Astronomy 1
Stellar Physics 1

Dr Morag Casey

Room 230b, Kelvin Building
m.casey@physics.gla.ac.uk

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Things you need to know. . .

11 lectures

- Dr Martin Hendry will take the second part of this course: ‘Stellar Physics II’.

Course Textbook. . .

- An Introduction to Modern Astrophysics

The important results are *numbered*. 
Things you need to know... 

Handouts

- Slides *with blanks* for the whole course are available *now* on Moodle.

Example Questions

- These come from the *A1 Problems Handbook*.
- Will go through some of them during lectures of the problems to illustrate the coursework.
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Measuring the Stars: Part I

What is a star?

- The (temporary) triumph of nuclear physics over gravity.
- What would the Universe be like without the pressure generated by nuclear reactions?
How do stars form?

- A cloud of gas collapses under its own gravity.
- As the cloud contracts, it heats up.
- The increase in temperature kickstarts nuclear reactions.
- These generate *pressure* which balances gravity and prevents further collapse.
Measuring the Stars: Part I

Black Holes

- If only gravity acts, the cloud would eventually collapse to form a black hole.
- The outward pressure from thermonuclear reactions stop the collapse at some radius.
- There is *equilibrium* for some time.
- Eventually, all the nuclear fuel is used up and the collapse continues.
  - See Stellar Physics II.
Measuring the Stars: Part I

Some Questions For SPI

- How big are stars?
- How luminous are stars?
- Are all stars the same age?
- What is the most important characteristic of a star?
- What keeps a star in temporary equilibrium?
- How are stars born?
- How do stars change as they age?
- How do stars die?
- Are stars fixed in space?
- How are stars distributed in space?
Measuring the Stars: Part I

How big are stars?

- Let’s start with the Sun.

\[
R_{\odot} \simeq 110 \times R_{\oplus} = 6.960 \times 10^8 \text{ m}
\]

- \(R_{\odot}\) is the radius of the Sun.
- \(R_{\oplus}\) is the radius of the Earth.
Measuring the Stars: Part I

How do we measure the radius?

- We can get $R$ from the *angular diameter* of the Sun if we know how far away it is.
- We need to know the distance $D$.
- For *small* angles:

$$\alpha \approx \frac{2R}{D} \quad (1)$$

- The units of $\alpha$ are *radians*. 
Measuring the Stars: Part I

**How big is the Sun?**

- Use equation 1.
- \( \alpha \approx 0.52^\circ \).
- Convert this to *radians*.
- \( D = 1 \text{ A.U.} = 1.496 \times 10^{11} \text{ m} \)
- So, \( R = 6.96 \times 10^8 \text{ m} \).
Measuring the Stars: Part I

The next ‘biggest’ star…

- Betelgeuse.
- $\alpha = 0.000014^\circ$

Other stars?

- We must deduce their size *indirectly*.
- This is because they are *very, very far away*.
Measuring the Stars: Part I

Other Stars

<table>
<thead>
<tr>
<th>Star</th>
<th>Radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>Betelgeuse</td>
<td>$\sim 320 R_\odot$</td>
</tr>
<tr>
<td>Aldebaran</td>
<td>$22 R_\odot$</td>
</tr>
<tr>
<td>Arcturus</td>
<td>$12 R_\odot$</td>
</tr>
<tr>
<td>Mira</td>
<td>$210 R_\odot$</td>
</tr>
</tbody>
</table>

- These are all much larger than the Sun; some are much smaller.
- There is a range of radii:

$$10^{-2} R_\odot < R < 10^3 R_\odot$$
Aside: How Empty is the Universe?

- Relative to their size, stars are the most isolated structures in the Universe.
- Diameter of the Sun $\sim 10^9 \, m$.
- Distance to the next nearest star $\sim 10^{16} \, m$.
- This is a factor of $10^7$ relative to distance (or $10^{21}$ relative to volume).
- If you were to model stars as grains of sand ($\sim 1 \, mm$), you would need to place the grains $\sim 10 \, km$ apart.
Stellar Masses

- We look at the dynamics of binary systems to determine mass.
- We find:

\[ 10^{-1} M_\odot < M < 50 M_\odot \]

- Why are there no stars with \( M \gg 50 M_\odot \)?
- Why are there no stars with \( M \ll 0.1 M_\odot \)?
Measuring the Stars: Part I

Stellar Luminosity

- How *bright* is a star?
- Define *luminosity*:

  **Luminosity is the energy output by a star per unit time**

- We measure luminosity in *Joules per second*: \( \text{Js}^{-1} \) or *Watts*: \( W \).
Lightbulbs and Luminosity

- Lightbulbs are rated in the units of power (Watts), e.g.: 60 W.
- You can buy energy saving lightbulbs that use less energy but appear just as bright, e.g.: 11 W.
- Incandescent light bulbs emit light at many wavelengths in the visible region but most of the ‘power’ is emitted in the infra-red region.
- Energy-saving lightbulbs emit light at only a few wavelengths in the visible region and have very little emission in the infra-red.
  - They appear as bright as incandescent bulbs because the eye cannot see infra-red.
  - Therefore, less power is needed to provide the same amount of visible light.
Measuring the Stars: Part I

Flux

- We measure stellar luminosity by observing a star’s flux and estimating its distance.

- Definition of flux:

**Flux is the power from a star passing through a unit area perpendicular to the direction of the star.**

- The Sun’s flux at the Earth is 1372 Wm$^{-2}$.
- This is known as the Solar Constant.
Measuring the Stars: Part I

Flux and Luminosity

- If a star radiates equally in all directions (isotropically), the flux must be the same over all parts of a sphere of radius $D$ centred on the star.

$$\text{Power through sphere} = \text{surface area of sphere} \times \text{flux}$$

$$= \ 4\pi D^2 \times F$$

$$= \ Luminosity \ L$$

- This is required by energy conservation.
Measuring the Stars: Part I

Flux and Luminosity

\[ L = 4\pi D^2 \times F \]  
(2)

For the Sun...

\[ F = 1372 \text{ Wm}^{-2} \]

and \[ D = 1.496 \times 10^{11} \text{ m} \]

\[ \Rightarrow L_{\odot} = 3.86 \times 10^{26} \text{ W} \]
Luminosities of Other Stars

\[ 10^{-4} L_\odot < L < 10^6 L_\odot \]

- This is a very large range – why?
- If we compare the least luminous star to a 100 W incandescent lightbulb, this range tells us that the most luminous star is the entire world’s power output.
Measuring the Stars: Part I

The Earth at Night

http://veimages.gsfc.nasa.gov/1438/earth_lights.gif
Measuring the Stars: Part II

Stellar Temperatures

- We can determine the surface temperature of a star using the relationship between colour and temperature.
- Stars are good approximations to black-bodies.

Some Laws

- Planck Radiation Law.
- Wien’s Displacement Law.
- Stefan-Boltzmann Law.
Planck Radiation Law

- Relates relative intensity to colour (i.e.: frequency or wavelength).

Fig. 18.2. Spectral intensity distribution of Planck’s black-body radiation as a function of wavelength for different temperatures. The maximum of the intensity shifts to shorter wavelengths as the black-body temperature increases.
Measuring the Stars: Part II

Planck Radiation Law

- The maximum in the Planck curve shifts to shorter wavelengths (higher frequencies) for higher temperatures.
- This means that a blue star like Sirius A is hotter than a red star like Betelgeuse.
- You can measure the surface temperature of a star by matching its spectrum to a Planck curve.
- This allows you to find the maximum wavelength.
Measuring the Stars: Part II

Black-body Radiation and the Planck Radiation Law

- Any object that totally absorbs all the radiation incident on it is called a **black-body**.
- If it absorbs radiation it will **heat up**.
- Therefore, it must **re-radiate** energy to **cool down**.
- Black-bodies must, therefore, be perfect emitters of radiation as well as perfect absorbers.
  - If a black-body is sufficiently hot, it will glow like a lightbulb (and not look black).
Black-body Radiation and the Planck Radiation Law

• The spectrum of radiation from a perfect black-body must depend only on the source’s temperature.
• Otherwise, some sources might be ‘more perfect’ than others.
• It must also have a continuous spectrum and follow a curve described by the Planck Radiation Law.
Black-body Radiation and the Planck Radiation Law

- Explicitly, the brightness of the surface of a black-body is given as:

\[ B(\nu) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1} \]

- You don’t need to memorise this!
- \( \nu \) is the frequency of the radiation.
  - Think about the last diagram.
Measuring the Stars: Part II

Black-body Radiation: Some Important Points

- ‘Continuous spectrum’: a smooth curve covering all frequencies.
  - Objects that emit at only a few wavelengths (‘discrete spectrum’) will not have Planck curves.

- All warm objects that absorb radiation also emit radiation.
  - Good absorbers are also good emitters.
Measuring the Stars: Part II

Black-body Radiation: Limiting Cases

- Low temperatures: Wien’s Law.
  - \( h\nu \gg kT \)
  - \( B(\nu) \approx 2h\nu^{-3}c^{-2}\exp(-h\nu/kT) \)
- High temperatures: Rayleigh-Jeans Law.
  - \( h\nu \ll kT \)
  - \( B(\nu) \sim 2kT(\nu/c)^2 \)
Measuring the Stars: Part II

Wien’s Displacement Law

- Once you have the maximum wavelength for a star, you can use Wien’s Displacement Law to find the star’s temperature.

\[ T \lambda_{\text{max}} = 0.0029 \text{ mK} \]  

\[ (3) \]

- \( T \) is the temperature of the body in Kelvin.
- \( \lambda_{\text{max}} \) is the peak emission wavelength.
- The units of the constant are metre Kelvin and not milliKelvin.
The Sun

- The Sun has a peak temperature of $T_\odot = 5780 \text{ K}$.
  - This is why it appears yellowish to us.
  - Compared to other stars, it is quite cool.

Temperatures of Other Stars

$$0.3T_\odot < T < 20T_\odot$$
Measuring the Stars: Part II

Stefan-Boltzmann Law

- The total power emitted by a star at all frequencies, per unit area, can be got by \textit{integrating} under a Planck curve.

\[ P_{\text{total}} = \frac{\pi^2 k^4 T^4}{60 \hbar^3 c^2} \]

- \( k = 1.381 \times 10^{-23} \text{ JK}^{-1} \) is Boltzmann’s constant.
- \( T \) is the temperature of the star in Kelvin.
- \( \hbar = \frac{h}{2\pi} \) and \( h = 6.626 \times 10^{-34} \text{ Js} \) is Planck’s constant.
- \( c = 3 \times 10^8 \text{ ms}^{-1} \) is the speed of light.
Stefan-Boltzmann Law

- This law can be written more simply:

\[ P_{total} = \sigma T^4 \]  \hspace{1cm} (4)

- \( \sigma = 5.67 \times 10^{-8} \text{ W}\text{m}^{-2}\text{K}^{-4} \) is the Stefan-Boltzmann constant.

Relate Temperature to Luminosity:

\[ L = 4\pi R^2 \sigma T^4 \]  \hspace{1cm} (5)
Prediction

- A star can vary in a number of ways.
  - Radius.
  - Mass.
  - Luminosity.
  - Surface Temperature.

