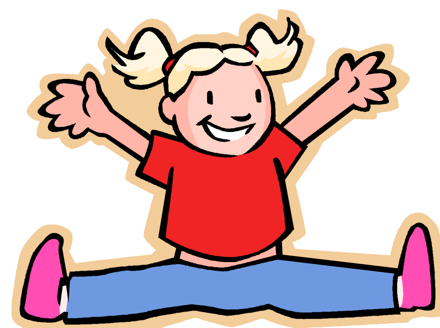
Motivation

- We want to understand the universe.

- The universe is complicated.
- Mathematical models describing the universe are frequently too complicated to solve analytically.
- We use numerical simulations to approximate solutions to mathematical models.



- We understand the universe a little better.

Motivation

- Numerical simulations allow us to play around with a system and observe how different initial conditions and different physical processes affect the system.
- Understand the consequences of physical laws.
- Help interpret observations/experiments.
- Predict new requirements for observations/experiments.

Motivation

- What happens if we change the initial conditions?
  - Distribution of particles (e.g. Maxwellian or power-law)?
  - Initial distribution parameters (e.g. position, energy spectra, temporal)?

- What happens if we add/remove physical processes?
  - Particle transport and/or diffusion.
  - Particle collisions.

- What happens if we interact with waves?
  - Electromagnetic waves.
  - Plasma waves.

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Basic Process

Program Description

- Pseudo code is a great way to plan your program before you start at the computer.
- Description of your code is there to help YOU (and others).

Initialise slide to 1

Introduce the topic to be studied

WHILE lecture not complete

    Increment the slide number

    Describe the concept on the slide

    IF additional explanation is required

        Draw on the whiteboard to assist the description

    END IF

END WHILE

Program Description Examples

Initialise variables
Get ready for the night out
Arrange transport
Meet friends at the pub
WHILE not drunk or tired or bored
    socialise
    IF thirsty or want alcohol then drink
    IF location has moved to club
        IF music is good then dance
    END IF
END WHILE
Return home
Drink water
Sleep

Initialise core variables
READ input spreadsheet of distributions
Set important variables
IF any variables are junk
    set them to zero
    flag the indices in the error array
END IF
FORALL distributions
    calculate distribution moments
END FORALL
FORALL distributions
    correlate moments
    IF correlation is high then generate fit
END FORALL
Plot relevant graphs.

Code Comments

- Have an initial description of what the program does including inputs and outputs.
- Include a sample call to the program.
- Add comments throughout the code to explain what the program is doing.

```
Function y=square(x)
% This program will find the
%  square of any number
% INPUTS:
%  x – the number that will be
%  squared
% OUTPUTS:
%  y – the value of x2
% Sample Call:
%  y = square(5);

y = x^2
%  This line finds the square of x
```

Numerical Approaces

- Test-particle
- Kinetic
- MHD

Propagation Example

- Usually we deal with **INITIAL VALUE PROBLEMS**. We know the initial value of our function at t=0. We then integrate over t to find the function at t=tmax.

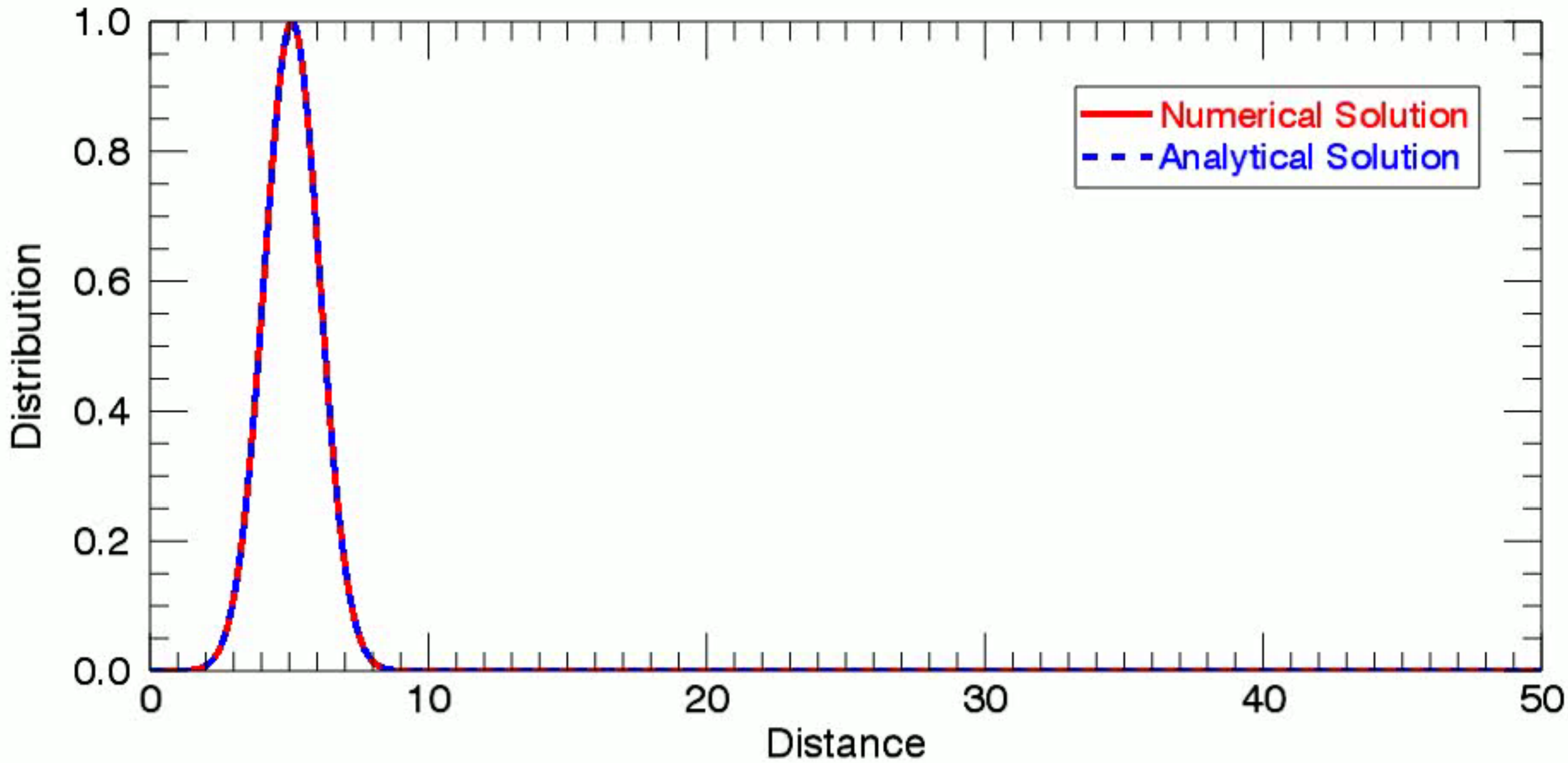
- Propagation Equation:

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$$

- Start with an initial condition:
- d/2 = stdev, x_0=mean.

$$u(x,t=0) = \exp\left(\frac{-(x-x_0)^2}{d^2}\right)$$

Propagation Example



Finite Difference

- We represent our function u using the notation u_j^n
- $t_{min} \leq t_n \leq t_{max}$ at $x_{min} \leq x_j \leq x_{max}$

- Using a simple Euler method we represent derivatives as

$$\frac{\partial u}{\partial t} = \frac{u_j^{n+1} - u_j^n}{\Delta t} \qquad \frac{\partial u}{\partial x} = \frac{u_{j+1}^n - u_j^n}{\Delta x}$$

$$\boxed{\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0} \Rightarrow \frac{u_j^{n+1} - u_j^n}{\Delta t} = -v\frac{u_{j+1}^n - u_j^n}{\Delta x}$$

Finite Difference

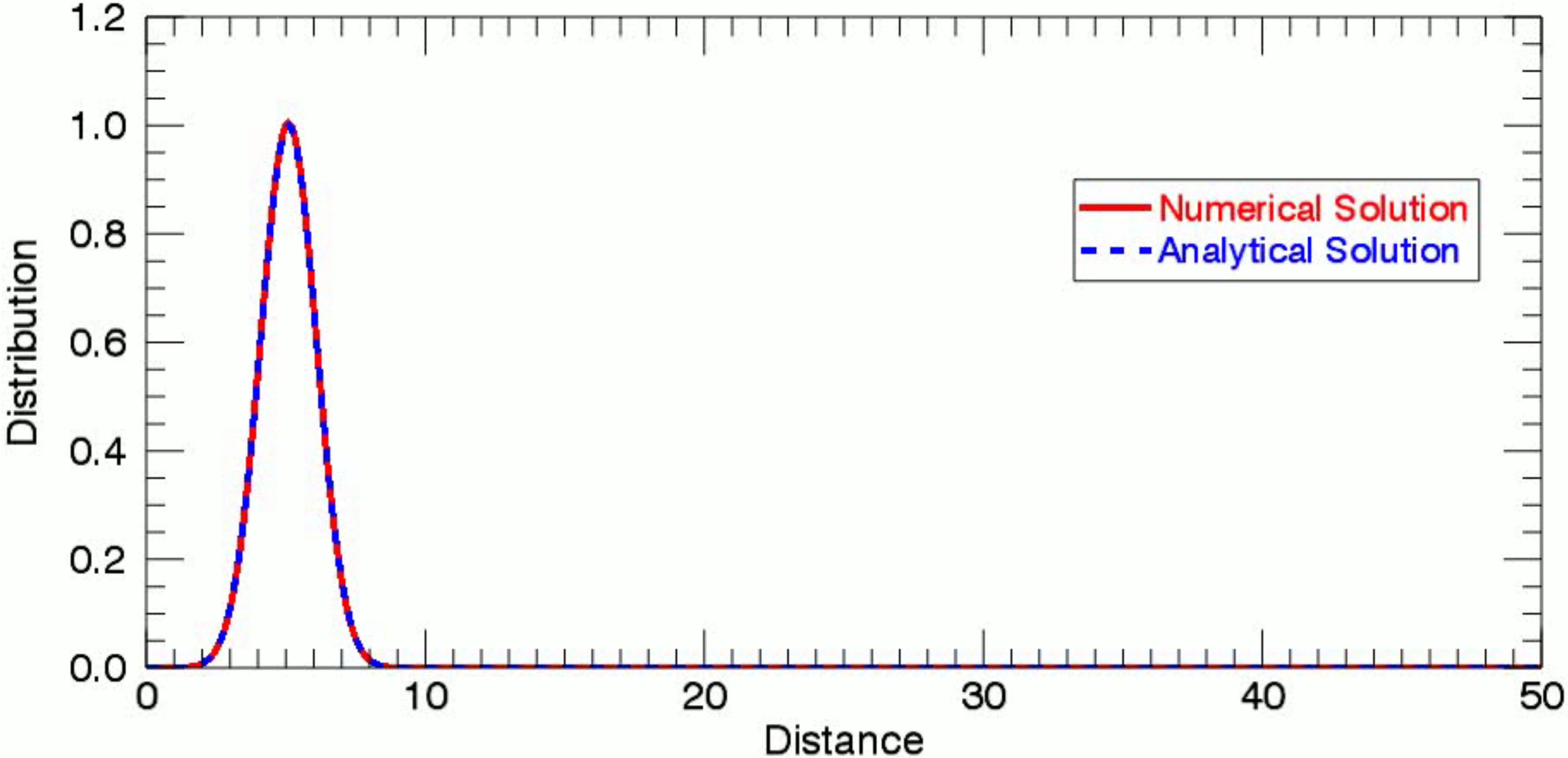
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v\,\frac{u_{j+1}^n - u_j^n}{\Delta x} \Rightarrow u_j^{n+1} = u_j^n - \frac{v\Delta t}{\Delta x}\left(u_{j+1}^n - u_j^n\right)$$

- We have $u_j^{n=0}$ and we want to have $u_j^{n=1}$

- We can use other more complicated schemes than the simple Euler scheme. For instance the central difference method would look like:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

Propagation Example



Numerical Stability

- What can causes this numerical instability?
- We can find out through the **von Neumann Stability Analysis**.
- If we assume that the system is stable in space and time we can find the solution or *eigenmodes* of the equations are of the form:

$$u_j^n = \xi^n e^{ikj\Delta x}$$

- where k is a real spatial wave number and ξ is complex. The system is unstable if $|\xi| > 1$ for some k where ξ is called the **amplification factor**.

