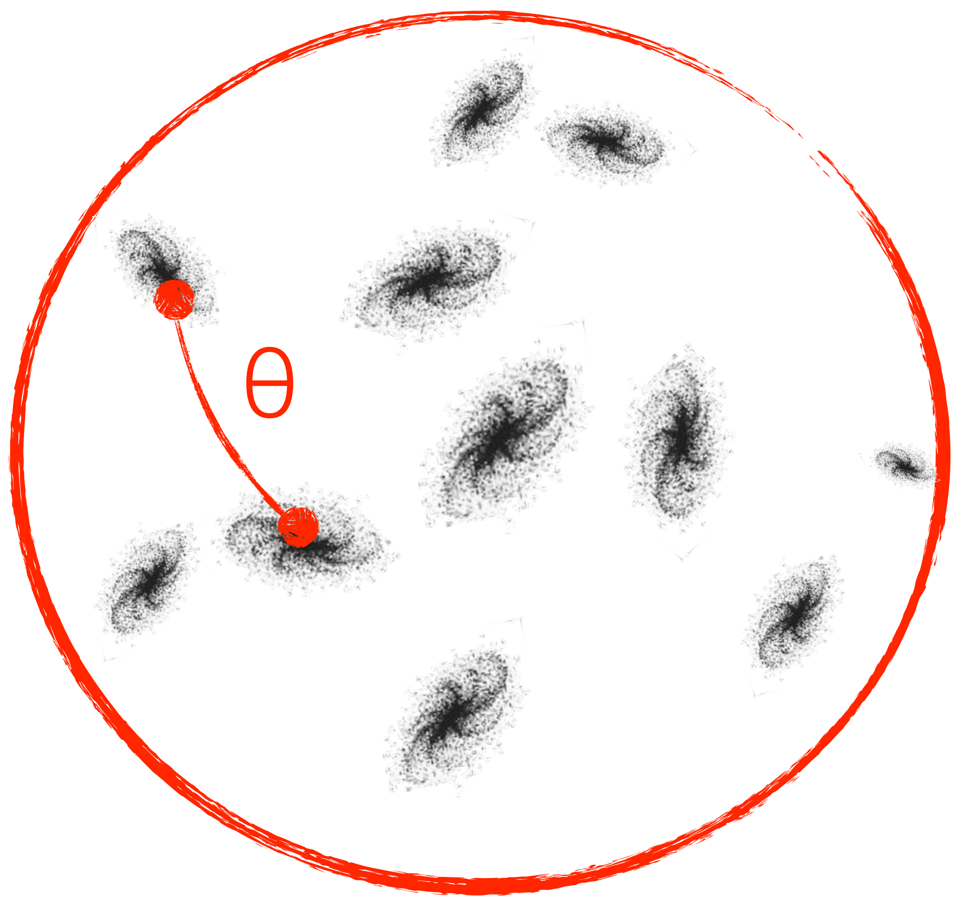
- the same for a 3-D universe (change Ω_2 to Ω_3)

Space-time

- If our metric is $$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\theta^2$$

- What's the distance and relative velocity between 2 galaxies separated by θ?

$$d = a\theta, \qquad \dot{d} = \dot{a}\theta$$



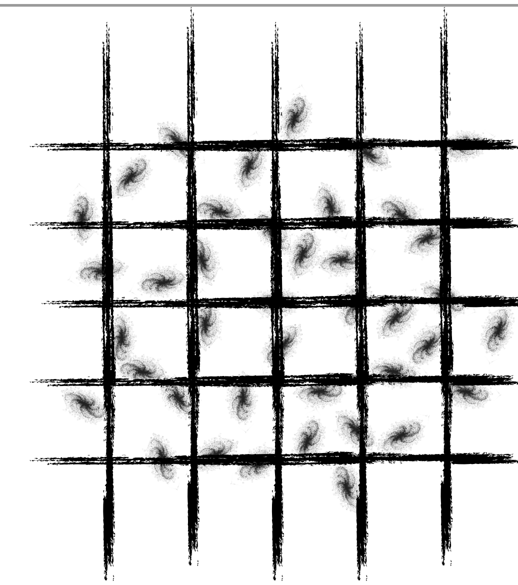
- Then again get the Hubble law

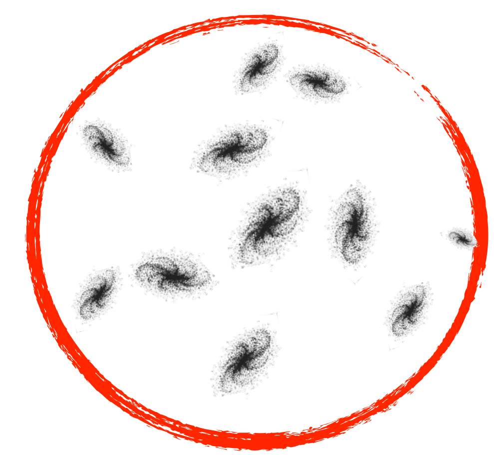
$$H(t) = \frac{\dot{a}}{a} = \frac{\dot{d}}{d}$$

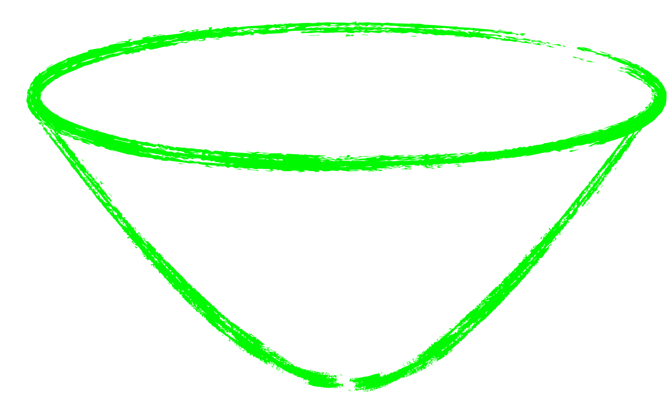
- Just the same as before (and true for all geometries)

The 3 cases

- Flat $$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$$
- Spherical $$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\Omega_3^2$$
- Hyperbolic $$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\mathcal{H}_3^2$$
- All satisfy Hubble's law
- Next stage is to use GR to generate equations of motion for a(t) - the same equations for the Newtonian 3 energy cases.







Thanks for your attention