- We predict that not all combinations of these four parameters are possible.
  - Small, hot stars are generally dimmer than large cool stars.
Measuring the Stars: Part II

Stefan-Boltzmann and other Laws

- Stefan-Boltzmann relates $L$, $R$ and $T$.
- We will see later that most stars also obey the *mass-luminosity* relation.

Example Question

- Question 4.9 from the Tutorial Book.
Stellar Types & Classification: Part I

Classifying Stars

- Stefan-Boltzmann’s law (equation 5) helps us classify stars.
  - If you choose *any two* of $L$, $R$ or $T$, you can find out the third.
- Take logarithms:

$$\log_{10} L = 4 \log_{10} T + 2 \log_{10} R + \text{constant}$$

- Plot a graph of $y = \log_{10} L$ against $x = -\log_{10} T$
  - Will explain later (‘Classification of Stars’) why we choose a negative sign for the x-axis.
Stellar Types & Classification: Part I

Plot of log $L$ against log $T$

- Notice that we are plotting *logarithms* of the values.
- Notice that we are plotting log $T$ *backwards*.
- Where do real stars fall on this plot?
Stellar Types & Classification: Part I

Hertzsprung and Russell

- Two astronomers independently devised ways to determine the luminosity of stars, $L$.
  - Ejnar HERTZSPRUNG (Denmark 1873 – 1967)
  - Henry Norris RUSSELL (USA, 1877 – 1957)
- They constructed the plot now known as an *Hertzsprung-Russell Diagram*.
  - We’ll call this an *HR-Diagram* for short.
Stellar Types & Classification: Part I

Hertzsprung-Russell Diagram
Stellar Types & Classification: Part I

Hertzsprung-Russell Diagram

- Stars are found to cluster in distinct areas.
  - 80% to 90% of stars lie on a strip called the Main Sequence.
  - There are other branches of stars: White Dwarfs, Giants and Supergiants.
  - Notice that temperature (x-axis) decreases as we go from left to right.

We use the HR-diagram to understand stellar evolution
Stellar Types & Classification: Part I

A real HR-Diagram

- 41,453 stars.
- Hipparcos satellite.
- Notice the Main Sequence and Giants branch.
- Axes are magnitude against colour index.
  - These are related to luminosity and temperature.

http://www.rssd.esa.int/index.php?project=HIPPARCOS
Stellar Types & Classification: Part I

Interpretation of the HR-Diagram

- We need to make some assumptions first:
  - HR-diagram samples a wide range of stellar ages.
  - As a star evolves, only certain combinations of $L$ and $T$ are allowed.
Stellar Types & Classification: Part I

What do these assumptions tell us?

- Stars follow ‘tracks’ on the HR-diagram during their lifetimes.
- Clusters in the HR-diagram are stars at similar stages of their lives.
- We are looking at a *snapshot* of the stellar population.
- The number of stars in each part of the HR-diagram is proportional to the duration of that stage of evolution.
Stellar Types & Classification: Part I

Problems?

- Some stars may *not* pass through certain stages.
- Some stages may be very hard to observe:
  - Take place over a very short period of time.
  - Involve very dim stars.
Individual Stars?

- Generally we don’t see individual stars evolving.
  - Stellar evolution takes too long.
- We do see *variable stars* and *supernovae*.
Stellar Types & Classification: Part I

The Main Sequence

- Consists of hydrogen burning stars like the Sun.
- 80% to 90% of stars in the HR-diagram are on the Main Sequence.
  - This means that 80% to 90% of stars spend their lives burning hydrogen.
- Stars do not evolve along the Main Sequence.
  - They drop on to it when hydrogen-burning starts and move off it when they run out of hydrogen.
The Main Sequence: Mass & Luminosity

- The location of a star on the main sequence is mainly determined by its mass.
  - Later on we’ll prove that $L \propto M^\alpha$.
- We can calculate a star’s lifetime on the main sequence:
  $$\text{Lifetime} = \frac{\text{energy available}}{\text{luminosity}} \propto \frac{M}{M^\alpha}$$
- The most massive stars spend the least amount of time on the main sequence.
  - Will come back to this in more detail later.
The Main Sequence: Limits

- The Main Sequence has a finite width due to small movements (left to right) of stars as they age.
- There are two cut-offs:
  - Top: very massive stars are also extremely luminous; this is the Eddington Limit.
  - Bottom: very low mass stars are not hot enough to begin nuclear reactions; they don’t shine.
We think of the HR-diagram as a plot of \( \log L \) against \( \log T \).

Can plot other things on these axes.
- Remember the diagram from the Hipparcos satellite.

Can use Absolute Magnitude, \( M \) to measure Luminosity, \( L \).

Remember how we relate *apparent magnitude* to *measured flux*:

\[
m_1 - m_2 = -2.5 \log_{10} \frac{F_1}{F_2}
\]  

(6)
Stellar Types & Classification: Part I

HR-Diagram: Technical Notes

- Absolute magnitude is the same as apparent magnitude at a distance of 10 parsecs.
- \( F = \frac{L}{4\pi R^2} \).
- So, if \( R \) is fixed, flux is proportional to luminosity.
- Therefore:

\[
M_1 - M_2 = -2.5 \log_{10} \frac{L_1}{L_2}
\]  
(7)
Stellar Types & Classification: Part I

HR-Diagram: Technical Notes

• We know that the absolute magnitude of the Sun is $M_\odot = 4.72$
• Therefore:

$$M = -2.5 \log_{10} L + 2.5 \log_{10} L_\odot + 4.72$$

• The last two terms are constant so we can say that:

$$M \propto -\log_{10} L$$
Stellar Types & Classification: Part I

HR-Diagram: Technical Notes

- On the x-axis of the Hipparcos HR-diagram we plotted *colour index* instead of \( \log_{10} T \).
- Colour Index tells you what colour an object is.
- And the colour of something tells you about its temperature.
- If you heat something up, it glows red, then yellow and, eventually blue and then white hot.
- The Sun, for example, is yellow because it is quite cool; a star like Sirius is blue-white because it is much hotter.
Red Giants

- Red giants are *cool* and *luminous*.
- Once a star on the Main Sequence has burned all the hydrogen in its core it evolves off the Main Sequence and becomes a Red Giant.
- The core contracts and heats up.
- The outer layers expand and cool.
  - \( L \sim 1000 \, L_\odot \) (due to high surface area).
  - \( T_{\text{effective}} \sim 4000 \, K \).
  - \( R \sim 70 \, R_\odot \).
Stellar Types & Classification: Part I

Red Giants

- Red Giants are found in the top right hand corner of the HR-diagram.
- They are in an advanced stage of nuclear burning (see later).
- This is a relatively brief stage.
  - Far fewer Red Giants are seen on the HR-diagram compared to stars in the Main Sequence.
Stellar Types & Classification: Part I

White Dwarfs

- Take a star that is not too massive. e.g.: the Sun.
- Eventually, it will become a Red Giant.
- Then, just before all its fuel is used up, it will shed up to 90% of its mass causing the formation of a planetary nebula.
- It will leave behind a hot, dense, small remnant called a White Dwarf.
  - $L \sim 0.01 L_{\odot}$ (none visible to the naked eye).
  - $T_{\text{effective}} \sim 16,000 \, K$.
  - $R \sim R_{\odot}/70$ (about earth-sized).
  - $Density \sim 10^{10} \, kg \, m^{-3}$.
- See also SPII.
White Dwarfs: Example

- Sirius B is the closest White Dwarf star.
- It is the companion star to Sirius A.
- Its existence was predicted from the wobble in the motion of Sirius A.
White Dwarfs: End State

- White Dwarfs don’t burn much nuclear fuel.
- They are just hot spheres cooling slowly.
- In about 5 billion years the Sun will become a White Dwarf.
  - This will happen after the Red Giant phase.
Variable Stars

• By ‘variable’ we mean that the star’s flux changes with time.
• We observe this by measuring apparent magnitude.
• It can be a real variation (because the star itself changes) or an apparent variation (e.g.: something moves in front of the star and blocks it from us).
• The variation can be *irregular* or *regular*. 
Irregular Variable Stars: Novae

- Novae are sometimes called *cataclysmic variables*.
- They flare in brightness irregularly, sometimes only once.
- Their luminosity may increase by a factor of 1000 over a period of one week and then decay slowly.
- All novae exist in *binary systems*.
Stellar Types & Classification: Part I

Irregular Variable Stars: Novae

- In a binary system, material is transferred from one star to the other.
- This causes a bright outburst.
- See also *Introduction to Cosmology* for information about Type I supernovae.
Irregular Variable Stars: T Tauri Stars

- This is a class of irregular variable stars named after their prototype \textit{T Tauri} in the constellation of Taurus.
- Their luminosity increases by a factor of three in a few days.
- These are very young stars (protostars).
- They are powered by gravitational energy as they contract and move towards the Main Sequence.
- See later and also \textit{SPII}.
Stellar Types & Classification: Part I

Irregular Variable Stars: Type II Supernovae

- The end state of a very massive star.
- After nuclear fuel is exhausted, the core collapses and the outer layers are blown off.
- A small dense neutron star remains, surrounded by expanding spheres of circumstellar matter.
Stellar Types & Classification: Part I

Irregular Variable Stars: Supernova 1987A

Supernova 1987A Rings

Hubble Space Telescope
Wide Field Planetary Camera 2
Regular Variable Stars

- Regular variables exhibit flux variations that follow a regular repeating pattern.
- Examples are: RR Lyrae, Miras, Cepheids.
- Look more closely at *Cepheids*.
Regular Variable Stars: Cepheids

- Named after the star $\delta$ Cephei which was first observed in 1786.
- Giant or supergiant stars – very luminous.
- Luminosity varies by factors of up to ten.
- Depending on the star, this variation repeats over periods between 1 and 100 days.
- Example: Polaris (the Pole Star) has a period of about 4 days and changes its luminosity by about 5% over that period.
Regular Variable Stars: Cepheids

- Cepheids *pulsate*.
- The *period* of this pulsation is *very* regular.
- Radial pulsations result in periodic changes of:
  - Luminosity.
  - Temperature.
  - Radius.
  - Velocity of the star’s surface.
Stellar Types & Classification: Part II

Regular Variable Stars

- $\delta$ Cephei.
Stellar Types & Classification: Part II

Regular Variable Stars: Cepheids

- Cepheid Variables lie in a particular region of the HR-diagram.
- This is called the *instability strip*.
- It is roughly at right angles to the Main Sequence.
- It is in the direction of the Giant branch.
Stellar Types & Classification: Part II

Regular Variable Stars: Instability Strip

![Graph showing Stellar Types and Classification]

- Instability Strip
- Cepheid variables
- RR Lyrae variables
- Main sequence
- Red giants

$L (L_{\odot})$ vs. $T (K)$
Regular Variable Stars: Cepheids

- Cepheids are very useful as *distance indicators*.
- This is because their period of variation depends only on the average luminosity.
- We can describe this relationship mathematically:

\[
M = -(1.8 + 2.4 \log_{10} P)
\]

\[
M = -(0.4 + 2.4 \log_{10} P)
\]