Numerical Stability

- Substitute the value of u back into the equation for the finite difference approximation and we get

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

$$\frac{\xi^{n+1} e^{ikj\Delta x} - \xi^n e^{ikj\Delta x}}{\Delta t} = -v \frac{\xi^n e^{ik(j+1)\Delta x} - \xi^n e^{ik(j-1)\Delta x}}{2\Delta x}$$

- Now divide through by the value of u and we are left with

$$\frac{\xi - 1}{\Delta t} = -v \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}$$

Numerical Stability

$$\frac{\xi - 1}{\Delta t} = -v \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}$$

- Rearrange the above equation and we obtain:

$$\xi = 1 - \frac{v\Delta t}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$

- Using the relation that $e^{ix} - e^{-ix} = 2i\sin(x)$ we obtain

$$\xi = 1 - i\frac{v\Delta t}{\Delta x}\sin(k\Delta x)$$

- The modulus of this function is always > 1 for *all* k.

Numerical Stability

- The von Neumann method is not rigorous but generally gives a good approximation for your scheme.
- It can be VERY useful when dealing with your code as it allows you to asses the stability criteria and tweak your timestep to make sure your system is stable.

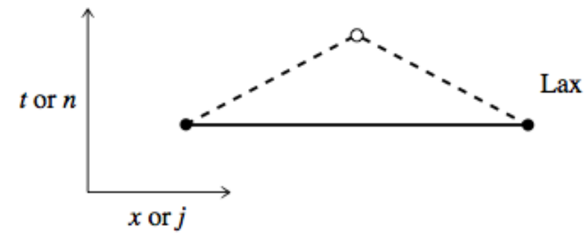
- Is the future bleak for our method? Will it always be unstable?

Lax Method

- Of course not! Recall $u_j^{n+1} = u_j^n - \frac{v\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$

- We can perform a simple change to FTCS. We replace

$$u_j^n \rightarrow 0.5\left(u_{j+1}^n + u_{j-1}^n\right)$$



$$u_j^{n+1} = 0.5(u_{j+1}^n - u_{j-1}^n) - \frac{v\Delta t}{2\Delta x} u_{j+1}^n - u_{j-1}^n$$

- We obtain an amplification factor such that:

$$\xi = \cos(k\Delta x) - i\frac{v\Delta t}{\Delta x}\sin(k\Delta x)$$

Courant Condition

$$\xi = \cos(k\Delta x) - i\frac{v\Delta t}{\Delta x}\sin(k\Delta x)$$

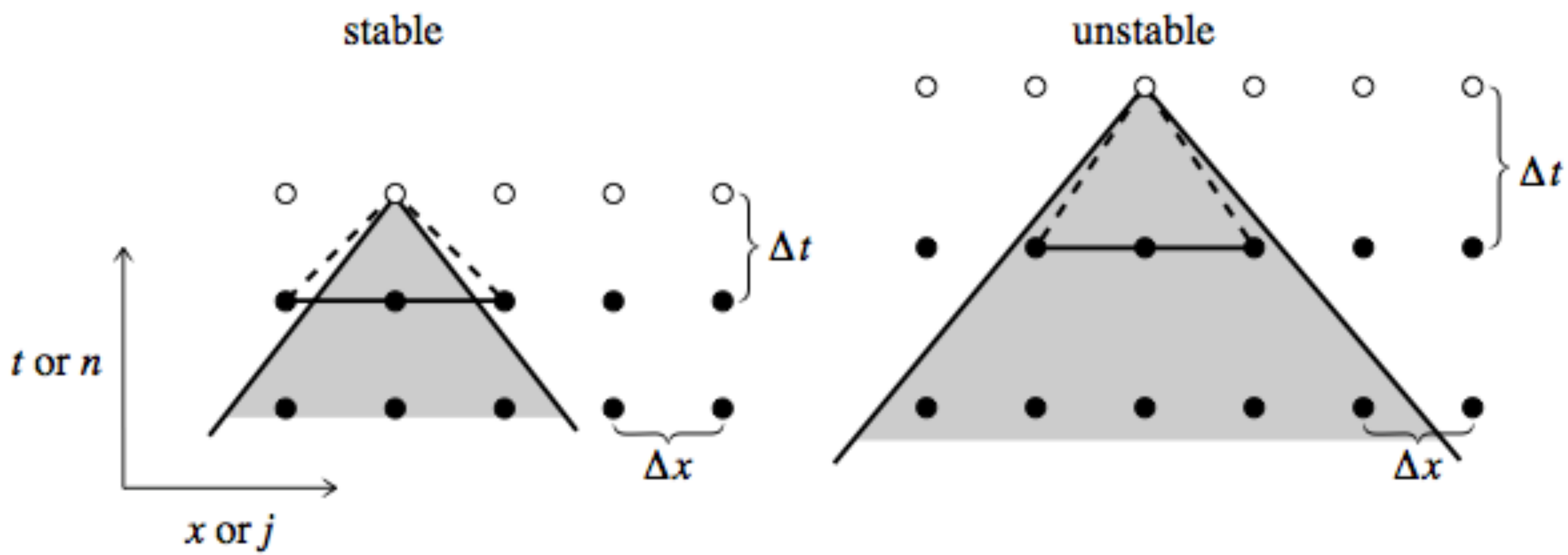
- We now have a system where if $|\xi|^2 < 1$ then we must have

$$\frac{|v|\Delta t}{\Delta x} \leq 1$$

- This is the Courant-Friedrichs-Lewy stability criterion or simply the **Courant condition** for short.
- What does this mean I hear you cry?

Courant Condition

- If you are travelling at a velocity v then you will go a distance vΔt in time Δt. If Δt is larger then you will travel a larger distance. You cannot travel farther in one step than the distance of a grid point Δx. Otherwise past information is not enough to evaluate your position.



LAX instability

- Unfortunately I have bent the truth. I have used the following condition for some of the previous simulations.

$$\frac{|v|\Delta t}{\Delta x} = 1$$

- When we have the condition that $\frac{|v|\Delta t}{\Delta x} < 1$ we get what is know as amplitude dissipation.

- This is because we have effectively added a diffusion term to the system (see Numerical Recipes 20.1).

LAX instability

- There is more than just amplitude error that we have to contend with. There is also a phase error that can cause problems.

- If $\frac{|v|\Delta t}{\Delta x} < 1$ then we can get a phase error in LAX.

- We can rewrite the previous equation for the amplification factor in a different way.

Amplitude Error

Phase Error

$$\xi = e^{-ik\Delta x} + i\left(1 - \frac{v\Delta t}{\Delta x}\right)\sin(k\Delta x)$$

Upwind Model

- What can we do about this?
- We can use a more simple model that **physically** deals with the problem better, the simple Euler upwind scheme we started with but including a little twist.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v \begin{cases} \dfrac{u_j^n - u_{j-1}^n}{\Delta x}, & v_j^n > 0 \\ \dfrac{u_{j+1}^n - u_j^n}{\Delta x}, & v_j^n < 0 \end{cases}$$

- When v is positive then u_{j+1} is affected but not u_{j-1}
- When v is negative then u_{j-1} is affected but not u_{j+1}

Upwind Stability

- Applying the same stability analysis to the upwind scheme we obtain the following amplification factor for constant v:

$$|\xi|^2 = 1 - 2\left|\frac{v\Delta t}{\Delta x}\right|\left(1 - \left|\frac{v\Delta t}{\Delta x}\right|\right)[1 - \cos(k\Delta x)]$$

- This satisfies the stability criteria of $|\xi|^2 < 1$ when we have the Courant condition (again).

Staggered Leapfrog

- Previously we were using methods that were first order accurate in time. What about a method that is second order accurate in time, the **staggered leapfrog**

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = -v \frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x}$$

- The staggered leaprog method is more stable but does not use the information at the last point in space and time.

Staggered Leapfrog

- The staggered leapfrog has a quadratic equation for the amplification factor that can be solved to give the following equation:

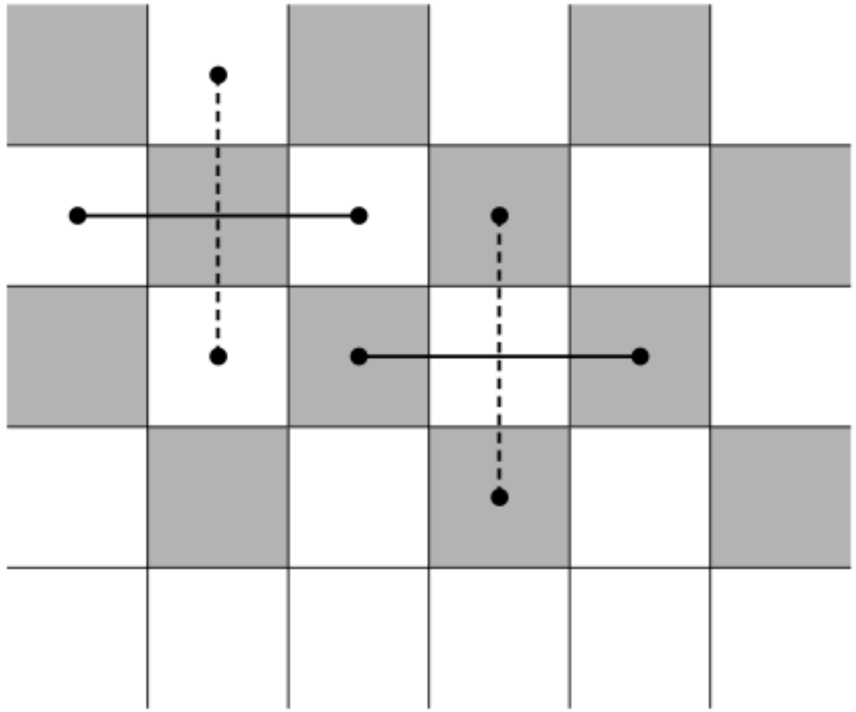
$$\xi = -i\frac{v\Delta t}{\Delta x}\sin(k\Delta x) \pm \sqrt{1-\left(\frac{v\Delta t}{\Delta x}\sin(k\Delta x)\right)^2}$$

- We actually have $|\xi|^2 = 1$ for any value of $v\Delta t \leq \Delta x$

- There is no amplitude dissipation!

Leapfrog Problems

- The staggered leapfrog is not the be-all and end-all of finite differencing mechanisms. It has problems with large gradients as points can become decoupled from each other.



- We can solve this problem using another advanced technique. See Numerical Recipes 20.1.

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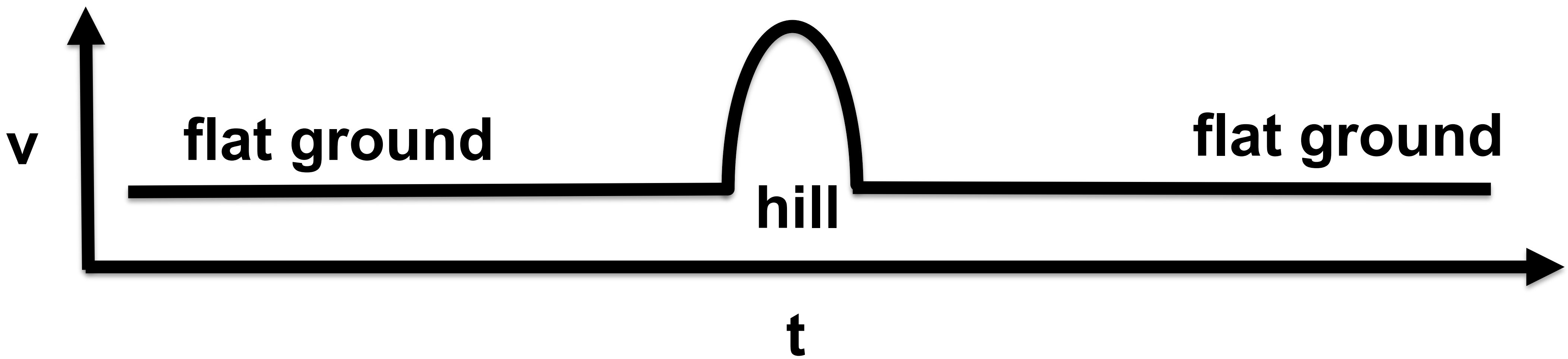
Using Stability Analysis

- What if our velocity was not a constant?

- We can use our stability analysis to make sure that our system is stable (to ensure that $v\Delta t \leq \Delta x$).
- This is VERY IMPORTANT when dealing with real world problems.