- \( P \) is the period of variation.
- \( M \) is the absolute magnitude of the star.
Regular Variable Stars: Cepheids

- The two period-luminosity relations on the previous slide tells us whether the star is:
  - massive, young and bright (first line).
  - older and smaller (second line).
- In all cases, these equations tell us that Cepheids lie on a straight line (the instability strip).
Stellar Types & Classification: Part II

Regular Variable Stars: Cepheids

Period–Luminosity Diagram

- \( \frac{L}{L_{\text{Sun}}} \) vs. \( \log(\text{Period}) \)
Regular Variable Stars: Cepheids

- If you know the period, you can get the luminosity.
- You can measure the flux of a star.
- You know how flux is related to luminosity:
  \[ F = \frac{L}{4\pi D^2} \]

- Therefore, you can calculate the distance to the star.
- Distance measurement is a major objective of the Hubble Space Telescope (and cosmology in general).
Stellar Types & Classification: Part II

Apparent Variables: Binary Systems

- Some stars in binary systems *appear* to vary in luminosity even though they are not of a type known to vary.
- Study of binary star systems is important for understanding some variables like *mass*.
- Treatment of the dynamics of binary systems is slightly harder than for the Solar System:
  - $M_{\odot} \gg M_{\text{planet}}$.
  - So the planets orbit about a ‘fixed’ centre.
- In a binary system, stars orbit about a point in between them.
Stellar Types & Classification: Part II

Apparent Variables: Binary Systems

- Newton’s and Kepler’s laws still apply.
- The *centre of mass* does not move (Newton I)
- Orbital motion is in a plane (conservation of angular momentum).
- Both bodies feel the same attractive force.
- If the orbits are *bound* (total energy is less than zero), each star describes an elliptical orbit about the centre of mass.
Stellar Types & Classification: Part II

Binary System Orbits
Stellar Types & Classification: Part II

Apparent Variables: Binary Systems

- Define the *common angular frequency*: \[ \omega = \frac{2 \pi}{P} \]

- Define the *total radius, R*: \[ R = r_1 + r_2 \]

- Can work out a relationship between period \( (P) \) and radius \( (R) \) based on the dynamics of the system.

![Diagram of binary system with common angular frequency and total radius defined.]
Stellar Types & Classification: Part II

Apparent Variables: Binary Systems

- For star 1:
  \[ m_1 \omega^2 r_1 = \frac{Gm_1 m_2}{R^2} \]
  \[ \Rightarrow \omega^2 r_1 = \frac{Gm_2}{R^2} \]

- For star 2:
  \[ m_2 \omega^2 r_2 = \frac{Gm_1 m_2}{R^2} \]
  \[ \Rightarrow \omega^2 r_2 = \frac{Gm_1}{R^2} \]

- Combine these two expressions to find the relationship between radius and period.
Stellar Types & Classification: Part II

Apparent Variables: Binary Systems

- Combine these two expressions to find the relationship between radius and period.

\[
\omega^2 (r_1 + r_2) = \frac{G(m_1 + m_2)}{R^2}
\]

\[\Rightarrow \omega^2 = \frac{G(m_1 + m_2)}{R^3}\]

\[\Rightarrow \frac{R^3}{P^2} = \frac{G(m_1 + m_2)}{4\pi^2}\]

- This is just Kepler’s third law.
Stellar Types & Classification: Part II

Apparent Variables: Binary Systems

- Kepler's third law was:

$$\frac{R^3}{P^2} = \frac{G(m_1 + m_2)}{4\pi^2}$$ (8)

- We measure *period* and *angular separation* (or $R$, if we know the distance to the binary system).

- We can then determine the total mass of the system: $m_1 + m_2$.

- To get the masses separately, we need to know $r_1$ and $r_2$. 
Stellar Types & Classification: Part II

Apparent Variables: Binary Systems

• Centre of mass dynamics tells us:

\[ m_1 r_1 = m_2 r_2 \]

• So, we can determine the masses:

\[ m_1 = \frac{(m_1 + m_2) r_2}{(r_1 + r_2)} \quad m_2 = \frac{(m_1 + m_2) r_1}{(r_1 + r_2)} \]
Stellar Types & Classification: Part II

Apparent Variables: Binary Systems

- We can make these measurements for *visual binaries* (where we can resolve the pair).
- We can also deduce masses even if we can’t resolve the two stars:
  - Spectroscopic binaries.
  - Eclipsing binaries.
Stellar Types & Classification: Part II

Apparent Variables: Spectroscopic Binary Systems
Stellar Types & Classification: Part II

Apparent Variables: Spectroscopic Binary Systems

- These are systems that cannot be resolved by telescope – separation is less than 0.5 arcseconds.
- We use *Doppler Shifts* of the stars’ spectral lines to determine period and velocity.
- Radial motion *away* from us (B in stage 1) shifts the wavelength to longer and redder wavelengths.
- Radial motion *towards* us (A in stage 1) shifts the wavelength to shorter and bluer wavelengths.
Stellar Types & Classification: Part II

Apparent Variables: Spectroscopic Binary Systems

- Define the change in wavelength $\Delta \lambda$ with relation to line-of-sight velocity $v$ due to the Doppler Effect:

$$\frac{\Delta \lambda}{\lambda_o} = \frac{v}{c}$$

- Our knowledge of binary system dynamics tells us:

$$\frac{\nu_1}{\nu_2} = \frac{r_1}{r_2} = \frac{m_2}{m_1}$$
Stellar Types & Classification: Part II

Apparent Variables: Spectroscopic Binary Systems

- Remember that for the individual stars we may write:

  \[ v_1 = r_1 \omega \quad \quad v_2 = r_2 \omega \]

- So we may determine the period, \( P \) from the velocities:

  \[ v_1 + v_2 = \frac{2\pi R}{P} \]
Apparent Variables: Spectroscopic Binary Systems

- In reality this it’s quite difficult to determine the period:
  - Orbits might be elliptical.
  - Orbit may be tilted out of the line of sight.
Stellar Types & Classification: Part II

Apparent Variables: Eclipsing Binary Systems

- **Partial eclipse**
  - Light variations due to star A passing in front of star B.

- **Total eclipse**
  - Light variations due to both stars A and B passing in front of each other.

**Orbital period**
- Time it takes for the system to complete one orbit.

**Time to cross disk or large star**
- Time it takes for one star to completely pass in front of the other.
Stellar Types & Classification: Part II

Apparent Variables: Eclipsing Binary Systems

- If the line-of-sight is almost in line with the orbital plane, the binary stars will periodically eclipse each other.
- We’ll see variation in the total light output of the system.
- Can use the shape of the light curve to work out the stellar radii and the orbit inclination.
- In turn, this allows us to find the mass of the system.

Example Question

- Question 4.10 from the Tutorial Book.
Different Stars

- We classify stars based on their **COLOUR**
- A star’s colour will depend on its temperature.
  - **Hot** stars appear blue.
  - **Cool** stars appear red.
- We define a star’s colour quantitatively by measuring its flux in different wavebands.
Stellar Types & Classification: Part III

Colour and Temperature

![Graph showing the relationship between intensity, wavelength, and temperature of stars. The graph compares blue and red filters for 20,000K and 3,000K stars.]
Stellar Types & Classification: Part III

Wavebands

- Three *filters* are used for measuring flux:
  - Ultraviolet, U: centred on \( \sim 350 \text{ nm} \).
  - Blue, B: centred on \( \sim 440 \text{ nm} \).
  - Visible, V: centred on \( \sim 550 \text{ nm} \) – yellow light.

Colour Index

- This is the difference in flux between the blue and visible bands:
  - “\( B - V \)"
Stellar Types & Classification: Part III

Colour Index

- We can quantify colour index:

\[ B - V = m_B - m_V = -2.5 \log_{10} \left( \frac{F_B}{F_V} \right) \]  

- \( m_B \) and \( m_V \) are the apparent magnitudes in the \( B \) and \( V \) bands respectively.
- \( F_B \) and \( F_V \) are the fluxes in the \( B \) and \( V \) bands respectively.
- Cooler (redder) stars will have greater \( B - V \) colour indices.\(^1\)

\(^1\) You might also see “\( U - B'' = m_U - m_B \).
Colour Index and the HR-Diagram

- Colour Index was the original quantity measured along the x-axis in the HR-diagram.
  - This is why log $T$ decreases from left to right.
- The exact relationship between Colour Index and effective temperature, $T_{\text{eff}}$, is not simple but we can state it approximately as:
  \[
  B - V \simeq \frac{7000\, K}{T_{\text{eff}}} - 0.56
  \]

- Colour Index is a straightforward, objective measure of colour.
- It is more accurate to determine the spectral type of a star.
Stellar Types & Classification: Part III

Spectral Type – History

• 1817: Fraunhofer noted that different stars had different spectra.
• 1890: Stars were classified in order of the line strength of hydrogen:
  A, B, C, … … M, N, O, P.
  • ‘A’ had the strongest line strength.
  • ’P’ had the weakest line strength.
Stellar Types & Classification: Part III

**Spectral Type – Current Scheme**

- We use the *Harvard Classification Scheme* nowadays.
- It reflects changes in other lines as well as hydrogen.
- It gives a sequence revealing *source temperature*:
  
  O, B, A, F, G, K, M  

  - ‘O’ stars are hot and blue
  - 'M' stars are cool and red.
  - Temperature increases from *right* to *left*.

- Under this scheme, the Sun is a G-type star.
Stellar Types & Classification: Part III

Spectral Type – Harvard Scheme

O, B, A, F, G, K, M

- **Oh Be A Fine Girl / Guy, Kiss Me**
- **Old Blind Astronomers From Glasgow Keep Mice**
Stellar Types & Classification: Part III

Harvard Subclasses

- Each type is subdivided into 10 subclasses.
- These reflect gradual temperature changes.
  
  ...A0, A1, A2... ...A7, A8, A9...

  - 0 represents the hotter end of a subclass.
  - 9 represents the cooler end of a subclass.
  - Choice of subtype depends on line ratios.

- Sun is a G2-type star.
- Other stars: Vega A0, Rigel B8, Polaris, F8, Betelgeuse M2.
Stellar Types & Classification: Part III

Spectral Type – Harvard Scheme
Stellar Types & Classification: Part III

Spectral Type – Harvard Scheme

- **O**: Hottest, blue-white stars. Few lines, Strong He II absorption (sometimes emission) lines.
- **B**: Hot, blue-white stars. He I absorption lines strongest at B2, HI I (Balmer) absorption lines becoming stronger.
- **A**: White stars. Balmer absorption lines strongest at A0, becoming weaker later.
- **F**: Yellow-white stars. Ca II lines continue to strengthen as Balmer lines continue to weaken. Neutral metal absorption lines (Fe I Cr I).
**Stellar Types & Classification: Part III**

### Spectral Type – Harvard Scheme

- **G**: Yellow stars. Solar-type spectra. Ca II lines continue to strengthen. Fe I and other neutral metal lines continue to strengthen.

- **K**: Cool orange stars. Ca II, H and K lines strongest at K0, becoming weaker later. Spectra dominated by metal absorption lines.

- **M**: Coolest red stars. Spectra dominated by molecular absorption bands, especially Titanium Oxide (TiO). Neutral metal absorption lines remain strong.
Stellar Types & Classification: Part III

Spectral Type and Luminosity

- Spectral type is a measure of temperature.
- Stars of a given spectral type can have a range of luminosities.
- Can further subdivide into *Luminosity Classes*.

MK Classification Scheme

- Scheme is due to Morgan and Keenan.
- It allows astronomers to place a star on the HR diagram solely on the basis of the star’s spectrum.
### Stellar Types & Classification: Part III

#### MK Classification Scheme

<table>
<thead>
<tr>
<th>Class</th>
<th>Type of Star</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ia</td>
<td>Luminous supergiants</td>
</tr>
<tr>
<td>Ib</td>
<td>Less luminous supergiants</td>
</tr>
<tr>
<td>II</td>
<td>Bright giants</td>
</tr>
<tr>
<td>III</td>
<td>Normal giants</td>
</tr>
<tr>
<td>IV</td>
<td>Subgiants</td>
</tr>
<tr>
<td>V</td>
<td>Main Sequence</td>
</tr>
<tr>
<td>VI</td>
<td>Subdwarfs</td>
</tr>
<tr>
<td>VII (D)</td>
<td>White Dwarfs</td>
</tr>
</tbody>
</table>
Stellar Types & Classification: Part III

MK Classification Scheme
Luminosity Classes

- Luminosity classes are determined (mainly) from the widths of spectral lines.
- Stars with similar temperatures can have different luminosity.
- This implies that they must have different radii, surface gravity and densities.
- All of these lead to a higher gas pressure.
- As temperature and pressure increase, atoms collide more frequently, broadening spectral lines.
  - The effect is seen particularly in hot, dense stars like White Dwarfs.
Mass-Luminosity Relationship

• Observations from binary systems (where masses can be determined) indicate a relationship between mass and luminosity.

• This is only true for Main Sequence stars.

• We find:

\[
\frac{L}{L_\odot} \sim \left( \frac{M}{M_\odot} \right)^\alpha
\]  

(11)

• The value of \( \alpha \) depends on the fit used in the data.
  
  • \( \alpha \approx 3.0 \) to 3.5
Stellar Types & Classification: Part III

Mass-Luminosity Relationship
Stellar Types & Classification: Part III

Mass-Luminosity Relationship: HR-diagram
Stellar Types & Classification: Part III

Mass-Luminosity Relationship

- The more massive a star, the more luminous it is.
- The luminosity is a very strong function of mass – this is because alpha is quite big.
  - The most massive stars are at the upper left hand part of the HR-diagram.
  - The least massive stars are at the bottom right.
  - The Sun, a G2V star, sits around the middle of the Main Sequence.
Stellar Types & Classification: Part III

Mass-Luminosity Relationship

- The relationship holds only for Main Sequence stars.
- It has great implications for how long stars live on the Main Sequence.
  - Massive stars have very short lifetimes because they burn up their fuel more quickly than less massive stars.

Example Question

- Question 4.23 from the Tutorial Book.
Stellar Atmospheres: Part I

Stellar Atmospheres

- We need 2 things to understand the nature of stellar atmospheres:
  - Vogt-Russell Theorem.
  - Spectroscopy.
Stellar Atmospheres: Part I

Vogt-Russell Theorem

- The nature of a star is defined uniquely by its mass and the distribution of elements within it.

Spectroscopy

- Method for determining the chemical composition of a star.
Stellar Atmospheres: Part I

The Sun

- We observe:
  - Photosphere.
  - Chromosphere.
  - Corona.
Stellar Atmospheres: Part I

Photosphere

- The Sun has no solid surface but becomes opaque to visible light at the photosphere.
- The photosphere is about 500 km thick and temperature, \( T \), varies throughout.
  - \( T_{\text{base}} \approx 6500 \text{ K} \)
  - \( T_{\text{top}} \approx 4400 \text{ K} \)
- Most light comes from a region where \( T \approx 5800 \text{ K} \) where the density is about one thousandth that of air.
- Find that the sun is \textit{not} a perfect black-body.
Stellar Atmospheres: Part I

Photosphere

http://sohowww.nascom.nasa.gov/
Chromosphere

- Get a little light from the next layer of gas.
  - This is the dim reddish-pink, seen during an eclipse.
- The chromosphere is about 2000 km thick.
- It is hottest furthest from the Sun: $T \sim 20000$ K.
Stellar Atmospheres: Part I

Chromosphere

Stellar Atmospheres: Part I

Corona

- Very hot: $T \sim 10^6$ K
- Outer atmosphere, extending as far as $\sim 3 R_\odot$.
- Together with the chromosphere, it contributes about $10^{-4} L_\odot$.
- Very diffuse and eventually leads to the Solar Wind.
  - Solar Wind stretches out to about 100 AU.
  - Its effect is seen on Earth as the Aurora.
Stellar Atmospheres: Part I

Corona

http://sohowww.nascom.nasa.gov/
Stellar Atmospheres: Part I

Corona

Stellar Atmospheres: Part I

Aurora Borealis over Alaska

Stellar Atmospheres: Part I

Spectroscopy

- 1802: Wollaston discovered *dark lines* in the Solar spectrum.
- 1814: Fraunhofer catalogued 475 of these lines.
  - They showed that the Sun’s light was *absorbed* at discrete wavelengths.
- 1860: Kirchoff and Bunsen published rules for the production of *spectral lines*.
Stellar Atmospheres: Part I

Kirchhoff’s Laws

- A hot dense gas produces a continuous spectrum with no spectral lines.
- A hot diffuse gas produces bright spectral lines – an emission spectrum.
Stellar Atmospheres: Part I

Spectra

- Continuous Spectrum
- Emission Spectrum
- Absorption Spectrum
Stellar Atmospheres: Part I

Solar Absorption Spectrum

Uses of Spectroscopy

- Each chemical element has its unique spectral line fingerprint.
- This allows you to determine the elements in a star’s atmosphere from its spectrum.

Strengths of Spectral Lines

- Strength tells us about stellar temperature.
- Depends on...  
  - Number of atoms present.
  - Temperature of the gas.
Stellar Atmospheres: Part I

Doppler Shifts

- Wavelengths and ‘widths’ of lines are affected by Doppler shifts due to the motion of the stellar atmosphere:
  - Bulk motions: can measure rotation, large-scale expansion or contraction (e.g.: Cepheids).
  - Random thermal motions: can measure temperature, pressure.
- It can be hard to disentangle all of these.
Stellar Atmospheres: Part I

Quantum Mechanics

- Need Quantum Mechanics to accurately understand spectral lines.
- Simplify using the Bohr Model.
  - This works well for hydrogen.
  - Sometimes for other elements too.
The Bohr Model

- Electrons, bound in atoms, are only allowed certain energies.
- These energies correspond to certain ‘orbits’ around the nucleus.
- These orbits must have angular momenta equal to an integer multiple of a universal constant: Planck’s constant.

\[ \hbar = \frac{h}{2\pi} \]

\[ h = 6.636 \times 10^{-34} \text{ J s} \]

- Only photons with energies corresponding to *differences* between energy levels can be emitted or absorbed.
The Bohr Model

- The energy of the $n^{th}$ level of hydrogen:

$$E_n \propto \frac{1}{n^2}$$

- For a transition between level $m$ and level $n$ require an energy difference:

$$\Delta E_{mn} \propto \left( \frac{1}{m^2} - \frac{1}{n^2} \right) \propto \frac{1}{\lambda}$$
Stellar Atmospheres: Part I

The Bohr Model

- This means we can calculate wavelength:

\[
\frac{1}{\lambda} = R_\infty \left( \frac{1}{m^2} - \frac{1}{n^2} \right)
\] (12)

- This is the Rydberg Equation.
- \(R_\infty = 1.097 \times 10^7 \text{ m}^{-1}\) is the Rydberg constant.
- Depending on the values of \(m\) and \(n\), we can construct several series of lines.
Stellar Atmospheres: Part I

Lyman Series (Ly)

- From $m = 2, 3, 4, \ldots$ to $n = 1$ (ground state).
- $Ly \alpha$: $m = 2$ to $n = 1$
- $Ly \beta$: $m = 3$ to $n = 1$

Balmer Series (H)

- From $m = 3, 4, \ldots$ to $n = 2$ (first excited state).
- $H \alpha$: $m = 3$ to $n = 2$
- $H \beta$: $m = 4$ to $n = 2$
Stellar Atmospheres: Part I

Bohr Model: Emission

- $m > n$
- Electron drops to a *lower* level.
- Photon is emitted.

Bohr Model: Absorption

- $m < n$
- Electron jumps to a *higher* level.
- Photon is needed for this to happen.
Stellar Atmospheres: Part I

Higher Order Series: hydrogen

- Paschen (P): to/from level 3.
- Brackett: to/from level 4.
The Sun

- Strongest visible line from solar hydrogen is the $H_\alpha$. 
- See this in absorption in the chromosphere.
- We see other atoms in absorption:
  - hydrogen, helium, magnesium, sodium, iron, chromium…
- There are about 250,000 lines in total.
  - Identify by heating elements in the lab and observing spectra.
Stellar Atmospheres: Part I

Hydrogen Alpha, $H_\alpha$

- $H_\alpha$ takes an electron from $m = 2$ to $n = 3$.
  - i.e.: the electron is already in an excited state.
- The electron gets from $m = 1$ to $n = 2$ by thermal excitation.
  - The chances of this depend on the Boltzmann factor.
Boltzmann Factor

- The probability of an electron occupying energy state $E$:

$$p(E) \propto \exp \left( - \frac{E}{kT} \right)$$ (13)

- $k = 1.381 \times 10^{-23} \, \text{JK}^{-1}$ is Boltzmann’s constant.
- States with $E \gg kT$ are very improbable.
Energy Difference

- We know:
  - 2 electrons can occupy the $n = 1$ shell.
  - 8 electrons can occupy the $n = 2$ shell.
- So we can calculate the energy difference between the two levels.

\[
\frac{N_2}{N_1} = \frac{8}{2} \exp \left(-\frac{\Delta E}{kT}\right)
\]

- $\Delta E$ is the energy difference between $n = 1$ and $n = 2$. 

Stellar Atmospheres: Part I
Energy Difference

- $\Delta E$ is the energy difference between $n = 1$ and $n = 2$.

\[
\Delta E = h\nu_{12} = \frac{hc}{\lambda_{12}} = h\epsilon_{\infty} \left( \frac{1}{1^2} - \frac{1}{2^2} \right) = 1.64 \times 10^{-18} \text{ J}
\]
Stellar Atmospheres: Part I

How many atoms?