- Consider a person riding their bike to work….

Using Stability Analysis



- We must keep $v\Delta t \leq \Delta x$
- For a constant Δt, it will have to be small to deal with the hill but that will be tedious to deal with the flat ground.
- If we evaluate $v\Delta t \leq \Delta x$ we can decease Δt when we get to the hill and we can increase it afterwards.

Other Methods

- We can use Implicit methods that go backwards in time to go forward. Uses a tri-diagonal matrix of the method to find u_j^{n+1} .

- Implicit schemes are more complicated to code but are generally stable in time for any time step.

-

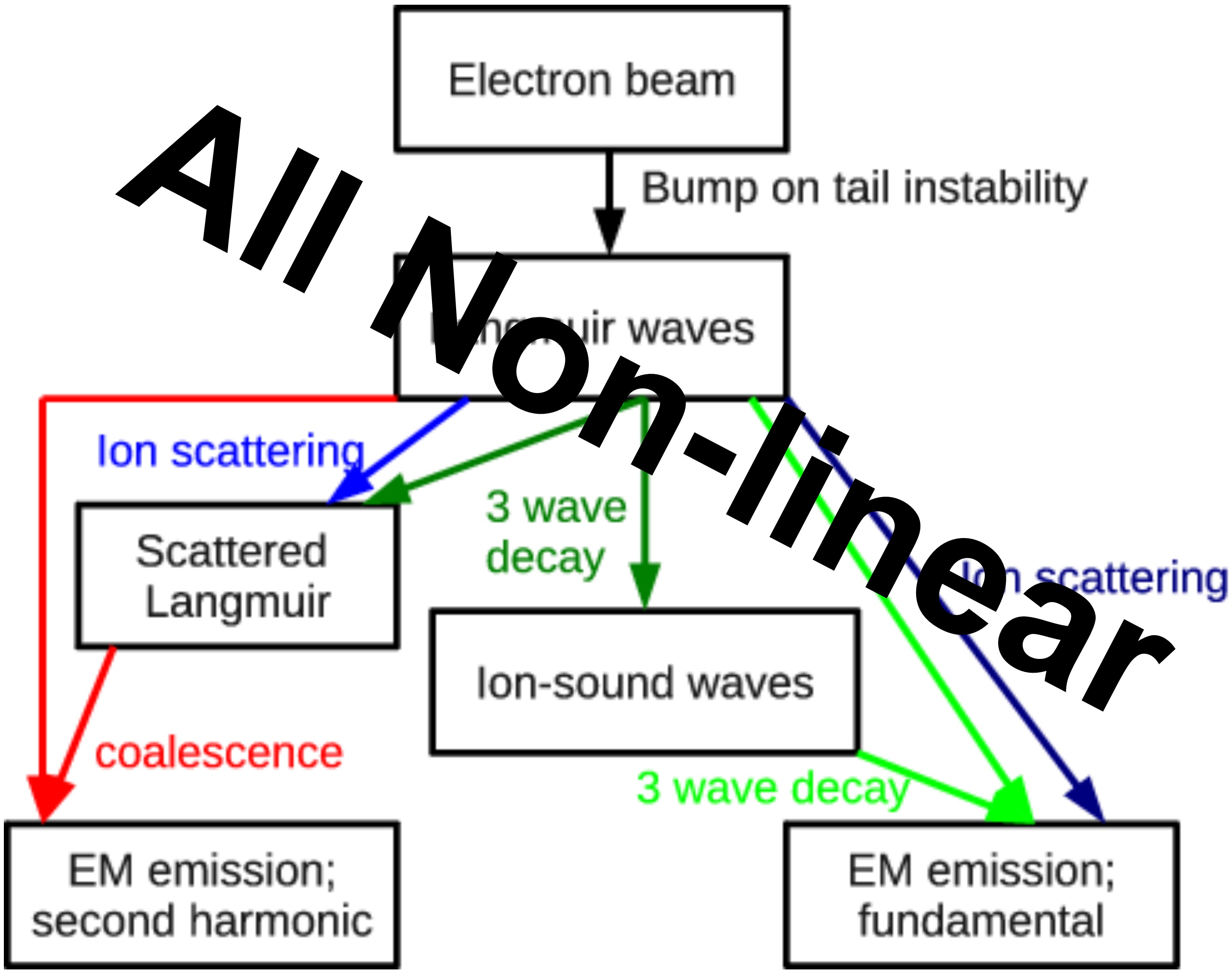
- Other techniques can be used such as the Crank-Nicolson scheme that are a hybrid between explicit and implicit schemes.

Radio Bursts

- Plasma emission is the generally accepted mechanism for non-thermal particles producing solar radio bursts *Ginzburg & Zhelezniakov (1958)*.

Coherent Process

Emission Process



Emission Process

- Most of the energy of the system is contained in the electrons.
- Some electron energy is deposited into the Langmuir waves.
- Some Langmuir wave energy is deposited into the radio waves.

- We can concentrate on the easier task of simulating the wave-particle interactions to understand the dynamics of the non-thermal particles.

Wave-particle Interaction

- Popular approach is the quasilinear approach that uses the WKB approximation (waves as particles).
- f(v,t) is the electron distribution function.
- W(v,t) is the Langmuir wave spectral energy density.

$$\frac{\partial f}{\partial t} = \frac{4\pi^2 e^2}{m^2} \frac{\partial}{\partial v} \left(\frac{W}{v} \frac{\partial f}{\partial v} \right)$$

$$\frac{\partial W}{\partial t} = \frac{\pi \omega_{pe}}{n_e} v^2 W \frac{\partial f}{\partial v}$$

$$\omega_{pe} = kv$$

Wave-particle Interaction

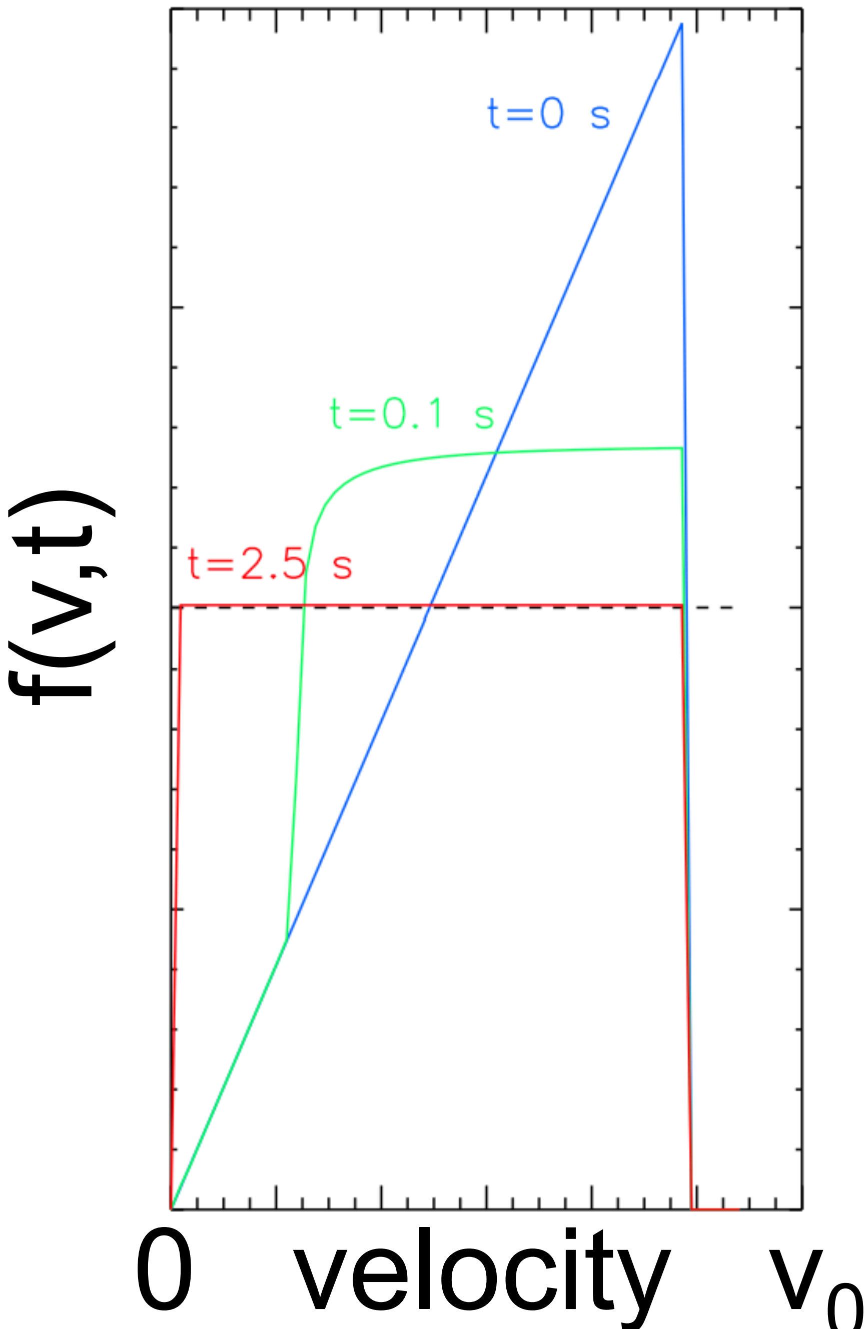
- Past work (e.g. Ryutov & Sagdeev, 1970, Vedenov & Ryutov, 1972, Mel'nik et al 1998, Kontar 2001) used the asymptotic solution for these equations assuming certain initial condition for f and W.

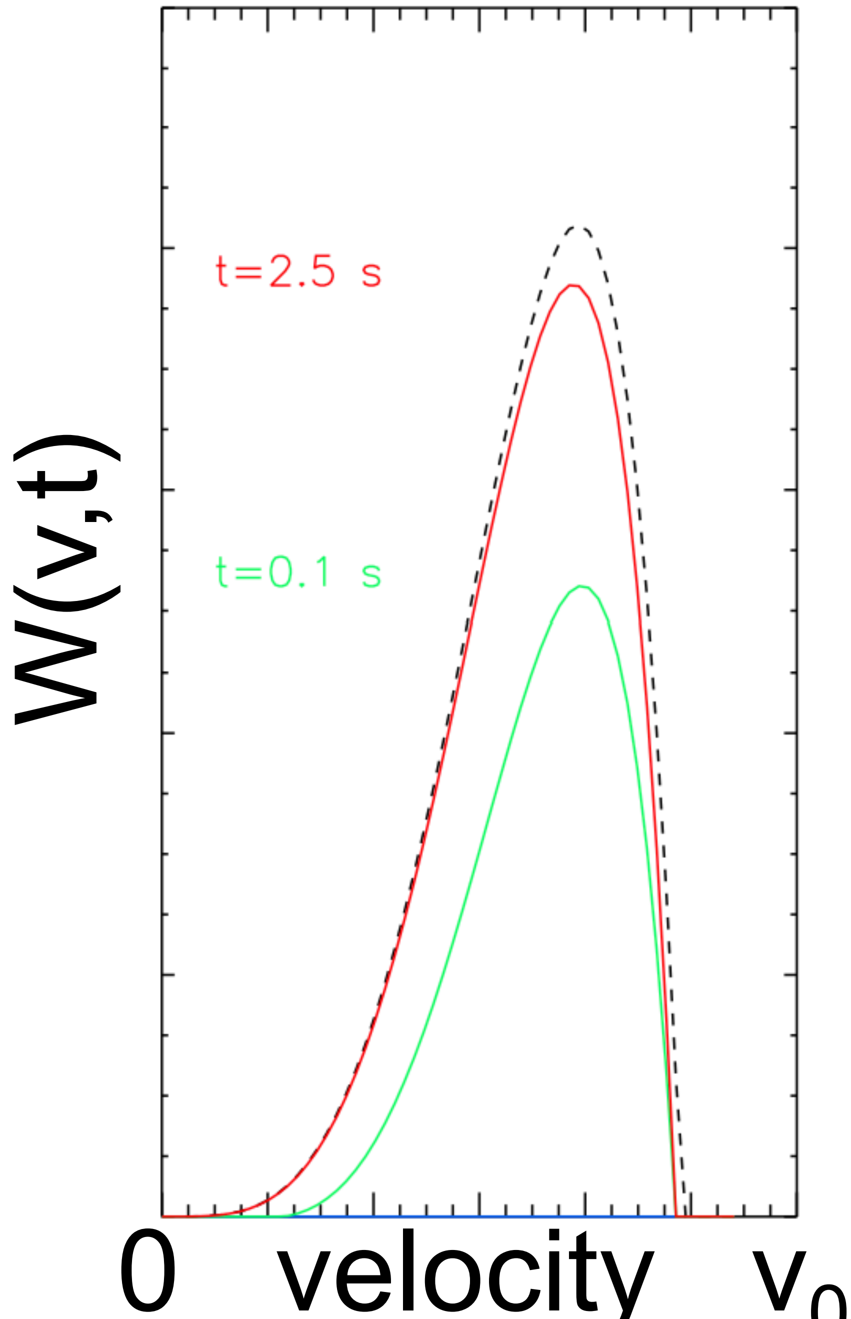
$$f(v,t=0) = \frac{2n_b v}{v_0^2},\ v < v_0 \qquad W(v,t=0) = \frac{\mathrm{T}}{2\pi^2 \lambda_D^2}$$

$$f(v,t=\infty) = \frac{n_b}{v_0},\ v < v_0$$

$$W(v,t=0) = \frac{\mathrm{m_e} n_b v^4}{v_0 \omega_{pe}} \int_0^v \left(1 - \frac{v}{v_0}\right) dv,\ v < v_0$$

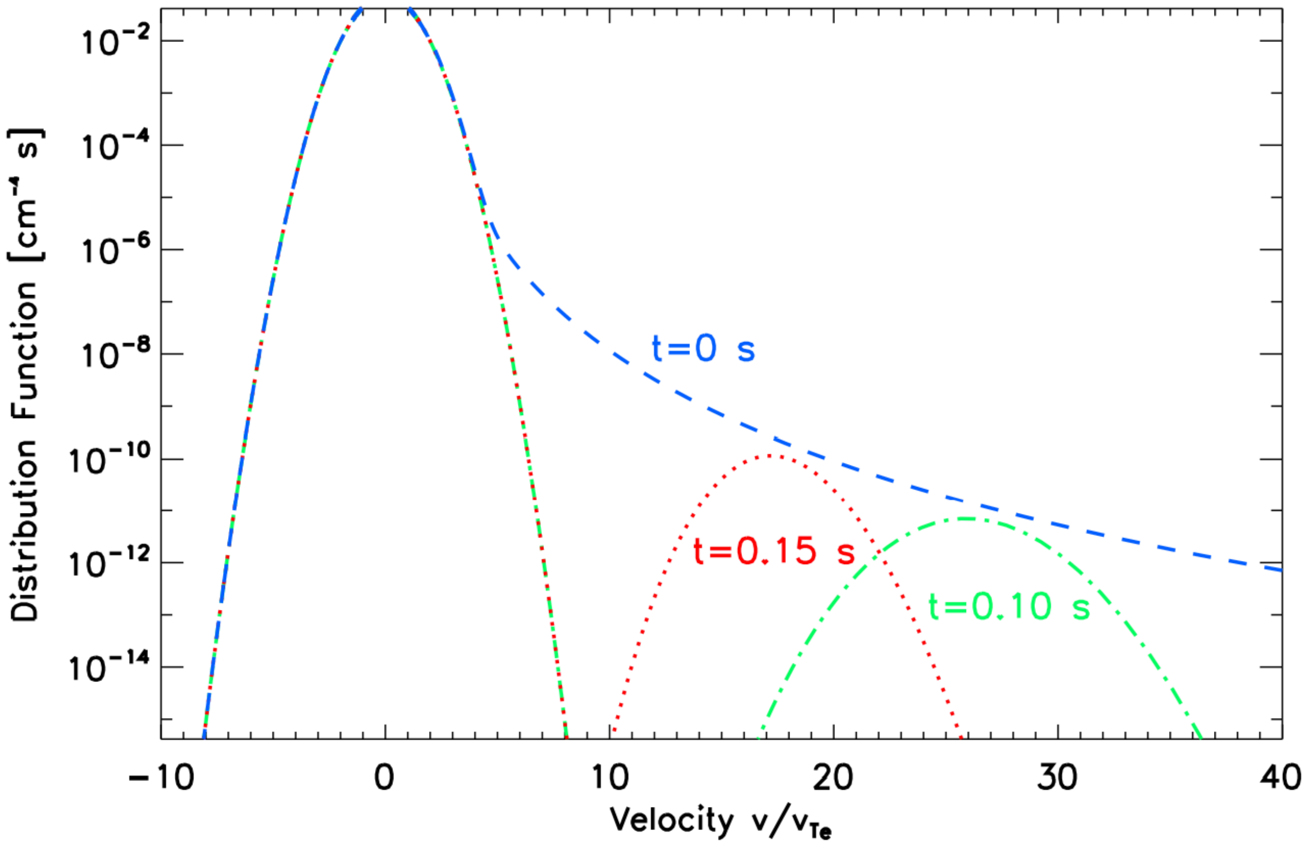
Quasilinear Interaction



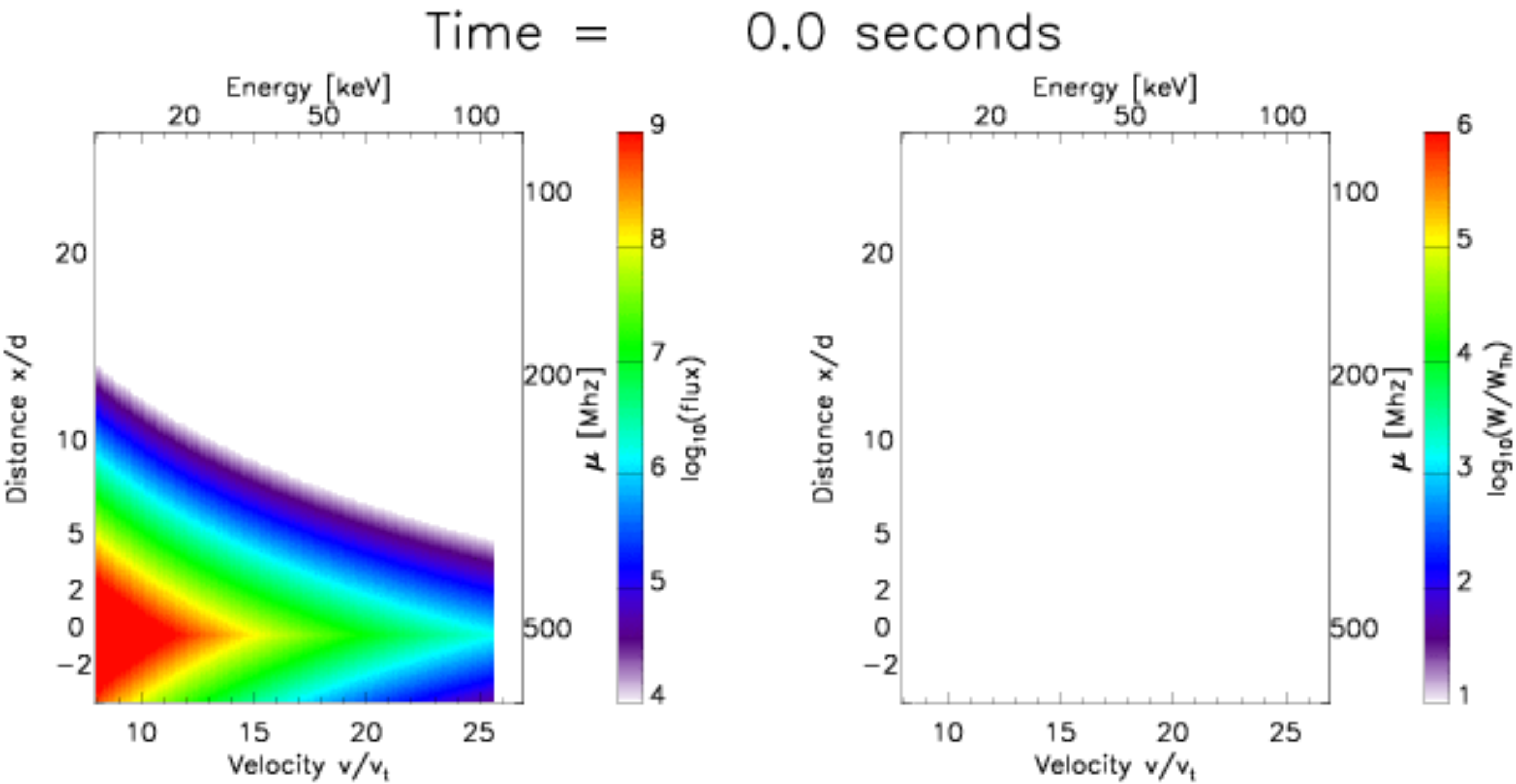


Quasilinear Interaction

- The bump-in-tail instability forms when an electron beam is injected at the Sun and streams away from the acceleration region.



Wave-Particle Instability



More Complicated Example

One dimensional quasilinear equations describing the kinetics of energetic electrons and Langmuir waves (e.g. Reid and Kontar 2015).

Radial Propagation and Expansion

Wave Generation and Absorption

$$\frac{\partial f}{\partial t} + \frac{v}{M(r)}\frac{\partial}{\partial r}M(r)f = \frac{4\pi^2 e^2}{m^2}\frac{\partial}{\partial v}\left(\frac{W}{v}\frac{\partial f}{\partial v}\right) - \gamma_c \frac{d}{dv}\left(\frac{f}{v^2}\right)$$

$$\frac{\partial W}{\partial t} + \frac{v^2}{L}\frac{\partial W}{\partial v} + \frac{3v_{Te}^2}{v}\frac{\partial W}{\partial x} = \frac{\pi\omega_{pe}}{n_e}v^2 W\frac{\partial f}{\partial v} - (\gamma_{cw} + \gamma_L)W + \gamma_s v f \log\left(\frac{v}{v_t}\right)$$

$$\omega_{pe} = kv$$

Background Plasma Inhomogeneity

Collisions

Spontaneous Emissions

Quasilinear time

- The characteristic time that is involved for the quasilinear interaction is related to the ratio of the beam density and the background density.

$$\tau_{ql} = \frac{n_e}{\pi \omega_{pe} n_b}$$

Sturrock's Dilemma

- A problem posed by Sturrock 1964 with the bump-in-tail instability is that all the energy from the particles will be deposited into the Langmuir waves over a very short distance.

- Consider: $n_e = 10^8$ cm^{-3}, $\omega_{pe} = 10^9$ s^{-1}, $n_b = 10^4$ cm^{-3}

- Quasilinear time is 10^{-6} s, much faster than any collisional damping rate of the waves by 10^{-2} s.

Sturrock's Dilemma

- We will lose ALL of the electron energy to Langmuir waves within a very small distance – metres!

- This is a large problem as we observe electron beams at the Earth, over distances of 1 AU.

- We need a way for the wave-particle interaction to be LESS efficient.

Beam-Plasma Structure

- One method to solve the dilemma is the beam-plasma structure. This was suggested analytically by Zheleznyakov & Zaitsev (1970) and further developed by Zaitsev et al. (1972).

- It was numerically worked on initially by Takakura & Shibahashi (1976); Magelssen & Smith (1977); Grognard (1985) but there have been many other authors that have worked on this subject.

- What is a beam-plasma structure?