- For $T \sim 6000$ K, we find:

$$\frac{N_2}{N_1} \sim 10^{-8}$$

- i.e.: 99,999,999 out of every 100 million hydrogen atoms is in the ground state.
- This transition (at this temperature) is very unlikely.
- We still see this line though because there is so much hydrogen in the Sun.
Introduction

Ionisation

- Under extreme conditions atoms can be *ionised*.
  - Atoms can lose electrons if they absorb enough energy from photons.
- For hydrogen, this requires a minimum *ionisation energy*, $\chi$.
- We can find $\chi$ by using the Rydberg Equation (12) with $n = \infty$.
  - This is the energy required to remove an electron from an energy level to infinity.
- When the electron has been knocked out of the atom, an *ion* is left.
Stellar Atmospheres: Part II

**Ionisation**

- Ionisation wavelength:

  \[
  \frac{1}{\lambda_\infty} = R_\infty \left( \frac{1}{m^2} - \frac{1}{\infty} \right) = \frac{R_\infty}{m^2}
  \]

- The ionisation energy is:

  \[
  \chi = \frac{hc}{\lambda_\infty} = \frac{hc}{m^2 R_\infty} = \frac{13.6}{m^2} \text{ eV}
  \]
Ionisation

• We normalised the previous equation to 13.6 eV because this is the first ionisation energy of hydrogen.
  • i.e.: This is the energy required to strip the hydrogen atom of its ground state electron.
• An eV is an *electron volt*.
  • $1\, eV = 1.602 \times 10^{-19}\, J$. 
How do ions form?

- Two processes:
  - Absorption of a photon.
  - Collision with another particle like an electron (scattering).
Absorption of a photon

- In a black-body with temperature $T$, there is a distribution of energies.
- The *average* photon has energy $E = kT$.
- Only ions with energies greater than the ionisation energy, $\chi$, can be ionised.
- In a distribution, this is always possible for *some photons*. 

Stellar Atmospheres: Part II
Absorption of a photon

Number of photons with Energy $E$

$E = kT$

These can ionise

Energy
Stellar Atmospheres: Part II

Scattering

- Collisions can occur with other particles, e.g.: electrons.
- There will be a similar distribution of energies.
  - The average this time will be $E = \frac{3}{2} kT$.
  - The fraction with $E > \chi$ will be higher.
Stellar Atmospheres: Part II

What proportion of atoms do we expect to be ionised?

- Need to find the equilibrium in the following reaction:

\[ \text{photon} + \text{atom} \rightleftharpoons \text{electron} + \text{ion} \]

- The answer to this is the Saha Equation:

\[
\frac{\text{number of ions}}{\text{number of atoms}} \approx \frac{10^{21} T^{3/2} \exp(-\chi/kT)}{\text{number of electrons}}
\]

- The rule of thumb is that 50% ionisation occurs when \( kT \sim \chi/18 \).
Stellar Atmospheres: Part II

The Solar Photosphere

- Temperature is $T \simeq 6000 \text{ K}$ so $18kT \simeq 9.3 \text{ eV}$.

<table>
<thead>
<tr>
<th>Element</th>
<th>Ionisation Energy $\chi$</th>
<th>Status in Photosphere</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>13.6 eV</td>
<td>a few ions</td>
</tr>
<tr>
<td>He</td>
<td>24.6 eV</td>
<td>effectively no ions</td>
</tr>
<tr>
<td>Na</td>
<td>5.14 eV</td>
<td>$\sim$ fully ionised</td>
</tr>
<tr>
<td>Fe</td>
<td>7.9 eV</td>
<td>$\sim$ fully ionised</td>
</tr>
</tbody>
</table>
Stellar Atmospheres: Part II

Line Strength

- Line strength depends on two things:
  - Element abundance.
  - Temperature.
- Temperature is critical because it determines:
  - The number of atoms in the correct state.
  - The number of photons with enough energy to cause transition.
- Line strength is, therefore, a good indicator of temperature.
Stellar Atmospheres: Part II

Temperature

- If a gas gets too hot, all the atoms may already be ionised.
  - There may not be any low-level electrons able to absorb energy.
  - There may not be any high-level electrons able to emit energy.
- If a gas is too cool, electrons may be in too low an energy state for a particular line.
  - Remember that Balmer absorption needs $n = 2$. 
Stellar Atmospheres: Part II

Temperature

- Line strengths of different elements vary with temperature in different ways.
- This is crucial to *stellar classification* – later.
- We will consider only *visible spectra*.
Hydrogen Lines

- Balmer series is strongest when $T \sim 10,000\, \text{K}$
- If it’s cooler, there are not enough photons to excite electrons.
- If it’s hotter, the atmosphere is fully ionised.
Stellar Atmospheres: Part II

Helium Lines

- For visible lines, require $T \gtrsim 10^4$ K.
- Best temperature is $T \sim 250,000$ K.
- Little is seen from the Sun in absorption although they are observed faintly in the upper chromosphere during eclipses.
- Helium ions can also be excited to a state that gives visual absorption lines if the temperature is $T \sim 350,000$ K – corona.
Stellar Atmospheres: Part II

Metal Lines

- ‘Metal’ means all elements after helium in the periodic table.
- These are very rare (≈ 0.1% of the stellar atmosphere).
- Metal lines only dominate at low temperatures where hydrogen and helium are ‘frozen out’.
- Strong lines from singly ionised calcium and iron are observed, as are those from singly and doubly ionised iron if it’s hot enough.
- The relative abundances of metals in all stars is fairly similar.
- The abundance of metals relative to hydrogen is very different in some stars, though.
Population I stars

- The ratio of metals to hydrogen and helium is very much like that found in the Sun.
- Young stars, generally made from material ejected from older stars.
- They’ve formed late in the evolution of the Galaxy and, for that reason, are found predominantly in the Galactic disk.
Stellar Atmospheres: Part II

Population II stars

- The ratio of metals to hydrogen and helium is 100 times less than that found in the Sun.
- These are ‘metal poor’ stars.
- They are old stars, formed before the Galaxy was a disk and, therefore, are found predominantly in the Galactic halo.
Stellar Atmospheres: Part II

Chemical Composition of the Universe

- Count atoms to find:

<table>
<thead>
<tr>
<th>Element</th>
<th>Percentage of Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>~ 85%</td>
</tr>
<tr>
<td>He</td>
<td>~ 15%</td>
</tr>
<tr>
<td>C, N, O, Ne</td>
<td>~ 0.1% each</td>
</tr>
<tr>
<td>Si, Mg, Fe, Al</td>
<td>~ 0.01% each</td>
</tr>
</tbody>
</table>
Stellar Atmospheres: Part II

Molecular Bands

- Molecules can form in the outer atmospheres of cool stars.
- The typical binding energies of molecules are $\sim 4 - 6$ eV.
- This means that the lines will ‘fade out’ if $T \gtrsim 5000$ K.
- We observe $TiO, ZrO, CN$ and, sometimes, even $H_2O$.

Example Question

- Question 4.17 from the Tutorial Book.
Stellar Models: Part I

Stellar Structure

- We want to describe the structure and evolution of stars theoretically using physics.
- What questions should we ask?
  - How do star form from protostars?
  - What governs their interior temperatures?
  - What are the possible energy sources?
  - What are the relevant nuclear reactions?
  - How long to stars spend on the Main Sequence?
- Afterwards, we can look at stellar death.
Protostars

- Remember the collapsing gas cloud from lecture 1.
- The cloud starts to *infall*.
  - This could be triggered by e.g.: a supernova blast wave, a spiral density wave (from a nearby galaxy) or a close-approach by another star.
- The cloud fragments and each fragment starts to collapse under the effects of gravity.
  - Collisions between the cloud's atoms become more frequent and the temperature rises.
Stellar Models: Part I

Protostars

- After a while, the infall of the gas cloud becomes more gradual.
  - The star is heading towards *hydrostatic equilibrium*.
- The cloud eventually becomes *opaque* and energy can no longer escape – the cloud has become a protostar.
- We can calculate how much energy is released and how long this takes to happen.
Stellar Models: Part I

Density of a Cloud of Gas

<table>
<thead>
<tr>
<th>Object</th>
<th>Number Density (particles/m³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth’s Atmosphere</td>
<td>$2.5 \times 10^{25} m^{-3}$</td>
</tr>
<tr>
<td>Good lab vacuum</td>
<td>$1 \times 10^{16} m^{-3}$</td>
</tr>
<tr>
<td>‘Dense’ interstellar cloud</td>
<td>$1 \times 10^{14} m^{-3}$</td>
</tr>
<tr>
<td>‘Average’ interstellar cloud</td>
<td>$1 \times 10^{13} m^{-3}$</td>
</tr>
</tbody>
</table>

- Because the density is so low, you need a lot of mass ($\gtrsim 100 M_\odot$) for collapse.
Stellar Models: Part I

Time for Each Stage of Collapse of a Protostar

![Diagram showing stages and times of protostar collapse]

- Stage 1: $2 \times 10^6$ yr
- Stage 2: $3 \times 10^4$ yr
- Stage 3: $10^5$ yr
- Stage 4/5: $10^7$ yr
Stellar Models: Part I

Stages of Collapse of a Protostar

- Stage 1: Interstellar cloud of gas.
- Stage 2: Cloud starts to fragment and collapse.
- Stage 3: Cloud continues contracting and heating up.
- Stage 4/5: Protostar has formed.
Stellar Models: Part I

Hayashi Tracks

Theoretical Hayashi Tracks of Protostars

9 solar mass star
5 solar mass star
1 solar mass star
0.5 solar mass star

MAIN SEQUENCE (V)

Effective Temperature, K
30,000 10,000 7,000 6,000 4,000

Absolute Magnitude, M_

Luminosity compared to Sun

Colour Index (B - V)
-0.5 0.0 +0.3 +0.6 +0.8 +0.9 +1.2

Spectral Class
O5 B0 A0 F0 G0 K0 M0
Stellar Models: Part I

Hayashi Tracks

- After Stage 4, the core of the protostar has reached $\sim 10^7$ K which is hot enough for nuclear fusion.
- On the HR-diagram it moves down (to lower luminosity) and left (to higher temperature) on what is known as a *Hayashi Track*.
- These tracks are *theoretical curves* – we can’t see the protostar at this point because it is hidden in an opaque shroud of dust.
- Eventually, the star reaches and joins the Main Sequence.
Energy Available in a Cloud of Gas

- Newton’s Law of Gravitation says that the *self-gravitational energy* of a sphere of mass $M$ and radius $R$ is:

$$
\Omega = -\frac{GM^2}{R}
$$

(14)

- Therefore if the cloud collapses from radius $R$ to a radius very much smaller than $R$ then the energy released must be of the order of $\Omega$. 
Stellar Models: Part I

Time for Collapse of a Protostar

- Let’s assume that the protostar had a steady luminosity $L$ during the time of collapse.
- The *time* this takes to happen can be calculated as:

$$
\tau_{KH} \sim \frac{\text{energy available}}{\text{energy used}} \sim \frac{GM^2}{R L}
$$

(15)

- This is called the *Kelvin Helmholtz timescale*. 