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Beam-Plasma Structure

- Consider the growth rate of Langmuir waves is related to

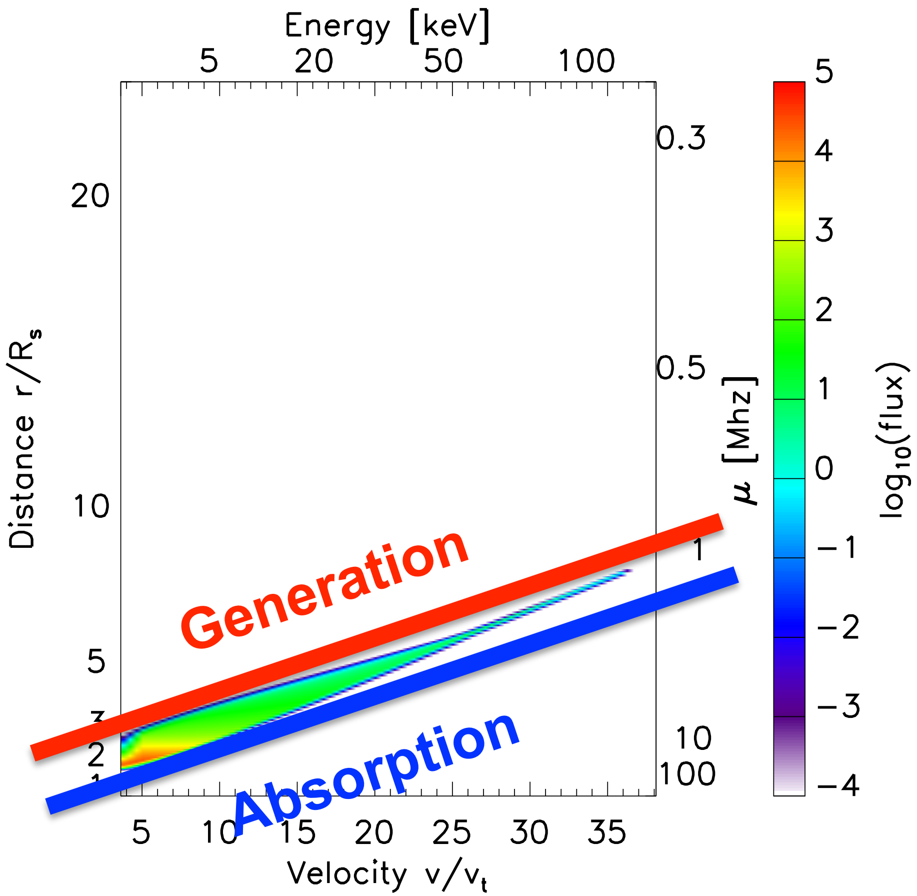
$$\frac{\partial W}{\partial t} = \frac{\pi \omega_{pe}}{n_e} v^2 W \frac{\partial f}{\partial v}$$

- The electron beam has a width in space. Langmuir waves are generated at the front of the beam and absorbed at the back of the beam.

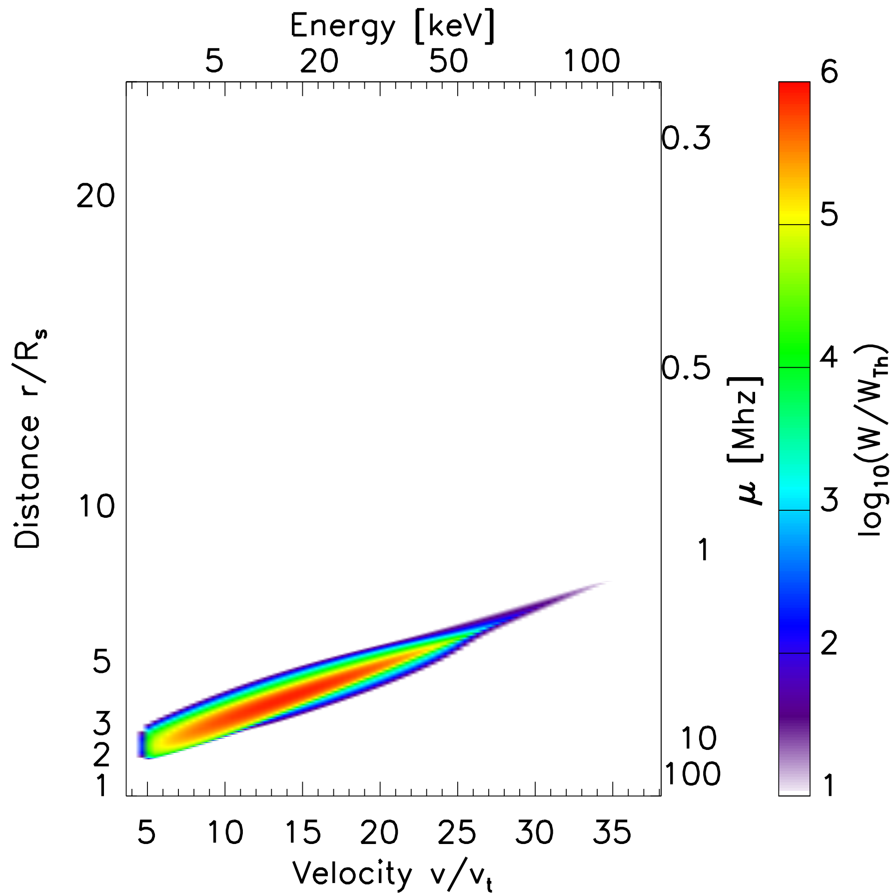
- Beam-plasma structure moves at the velocity of the electrons despite the Langmuir wave group velocity being low.

Beam-Plasma Structure

Time = 25.0 seconds



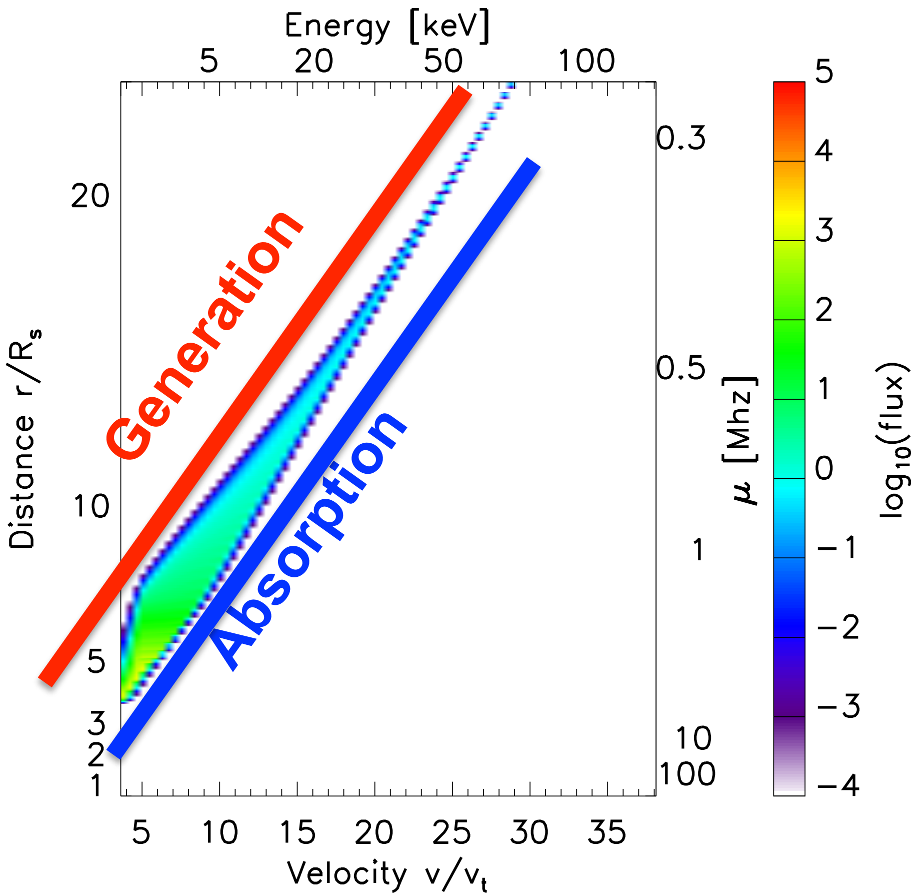
ELECTRON BEAM DISTRIBUTION

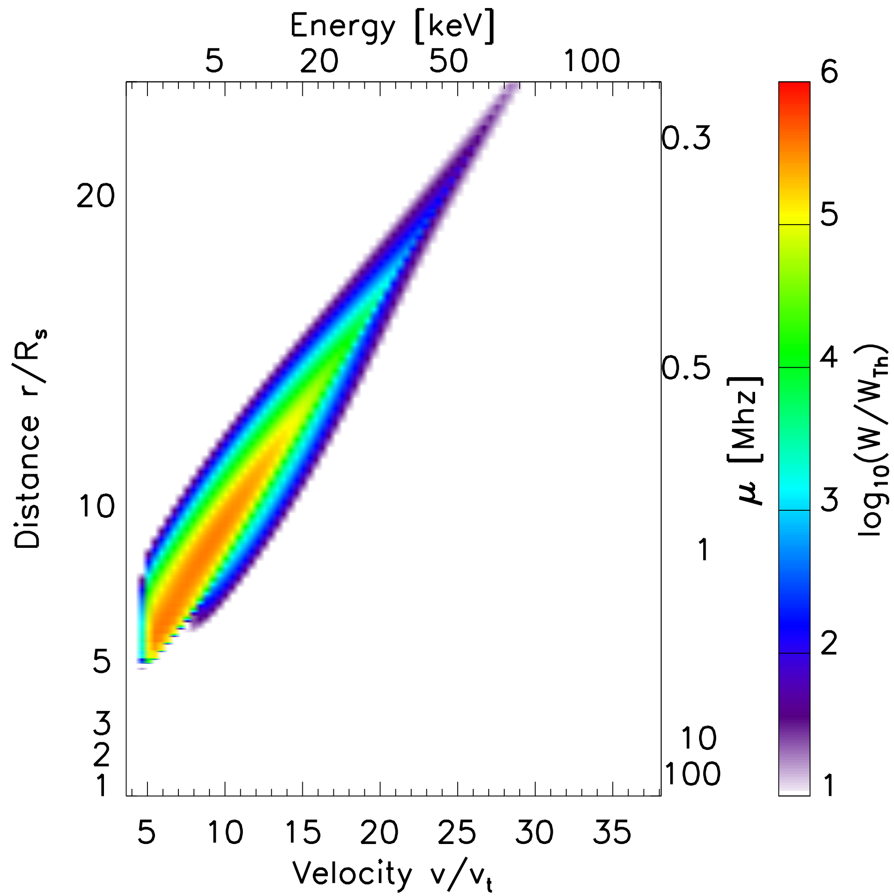


LANGMUIR WAVE DISTRIBUTION

Beam-Plasma Structure

Time = 100.0 seconds





ELECTRON BEAM DISTRIBUTION

LANGMUIR WAVE DISTRIBUTION

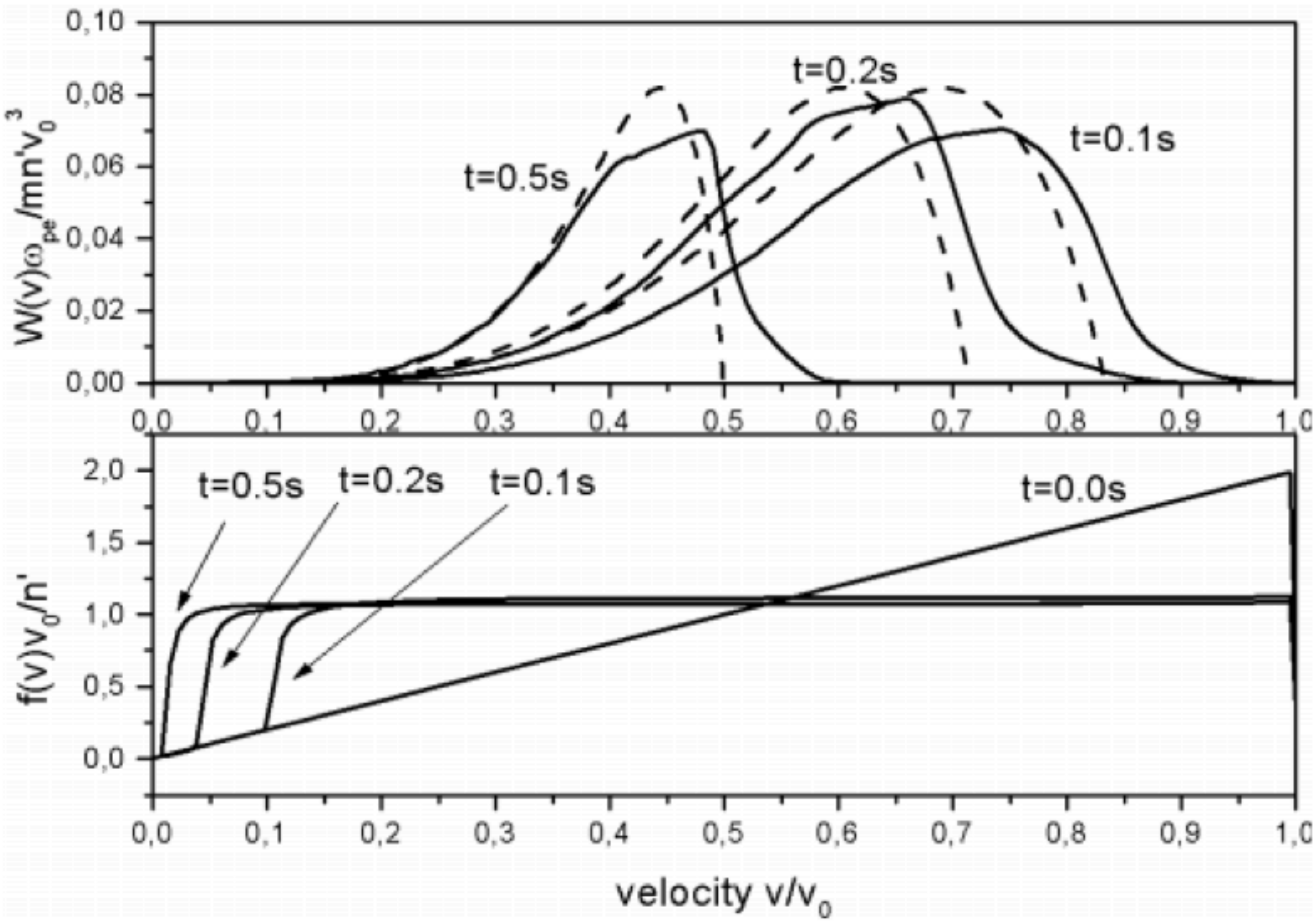
Density Gradients

- The background density gradient is also important in reducing the amount of Langmuir waves that are generated by the electron beam.

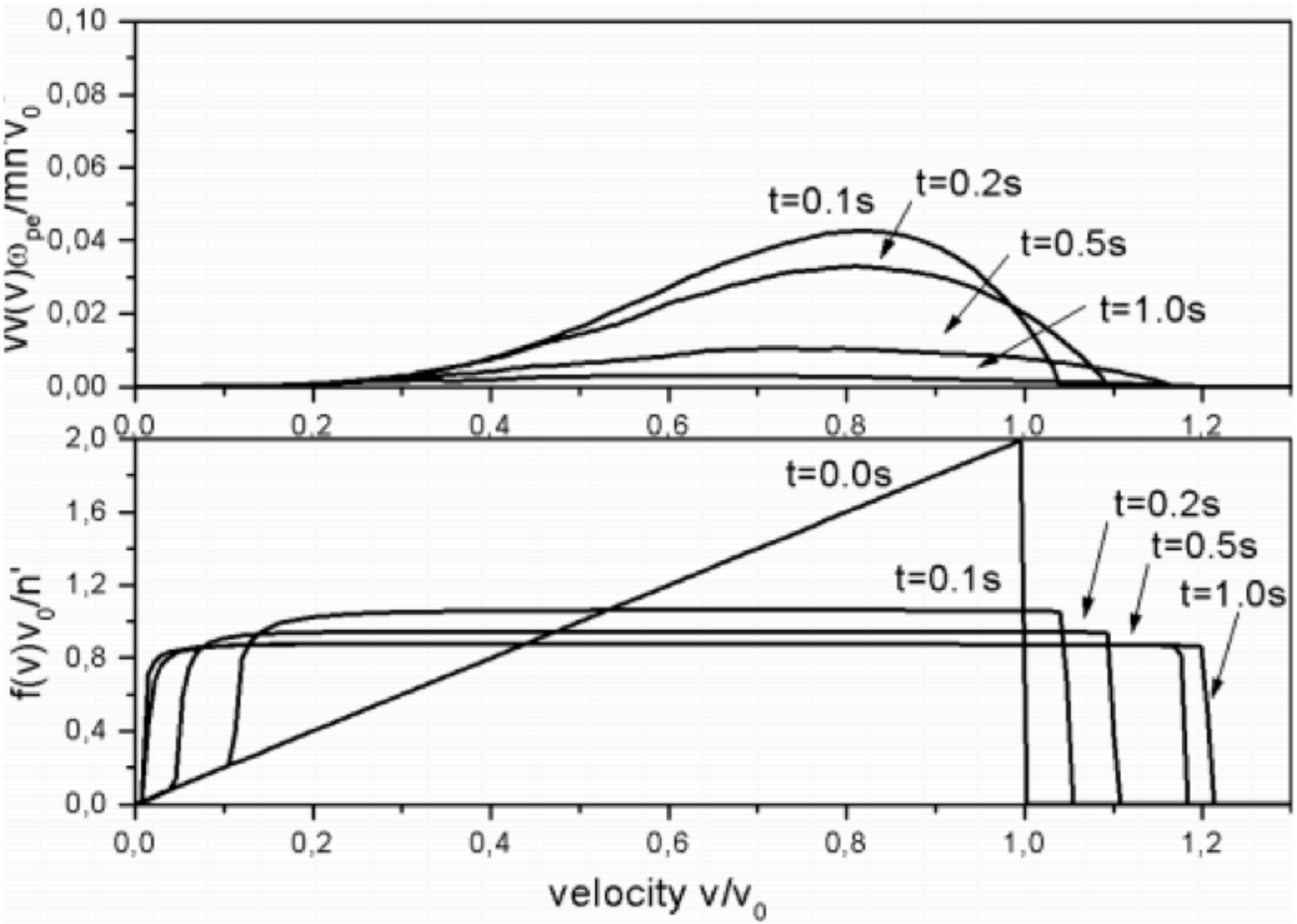
- Refraction.

- Nice paper by Eduard (Kontar 2001) that demonstrates this….

Negative Density Gradients



Positive Density Gradients



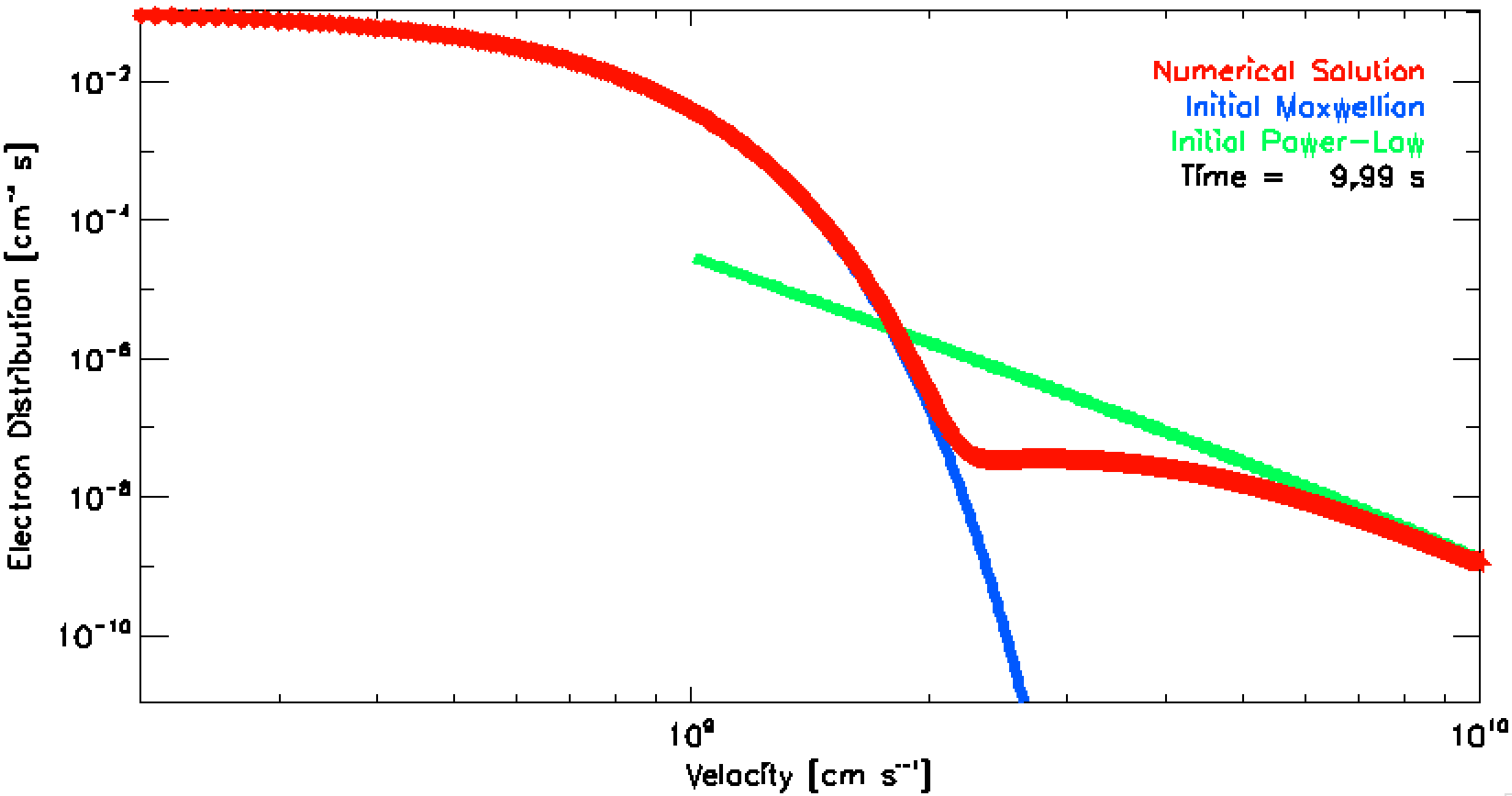
Coulomb Collisions

- Another important physical process that can be simulated is Coulomb collisions between the electrons in the beam and the ions in the background plasma.

- Proportional to the background plasma density.

$$\frac{\partial f}{\partial t} = \frac{4\pi e^4 n_e \ln\Lambda}{m_e^2} \frac{\partial}{\partial v}\left(\frac{f}{v^2} + \frac{v_{te}^2}{v^3}\frac{\partial f}{\partial v}\right)$$

Coulomb Collisions

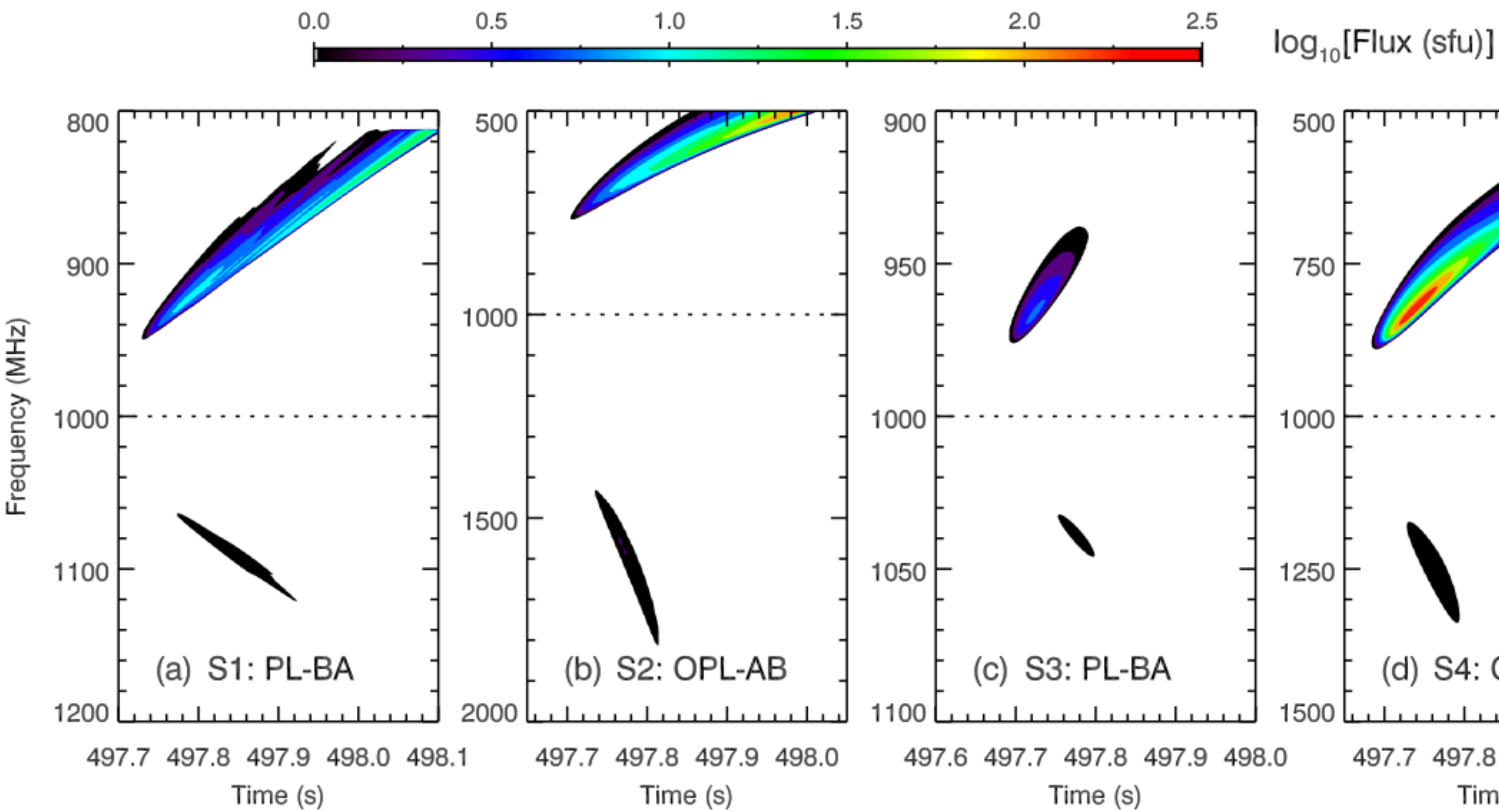


Coulomb Collisions

- Collisions affect the non-thermal particles when they are in the low corona or the chromosphere.