Stellar Models: Part I

Kelvin-Helmholtz Timescales

- The length of time that a star is a protostar is quite short by cosmic standards:
  - $15 \, M_\odot$ protostar – $10^5$ years.
  - $1 \, M_\odot$ protostar – $10^7$ years.
- The short lifetime is the reason that we don’t observe very many.
Stellar Models: Part I

Different Masses

• For a $1 \, M_\odot$ star, after about $10^4$ years:
  • $T = 2000 - 3000$ K.
  • $R \simeq R_\odot$.
  • $L \simeq L_\odot$ (from Stefan’s Law).

• For a $0.08 \, M_\odot$ star, the pressure and temperature are never high enough to start fusion.
  • A Brown Dwarf forms.
  • The protostar still shines for $\tau_{KH}$ though.
  • Note that the mass of Jupiter is $\sim 0.001 \, M_\odot$. 
Stellar Models: Part I

Pre-Main Sequence Stars

- All have distinctive characteristics:
  - Unstable luminosity.
  - Eject lots of gas.
  - Surrounded by warm clouds (from which they formed).
  - T Tauri stars are a good example of pre-MS stars.

- Young stars also exhibit ‘bipolar outflows’.
  - These are jets of material in opposite directions over a distance scale of about 1 lightyear.
Emission Nebulae

- Young stars are also often surrounded by *emission nebulae*.
  - These are known as *HII regions*.
- The UV light from a hot star (≈ 20,000 K) sweeps out a cavity and ionises the surrounding hydrogen.
- Recombination produces $H_{\alpha}$ and other emissions as well as a red glow.
- The Orion, Eagle, Trifid and Lagoon nebulae are all examples of emission nebulae.
Stellar Models: Part I

Orion Nebula (M42)
The Virial Theorem states that the total kinetic energy of a stable, self-gravitating mass distribution is negative one half the total gravitational potential energy:

\[ KE = -\frac{1}{2} PE \]  

This allows us to calculate the change in kinetic energy or the heat available in the protostar.
Temperature of the Interior of a Protostar

\[ \Omega = -\frac{GM^2}{R} \]

Virial Theorem:

\[ \Delta K = -\frac{\Delta \Omega}{2} \]

\[ \Rightarrow \Delta K = \frac{GM}{R} \]

- Differentiate with respect to mass because that’s what’s changing over star’s evolution.
Stellar Models: Part I

Temperature of the Interior of a Protostar

- Assuming that about half energy available goes into heat and the other half into radiation, the energy available per each hydrogen atom is therefore:

\[ E = \frac{GMm_p}{2R} \]

- Thermal Physics tells us about the amount of heat energy in a gas:

\[ E = 2 \times \frac{3}{2}kT = 3kT \]

- There is \(3kT/2\) per electron as well as per proton so multiply by two.
- \( k = 1.38 \times 10^{-23}\, JK^{-1} \) is Boltzmann’s constant.
Temperature of the Interior of a Protostar

- Equate the previous two equations to find the temperature of the protostar at radius $R$:

$$\frac{GMm_p}{2R} = 3kT$$

$$\Rightarrow T = \frac{Gm_pm_p}{6kR}$$

- For a $1 \, M_\odot$ protostar, this gives $T \sim 4 \times 10^6 \, \text{K}$.

- Eventually this is high enough, and the matter dense enough, for nuclear fusion of hydrogen to helium to start.

- Fusion maintains the high temperature while preventing further contraction – the protostar is a star.
Example Question

- Question 4.15 from the Tutorial Book.
Solar Energy Requirements

- We know the mass of the Sun: $M_\odot = 2 \times 10^{30}$ kg.
- We know that it radiates with a luminosity: $L_\odot = 4 \times 10^{26}$ W.
- We know how old it is: $\tau_\odot = 4 \times 10^9$ years.
  - This is equivalent to $\tau_\odot = 1 \times 10^{17}$ seconds.
  - We know this from the fossil record.
- This means we can work out how much energy is required to keep the Sun shining.
Solar Energy Requirements

- Define the energy source of the Sun in terms of Joules per kilogram.
  - Call this $\varepsilon$.
- We know that:
  \[
  \varepsilon > \frac{L_\odot \tau_\odot}{M_\odot} \sim 1 \times 10^{13} \text{ J kg}^{-1}
  \]
- We need to find a source for all this energy.
Stellar Models: Part II

Possible Solar Energy Sources

- Chemical?
- Gravity?
- Annihilation of matter?
- Something else?
Chemical Reactions as a Possible Source of Solar Energy

- Chemical energy is released when atomic electrons are ‘re-arranged’ in the atom.
- The typical maximum energy of such a rearrangement is about 1 electron Volt (1 eV).
  \[ 1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}. \]
- Compare this to the ‘average’ atomic mass, a proton.
  \[ m_p \sim 1.7 \times 10^{-27} \text{ kg}. \]
**Stellar Models: Part II**

**Chemical Reactions as a Possible Source of Solar Energy**

- Therefore, the energy available per unit mass is:

\[
\varepsilon_{\text{chemical}} < \frac{1.6 \times 10^{-19}}{1.7 \times 10^{-27}} \sim 1 \times 10^8 \text{ Jkg}^{-1}.
\]

- Compare this to the Sun’s luminosity and mass to calculate the Sun’s lifetime:

\[
\tau_{\text{chemical}} \sim \frac{M_\odot \varepsilon_{\text{chemical}}}{L_\odot} \sim 50,000 \text{ years}
\]

- This is 100,000 times smaller than the observed lifetime of the Sun – 4 \times 10^9 years – not chemical reactions!
Gravity as a Possible Source of Solar Energy

- This is what Lord Kelvin thought powered the Sun.
  - This is one of the reasons that the Kelvin-Helmholtz timescale is so named.
- We saw in the last lecture that we could equate gravitational energy to thermal energy:
  \[ 3kT = \frac{G M m_p}{2R} \]
- To find the \( \epsilon_{gravity} \), the energy per unit mass, take the right hand side of the equation and divide by \( m_p \).
Stellar Models: Part II

Gravity as a Possible Source of Solar Energy

• Find $\varepsilon_{\text{gravity}}$:

$$\varepsilon_{\text{gravity}} \sim \frac{GM}{2R} \sim 5 \times 10^{10} \text{ Jkg}^{-1}$$

• This allows us to find the lifetime of the Sun as before:

$$\tau_{\text{gravity}} \sim \frac{M_{\odot} \varepsilon_{\text{gravity}}}{L_{\odot}} \sim 8 \times 10^6 \text{ years}$$

• This is 200 times smaller than the observed lifetime of the Sun – 4 $\times$ 10$^9$ years – not gravitational collapse!
Matter Annihilation as a Possible Source of Solar Energy

- Einstein discovered the equivalence of mass and energy:
  \[ E = mc^2 \]

- For the total conversion of mass to energy,
  \[ \epsilon_{Einstein} \sim \frac{mc^2}{m} \sim 9 \times 10^{16} \text{ J kg}^{-1} \]
Stellar Models: Part II

**Matter Annihilation as a Possible Source of Solar Energy**

- This allows us to find the lifetime of the Sun as before:
  \[
  \tau_{Einstein} \sim \frac{M_\odot \varepsilon_{Einstein}}{L_\odot} \sim 3 \times 10^{13} \text{ years}
  \]

- This is 10,000 times *bigger* than the observed lifetime of the Sun – 4 \times 10^9 years so is plausible from a mathematical standpoint.

- **However**, it requires half the Sun to be matter and the other half to be anti-matter.
  - *We don’t* observe gamma-ray lines from matter-anti-matter destruction.
  - It’s not matter annihilation that powers the Sun!
Stellar Models: Part II

Nuclear Fusion

- Nuclear fusion reactions are actually what power the Sun.
- Under conditions of very high temperature and very high density, several light nuclei can fuse to form a single nucleus.
- The newly formed nucleus has a slightly lower mass than the combination of the separate nuclei.
- This mass deficit – $\Delta m$ – is released as fusion energy.

$$\Delta E = \Delta m c^2$$ (17)
Aside: Nuclear Weapons

- There are many stages of explosion in an hydrogen bomb detonation, some of which are *fission* reactions.
  - The *fusion* reaction is when tritium and deuterium, both forms of hydrogen, combine to form the unstable helium-5 nucleus.
  - The helium-5 nucleus undergoes fission into a helium-4 nucleus and a neutron, releasing energy.
  - The neutrons bombard uranium-238, causing another fission reaction.
- Energy comes from all of these reactions, fission and fusion alike, because of the *mass deficit* described by equation 17.
Stellar Models: Part II

Aside: Fusion versus Fission in Nuclear Weapons

- Fusion is a relatively ‘clean’ reaction in that it does not release radioactive by-products.
- Fission of heavy nuclei produces large amounts of radioactive by-products.
- Beta and gamma radiation are also emitted as by-products of nuclear reactions.
Stellar Models: Part II

Fusion in the Stars

- In the Sun the fusion reaction that produces energy is when 4 hydrogen nuclei (protons) fuse to form helium-4 (an alpha particle).
- Energy is released in the form of radiation (gamma-rays) and neutrinos.
How much energy is available for stellar fusion?

... or can fusion power the Sun?

Calculate mass deficit, $\Delta m$

- The fusion reaction is when four protons are converted to one helium nucleus.
- Therefore, need to know:
  - The mass of four protons.
  - The mass of one helium-4 nucleus.
Stellar Models: Part II

Mass of four protons

- Mass of one proton: \( m_p = 1.67262158 \times 10^{-27} \text{ kg} \)
- Mass of four protons: \( 4 \times m_p = 6.69048632 \times 10^{-27} \text{ kg} \)

Mass of a helium-4 nucleus

- Helium-4 is an alpha-particle: \( m_\alpha = 6.6446565 \times 10^{-27} \text{ kg} \)
Mass Deficit in H-He Reaction

- Find the difference between the two masses:

\[ \Delta m_{H-He} = 0.04582982 \times 10^{-27} \text{ kg} \]

Energy available in one H-He Reaction

- Can calculate energy using Einstein’s mass-energy equivalence relation:

\[ \Delta E_{H-He} = 0.04582982 \times 10^{-27} \times c^2 \]
\[ = 4.127 \times 10^{-12} \text{ J} \]
Fusion as a Possible Source of Solar Energy

- Find $\varepsilon_{\text{fusion}}$:
  \[ \varepsilon_{\text{fusion}} \sim \frac{\Delta H_{\text{He}}}{m_p} \sim 2.47 \times 10^{15} \text{ J kg}^{-1} \]

- This allows us to find the lifetime of the Sun as before:
  \[ \tau_{\text{fusion}} \sim \frac{M_\odot \varepsilon_{\text{fusion}}}{L_\odot} \sim 4 \times 10^{11} \text{ years} \]

- This is 100 times bigger than the observed lifetime of the Sun – $4 \times 10^9$ years – plausible!
Hydrogen-Helium Fusion: The p-p Chain

- When we calculated the energy available for fusion before we assumed that *all* of the Sun’s hydrogen was converted into helium.
- The direct collision of 4 protons at once is incredibly unlikely.
- Hydrogen-helium fusion is possible with a succession of binary collisions.
  - This is called the *p-p chain*. 
Stellar Models: Part II

Hydrogen-Helium Fusion: The p-p Chain
The p-p Chain: Three Reactions

1. Two mass-1 isotopes of hydrogen (protons) undergo a simultaneous fusion and beta decay, producing a positron (beta particle), a neutrino and a mass-2 isotope of hydrogen (deuterium).

\[ ^1 H + ^1 H \rightarrow \beta^+ + \nu + ^2 H \]

2. The deuterium reacts with another mass-1 isotope of hydrogen to produce helium-3 and a gamma-ray (photon).

\[ ^2 H + ^1 H \rightarrow ^3 He + \gamma \]
Stellar Models: Part II

The p-p Chain: Three Reactions

Two helium-3 isotopes produced in separate implementations of the first two stages fuse to form helium-4 plus two mass-1 hydrogen isotopes (protons).

\[ ^3\text{He} + ^3\text{He} \rightarrow ^4\text{He} + 2\, ^1\text{H} \]

The p-p Chain: Net Reaction

\[ 4\, ^1\text{H} \rightarrow ^4\text{He} + 2\beta^+ + 2\nu + \gamma \]
The p-p Chain in Stars

- In stars of $\sim 1\,M_\odot$, 85% of nuclear reactions occur via the p-p chain.
- The other 15% of reactions are the temporary creation of heavier elements:
  - $^7\text{Be}$ and $^7\text{Li}$ (beryllium and lithium).
- In more massive stars, the core temperature is higher ($\gtrsim 1.6 \times 10^7\,K$) and other reactions are possible:
  - This is the CNO cycle.
Stellar Models: Part II

The CNO Cycle
# Stellar Models: Part II

## The CNO Cycle: Reactions

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{12}\text{C} + ^{1}\text{H}$ → $^{13}\text{N} + \gamma$</td>
<td></td>
</tr>
<tr>
<td>$^{13}\text{N}$ → $^{13}\text{C} + \beta^+ + \nu$</td>
<td></td>
</tr>
<tr>
<td>$^{13}\text{C} + ^{1}\text{H}$ → $^{14}\text{N} + \gamma$</td>
<td></td>
</tr>
<tr>
<td>$^{14}\text{N} + ^{1}\text{H}$ → $^{15}\text{O} + \gamma$</td>
<td></td>
</tr>
<tr>
<td>$^{15}\text{O}$ → $^{15}\text{N} + \beta^+ + \nu$</td>
<td></td>
</tr>
<tr>
<td>$^{15}\text{N} + ^{1}\text{H}$ → $^{12}\text{C} + ^{4}\text{He}$</td>
<td></td>
</tr>
</tbody>
</table>
Stellar Models: Part II

The CNO Cycle

- $^{12}C$ is needed for the cycle.
- It works like a catalyst but is not used up itself.
- You do \textit{not} need to memorise the CNO-cycle reactions!
Stellar Models: Part II

Energy Creation Rates

\[ \log \left( \frac{\varepsilon}{\rho x^2} \right) \text{ m}\text{s}^{-2} \text{kg}^2 \]

\[ \varepsilon \approx T^{17} \]

\[ \varepsilon \approx T^4 \]

CNO

PP

Sun

\( T_6 (\text{K}) \)

0 5 10 15 20 25 30 35

-6 -4 -2 0
Stellar Models: Part II

Energy Creation Rates: pp-Chain or CNO-cycle?

- The energy creation rates for the 2 chains depend strongly on temperature.
- However, for the CNO-cycle to work at all, carbon, nitrogen and oxygen must be present – this will depend on the star type.
- At lower masses and, therefore, temperatures, the pp-chain dominates.
- At higher temperatures, there is a sudden transition to dominance by the CNO-cycle.
- The energy production rate varies strongly with temperature for the CNO-cycle and so is more important for heavier stars.
  - Heavier stars have higher interior temperatures.
Stellar Models: Part III

Modelling Stars as Gases

• The mean density of a star is about 1.4 times that of water.
• The central density (in the core) is about 150 times that of water.
• Nonetheless, we can model a stellar atmosphere as if it were a gas.
  • Why is this so?
Modelling Stars as Gases

- At high temperatures, hydrogen is ionised into protons and electrons.
- These are tiny compared to hydrogen atoms.
- Compare the density of hydrogen atoms to the density of protons.
Stellar Models: Part III

Hydrogen Atom Density

- Size of hydrogen atom: $\sim 1 \times 10^{-10} \text{m}$.
- The equivalent volume: $\sim 1 \times 10^{-30} \text{m}^3$.
- The equivalent number density: $n_H \sim 1 \times 10^{30} \text{m}^{-3}$.
- The equivalent mass density: $n_H m_p \sim 1.7 \times 10^3 \text{kg m}^{-3}$.

Proton Density

- Size of proton: $\sim 1 \times 10^{-15} \text{m}$.
- The equivalent volume: $\sim 1 \times 10^{-45} \text{m}^3$.
- The equivalent number density: $n_p \sim 1 \times 10^{45} \text{m}^{-3}$.
- The equivalent mass density: $n_p m_p \sim 1.7 \times 10^{18} \text{kg m}^{-3}$.
Stellar Models: Part III

Modelling Stars as Gases

- There is a factor of $10^{15}$ difference in density between protons and hydrogen atoms:
  - The density of protons is $\sim 1.7 \times 10^{18} \text{ kg m}^{-3}$.
  - The density of hydrogen atoms is $\sim 1.7 \times 10^{3} \text{ kg m}^{-3}$.
- Thus, we can treat the stellar atmosphere (which is mostly hydrogen atoms) as a gas.
Stellar Models: Part III

Interior of a Star with $M \lesssim 1.5M_{\odot}$
Stellar Models: Part III

Interior of a Star with $M \lesssim 1.5 M_\odot$

- In stars like the Sun, photons collide with each other every millimetre or so.
- It takes them about 50,000 years to escape the Sun.
- Hydrogen is converted to helium in the core; very little helium escapes.
- Both radiation and convection transport heat energy from the core to the surface.
Stellar Models: Part III

Interior of a Star with $M \gtrsim 6M_\odot$
Stellar Models: Part III

Equilibrium Equations

- The heat from the core prevents the star collapsing further due to gravity.
- It provides a pressure that keeps the star in hydrostatic equilibrium.
- Simple models allow us to find the central temperatures and pressures.
Equilibrium Equations

Segment of a star with slab, surface area A

Slab seen side-on
Equilibrium Equations

• Take a small slab of a star, surface area $A$, thickness $dr$, at a distance $r$ from the centre.
• $M(r)$ is the mass of the star inside radius $r$.
• $\rho(r)$ is the density of the star at radius $r$.
• We can calculate the mass of the slab:

$$dm = \rho(r)A dr$$
Stellar Models: Part III

Equilibrium Equations

- Now that we know the mass we can calculate the forces on the slab due to gravity:

\[
F_{\text{gravity}} = -\frac{GM(r)dm}{r^2} = -\frac{GM(r)\rho(r)Adr}{r^2}
\]

- This inward force is exactly balanced by the slightly greater pressure on the bottom of the slab:

\[
F_{\text{pressure}} = (\text{force on top}) - (\text{force on bottom}) = AdP
\]
Stellar Models: Part III

Equilibrium Equations

- In equilibrium, the slab does not move:

\[ F_{\text{pressure}} = F_{\text{gravity}} \]

\[ \Rightarrow AdP = -\frac{GM(r)\rho(r)Adr}{r^2} \]

\[ \Rightarrow \frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \]

- Note the negative sign: pressure drops as radius increases.
Hydrostatic Equilibrium

- The equation of hydrostatic equilibrium is:

\[
\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \quad (18)
\]

- With a few more assumptions, we can use this to work out the central pressure and temperature of a star.
Stellar Models: Part III

Central Pressure

- We model the star as having a uniform density, \( \rho_{av} \):

\[
M(r) = \frac{4}{3} \pi r^3 \rho_{av}
\]

- Insert this into equation 18:

\[
\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}
\]

\[
= -\frac{4}{3} \pi G \rho_{av}^2 r
\]

- Integrate to find the central pressure.
Central Pressure

- Pressure is $P_o$ at centre and zero at the surface of the star.
- At the centre $r = 0$ and at the surface $r = R$.

\[
\int_{0}^{P_o} dP = -\frac{4}{3} \pi G \rho_{av}^2 \int_{0}^{R} r dr
\]

\[
\Rightarrow [P]_{P_o}^{0} = -\frac{4}{3} \pi G \rho_{av}^2 \left[ \frac{r^2}{2} \right]_{0}^{R}
\]

\[
\Rightarrow [0 - P_o] = - \left[ \frac{4}{3} \pi G \rho_{av}^2 \frac{R^2}{2} - 0 \right]
\]

\[
\Rightarrow P_o = \frac{4}{3} \pi G \rho_{av}^2 \frac{R^2}{2}
\]
Central Pressure of the Sun

- Use an average density for the Sun:

  \[ \rho_{av} = \frac{M_{\odot}}{\frac{4}{3} \pi R_{\odot}^3} = 1409 \text{ kgm}^{-3} \]

- This implies a central pressure that is about a billion times higher than air pressure (at sea level):

  \[ P_o = 1.3 \times 10^{14} \text{ Nm}^{-2} \]
Stellar Models: Part III

Central Pressure of the Sun

- Better models of the stellar interior allow for density to increase towards the centre.
- They predict:
  - $P_o = 1 \times 10^{16} \text{Nm}^{-2}$
  - $\rho_o = 1 \times 10^5 \text{kgm}^{-3}$ (about 10 times the density of lead).
Central Temperature of the Sun

- We assume an *equation of state* to describe the gaseous stellar atmosphere.
- This is the *Ideal Gas Law*:
  \[ PV = nRT \]  \hspace{1cm} (19)

- \( P \) is pressure and \( V \) is volume.
- \( n \) is the number of moles of the gas.
- \( R = N_A k \) is the ideal gas constant.
  - \( N_A \) is Avogadro’s constant and \( k \) is Boltzmann’s constant.
Central Temperature of the Sun

- Re-write the Ideal Gas Law:

\[ P = \frac{nN_A kT}{V} \]

\[ = \frac{\rho}{\mu} kT \]

- \( \rho = \frac{M}{V} = \frac{\mu n N_A}{V} \) is the density.
  - \( \mu \) is the mean molecular weight of atoms present in the atmosphere.
Central Temperature of the Sun

- **Rearrange:**

\[ T_o = \frac{\mu P_o}{k \rho_{av}} = 10^7 \, K \]

- We assumed constant density here \((\rho_{av})\) but better models give approximately the same answer.
  - Changes to \(P_o\) and \(\rho\) in more sophisticated density models offset each other in this calculation.