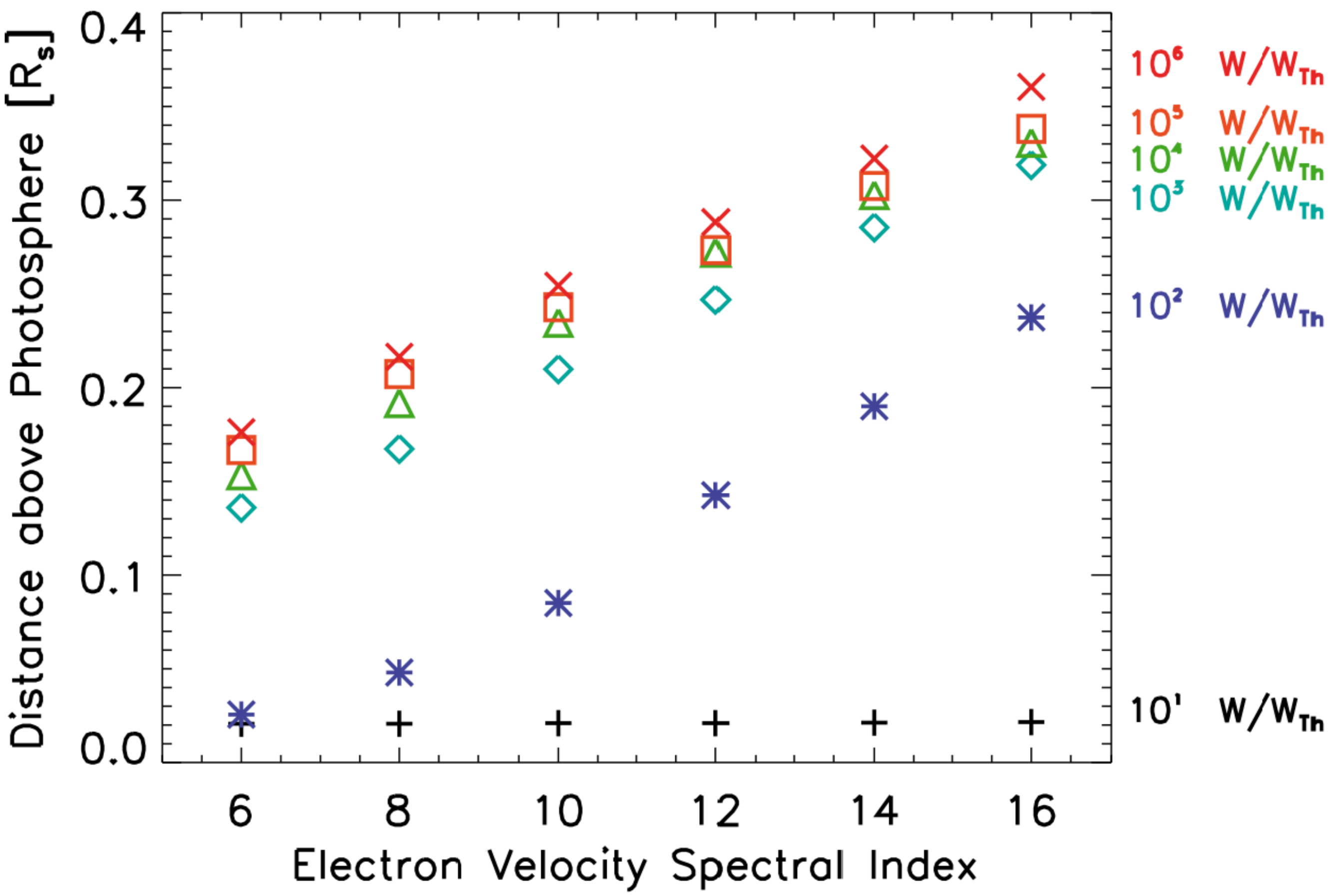
- We can treat the particles as collisionless when they are in the solar wind – background density is too low so the mean-free-path becomes very high.