Example Question

- Question 4.18 from the Tutorial Book.
Post Main Sequence Evolution

Mass-Luminosity Relationship Revisited

• Recall the relationship for Main Sequence stars:

\[ L \propto M^{3.5} \]  

(20)

• Why is the value of the exponent 3.5?
• The full theory is complicated but we can approximate it using the results from the last lecture.
Post Main Sequence Evolution

Mass-Luminosity Relationship Revisited

\[ T_o \propto \frac{P_o}{\rho_{av}} \]

\[ \Rightarrow T_o \propto \rho_{av}^2 R^2 \]

\[ \Rightarrow T_o \propto \rho_{av} R^2 \]

\[ \Rightarrow T_o \propto \frac{M}{R} \]

- i.e.: The core temperature is proportional to \( M/R \).
Mass-Luminosity Relationship Revisited

- Let’s assume that surface temperature is proportional to core temperature.
  - Actually, it depends slightly on $R$.
- Remember Stefan’s Law:

\[
L = 4\pi R^2 \sigma T^4_E
\]

\[\Rightarrow L \propto \frac{R^2 M^4}{R^4} \]

\[\Rightarrow L \propto \frac{M^4}{R^2} \]

- Assume constant density.
Post Main Sequence Evolution

Mass-Luminosity Relationship Revisited

- We assumed constant density:

\[ M \propto R^3 \]

\[ \Rightarrow R^2 \propto M^{2/3} \]

- Substitute this into the previous result:

\[ L \propto M^{3.3} \]

- This is quite a good fit, considering the approximations.
Post Main Sequence Evolution

Mass-Luminosity Relationship: Implications

• The most massive stars are also the most luminous.
• If the amount of fuel available is proportional to the mass, then we can work out the Main Sequence lifetime of a star:

\[ \tau_{MS} \propto \frac{M}{L} \]

\[ \Rightarrow \tau_{MS} \propto M^{-2.5} \]

• This means that more massive stars evolve faster.
  • More massive stars will spend less time on the Main Sequence.
Post Main Sequence Evolution

Cluster Evolution

- We assume that stars in a cluster are...
  - All the same age.
  - Different masses.
- The more massive stars will evolve off the Main Sequence first.
Post Main Sequence Evolution

Main-Sequence Turnoff

[Graph showing the evolution of stars from the main sequence to red giants, with labels for different masses and temperatures.]
Post Main Sequence Evolution

Cluster Evolution

• Let’s assume that stars leave the Main Sequence when they have converted 20% of the hydrogen fuel to helium.

• Let’s assume that this releases $\Delta E$ Joules of energy for every kilogram of hydrogen used.

\[
L = \frac{\text{total energy released}}{\text{Main Sequence lifetime}}
\]

\[
\Rightarrow L = \frac{0.2M \Delta E}{\tau_{MS}}
\]

• We can get $L$ from the mass-luminosity relationship.

• Then we can work out the lifetime spent on the Main Sequence, $\tau_{MS}$.
Cluster Evolution

- Let’s avoid using all the niggly constants in these equations and relate the Main Sequence lifetime for our star to something we know about – the Sun.

\[ \tau_{MS} = \frac{0.2M\Delta E}{L} \]

\[ \tau_{MS\odot} = \frac{0.2M_\odot\Delta E}{L_\odot} \]

- Divide the top equation by the bottom one:

\[ \frac{\tau_{MS}}{\tau_{MS\odot}} = \left( \frac{M}{M_\odot} \right) \left( \frac{L_\odot}{L} \right) \]
Post Main Sequence Evolution

Cluster Evolution

- Now use the mass-luminosity relationship:

\[
\tau_{MS} = \tau_{MS\odot} \left(\frac{M}{M\odot}\right) \left(\frac{L}{L\odot}\right)
\]

\[
\Rightarrow \tau_{MS} = \tau_{MS\odot} \left(\frac{M}{M\odot}\right) \left(\frac{M\odot}{M}\right)^{3.5}
\]

\[
\Rightarrow \tau_{MS} = \tau_{MS\odot} \left(\frac{M}{M\odot}\right)^{-2.5}
\]

\[
\Rightarrow \tau_{MS} = \tau_{MS\odot} \left(\frac{M\odot}{M}\right)^{2.5}
\]
Post Main Sequence Evolution

Cluster Evolution

- We can work out the Main Sequence lifetime of the Sun:

\[ \tau_{MS\odot} = \frac{0.2 M_\odot \Delta E_{H-He}}{L_\odot} \]

\[ \Rightarrow \tau_{MS\odot} = 2 \times 10^{10} \text{ years}. \]

- The previous result shows that a star 4 times as massive as the sun has a Main Sequence lifetime that is 32 times shorter.

Example Question

- Question 4.19 from the Tutorial Book.
Post Main Sequence Evolution

Ages of Star Clusters

- We know the relationship between age (in years) and mass:

\[
\tau_{MS} = 2 \times 10^{10} \left( \frac{M_{\odot}}{M} \right)^{2.5}
\]  

(21)

- To find the age of a cluster, we look for the highest mass star still on the Main Sequence.
- This would be a star near the top-left hand side of the HR-diagram.
- This star will allow us to determine the age of the cluster.
  - The ages of clusters are important for determinations of the age of the Universe.
Post Main Sequence Evolution

Young versus Old

![Diagrams showing luminosity versus temperature for M67 and M4 clusters.](image)
Post Main Sequence Evolution

Young *versus* Old

- M67 is a young (open) cluster.
- M4 is an old (globular) cluster.
- The Main Sequence lifetime is significantly shorter for the older cluster.
- The location of the turnoff differs precisely because of the relationship derived in equation 21.
- Can use the luminosity and temperature of stars at the turnoff point to date the clusters.
Evolution off the Main Sequence

- The details can get a bit complicated so we take a broad view:
  - Low mass stars ($\sim 1 M_\odot$).
  - High mass stars ($\gtrsim 8 M_\odot$).
Evolution of Low Mass Stars ($\sim 1 \, M_\odot$)

- Eventually H-He reactions runs out of fuel in the core.
- Then the pressure drops and the core collapses.
  - The temperature of the core increases to $\sim 10^8$ K.
- The helium starts to fuse to carbon in the core.
  - This is the *Triple Alpha* process.
Post Main Sequence Evolution

Triple Alpha Process

\[ ^4\text{He} + ^4\text{He} \rightarrow ^8\text{Be} + \text{energy} \]
\[ ^8\text{Be} + ^4\text{He} \rightarrow ^{12}\text{C} + \text{energy} \]
\[ ^{12}\text{C} + ^4\text{He} \rightarrow ^{16}\text{C} + \text{energy} \]

- The \(^4\text{He}\) nucleus is also known as an \(\alpha - \text{particle}\).
Post Main Sequence Evolution

Helium Fusion in a Low Mass Star ($\sim 1 \, M_\odot$)

- Helium fusion is a highly *exothermic* process.
  - The temperature of the core climbs further.
- Un-burned hydrogen around the core starts to fuse.
  - Hydrogen is converted to helium again.
- All this extra energy causes the luminosity to increase.
Post Main Sequence Evolution

Red Giants

- The increased luminosity provides extra radiation pressure.
  - The outer envelope of gas inflates and cools.
  - For a star of $\sim 1 \, M_{\odot}$, this equates to a radius of $\sim 100 \, R_{\odot}$.
- A high luminosity star with cool outer regions is called a Red Giant.
Post Main Sequence Evolution

Red Supergiants

• When the helium core burning stops there is a further collapse and rise in temperature.
• The shell of helium around the core starts to burn.
• The star becomes a Red Supergiant.
Post Main Sequence Evolution

Inside an Evolved Low Mass Star (\(\sim 1 \, M_\odot\))
Post Main Sequence Evolution

Instability

- The star is now unstable.
  - It pulses slowly and its luminosity increases by a factor of about 10 over a period of about 5000 years.
- The outer layers separate and are blown off.
  - A planetary nebula is formed and the hot core is exposed.
  - This process accounts for about half the star’s mass.
Post Main Sequence Evolution

White Dwarfs

- When all the helium fuel is used up, the star is supported by *degeneracy pressure*.
- It slowly cools to become a *White Dwarf*.
Post Main Sequence Evolution

Evolutionary Track of a Low Mass Star
Post Main Sequence Evolution

Planetary Nebula NGC 2440 with White Dwarf Visible

http://apod.nasa.gov/apod/ap070215.html
Post Main Sequence Evolution

Evolution of High Mass Stars ($\gtrsim 8 M_\odot$)

- Evolution is similar to that of low mass stars but there is added gravitational pressure due to the higher mass.
  - Higher temperatures are possible.
  - More (different) nuclear reactions are possible.
- After helium is used up, the core collapses until degeneracy sets in to stop star shrinking.
  - Temperature goes up and helium-carbon fusion starts.
- This cycle repeats through heavier and heavier elements up to iron.
  - The star consists of concentric shells – the onion model of the interior.
Post Main Sequence Evolution

Final Structure of a Supermassive Star

- H Burning Shell
- He Burning Shell
- C Burning Shell
- Ne Burning Shell
- O Burning Shell
- Si Burning Shell

Core Radius: $\sim 1 \, R_{\text{earth}}$

Envelope Radius: $\sim 5 \, \text{AU}$
Post Main Sequence Evolution

The Iron Catastrophe

![Diagram showing binding energy per nuclear particle vs mass number, highlighting the transition from hydrogen fusion to iron and then to uranium fission.](image-url)
Post Main Sequence Evolution

The Iron Catastrophe

- Elements can only fuse in the core if there is a nett drop in potential energy of the protons during the reaction.
- After a certain point (iron) fusion stops producing energy and starts absorbing it instead.
- There is a catastrophic drop in core pressure.
  - The core collapses in $\sim 0.25 \text{ s}$.
  - The star explodes in a supernova.
## Post Main Sequence Evolution

### Evolutionary Sequence of a High Mass Star ($\gtrsim 8 \, M_\odot$)

<table>
<thead>
<tr>
<th>Event</th>
<th>Temperature (K)</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen Burning</td>
<td>$4 \times 10^7$</td>
<td>$7 \times 10^6$ years (MS)</td>
</tr>
<tr>
<td>Helium Burning</td>
<td>$2 \times 10^8$</td>
<td>$7 \times 10^5$ years</td>
</tr>
<tr>
<td>Carbon Burning</td>
<td>$6 \times 10^8$</td>
<td>600 years</td>
</tr>
<tr>
<td>Neon Burning</td>
<td>$1.2 \times 10^9$</td>
<td>1 year</td>
</tr>
<tr>
<td>Oxygen Burning</td>
<td>$1.5 \times 10^9$</td>
<td>6 months</td>
</tr>
<tr>
<td>Silicon Burning</td>
<td>$2.7 \times 10^9$</td>
<td>1 day</td>
</tr>
<tr>
<td>Core Collapse</td>
<td>$5.4 \times 10^9$</td>
<td>0.25 second</td>
</tr>
<tr>
<td>Supernova ...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Post Main Sequence Evolution

Supernovae

- Eventually the core has no more nuclear fuel left to sustain the high temperature and pressure.
- The core is too massive to be supported by degeneracy pressure.
- The core collapses, forming a Neutron star or a Black Hole.
- The outer layers of the star are blown off – the supernova remnant.
Post Main Sequence Evolution

RCW 86: Supernova in 185 AD; Remnant Observed in 2006 AD

Kelvin Building – Lecture Room 222, 1959