Bi-directional T3s



Li et al 2012

Starting Frequency

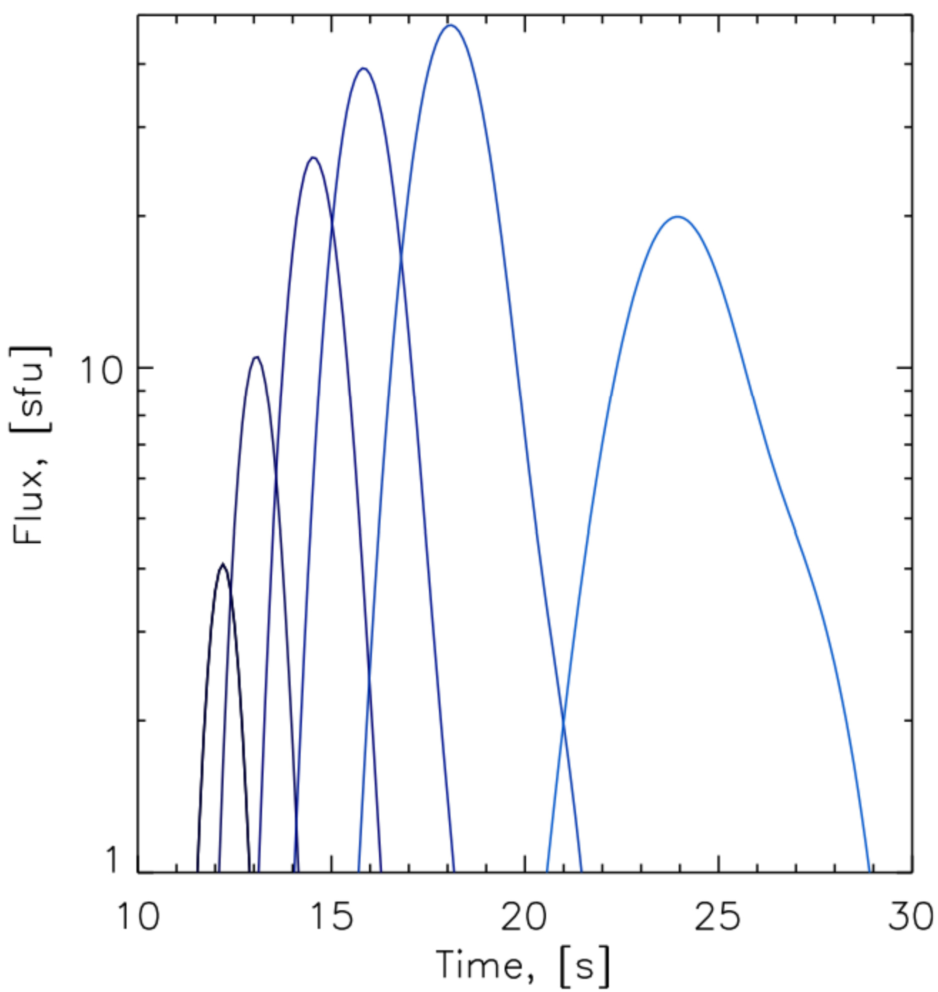


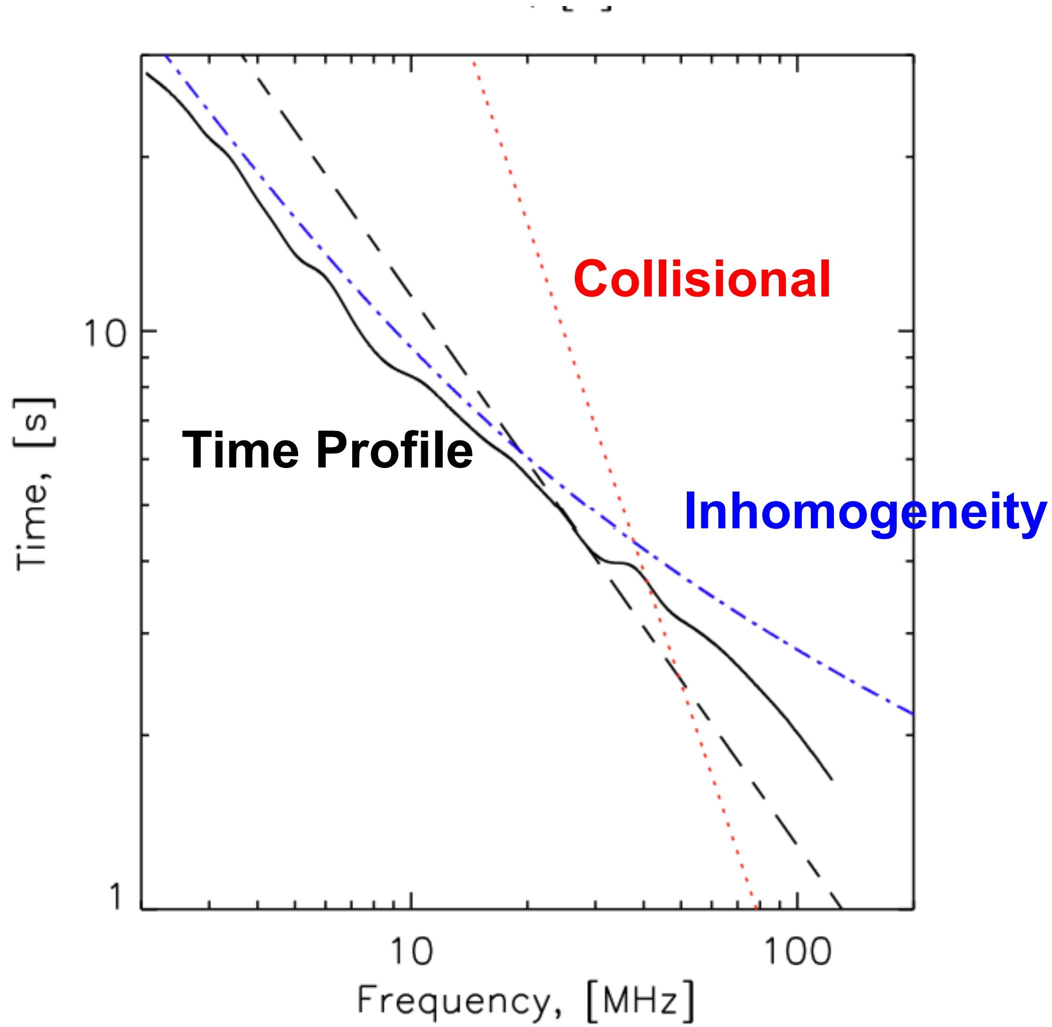
$$h_{typeIII} = d\alpha + h_{acc}$$

Reid et al 2011

Temporal Profile

Time profile of the radio waves dominated by the inhomogeneity of the Solar Wind

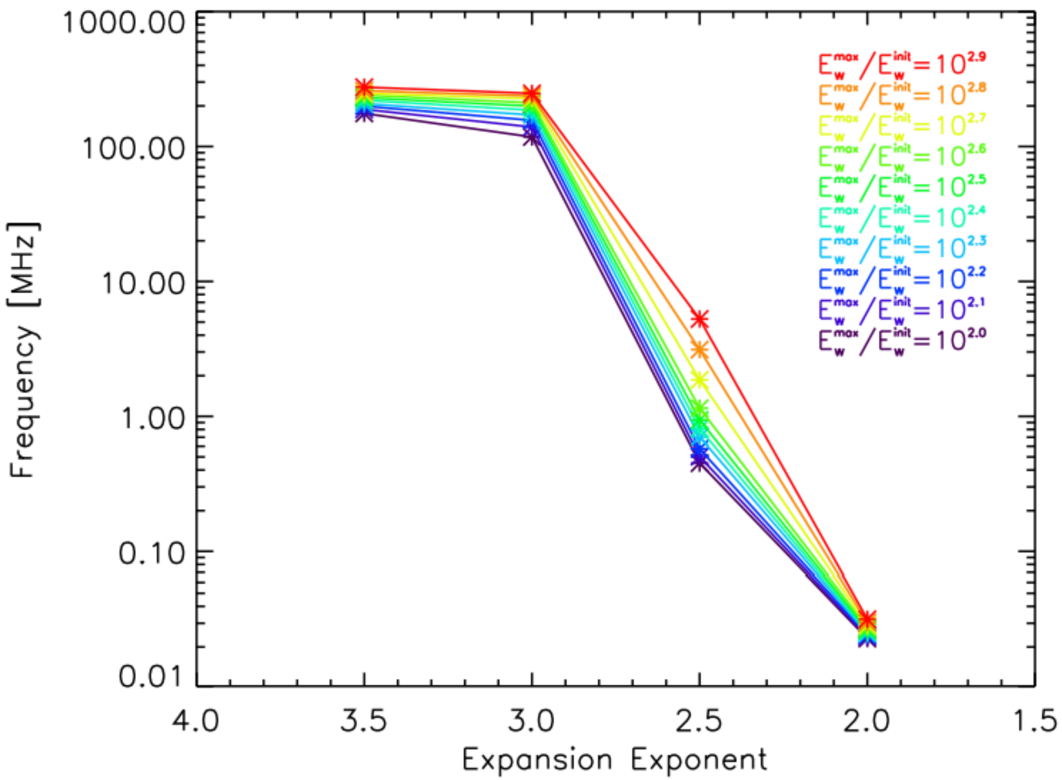


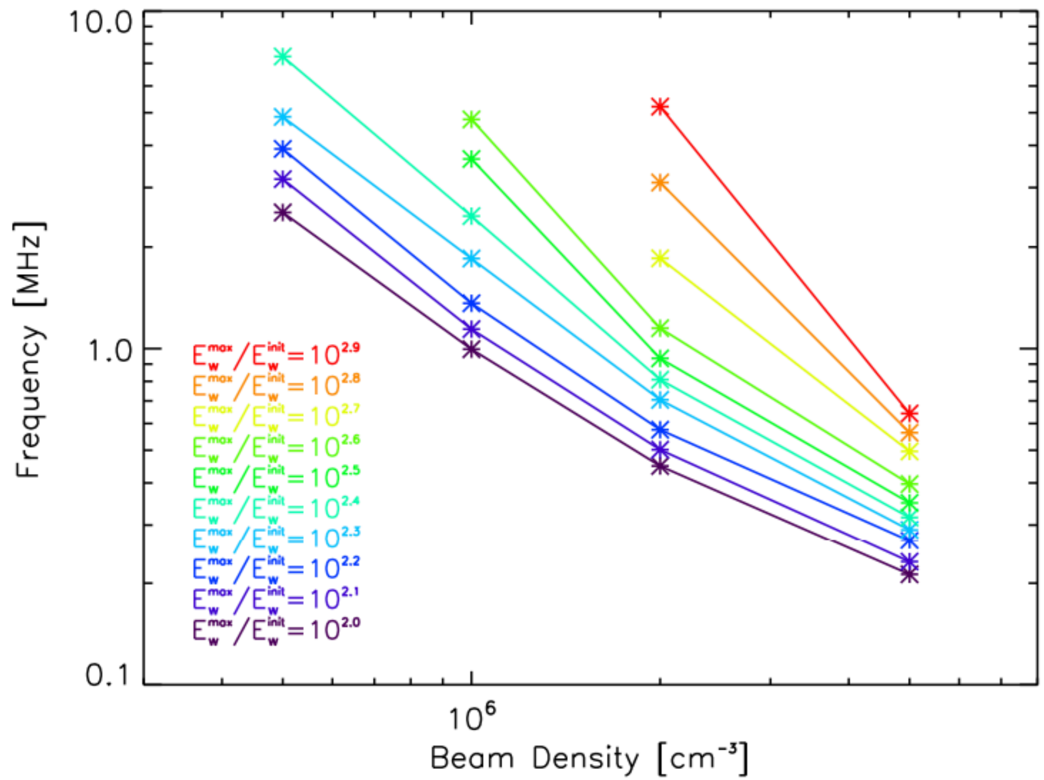


Ratcliffe et al 2014

Stopping Frequency

The production of Langmuir waves is related to the number of electrons. By increasing the expansion of the magnetic field you increase the stopping frequency. Similarly, less intense or dense beams have less stopping frequency.

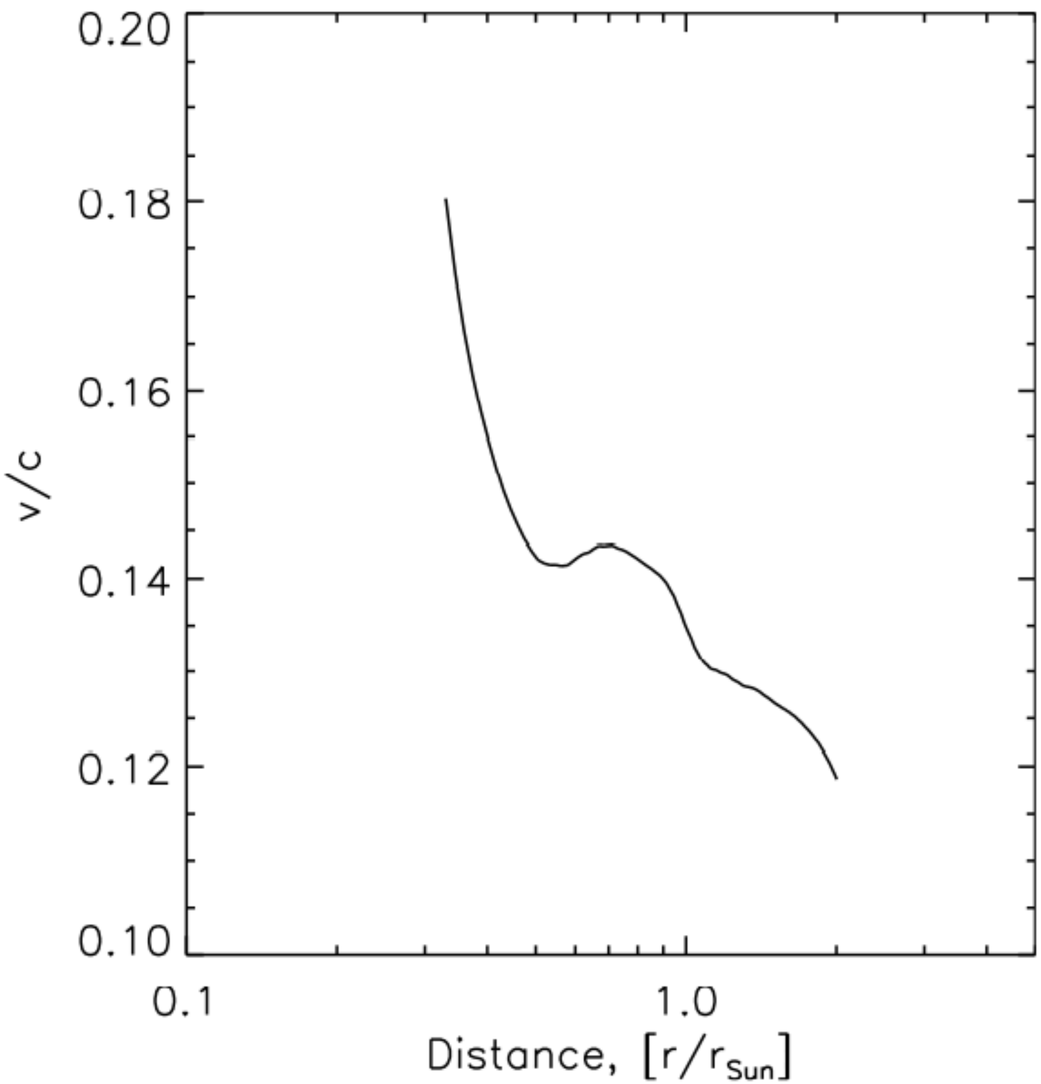




Reid & Kontar 2015

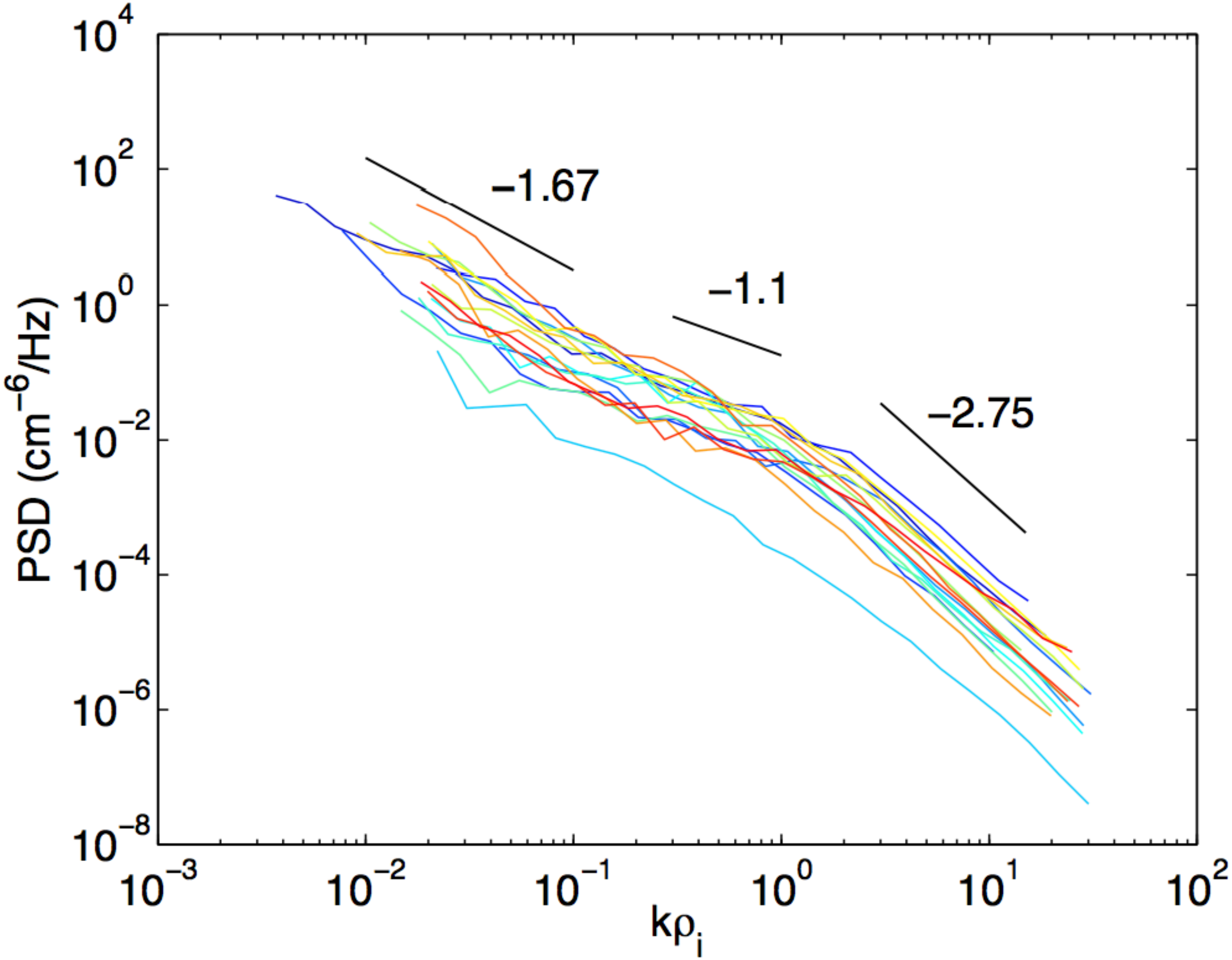
Exciter Velocity

- Excited velocity deduced from the simulations is not constant but decreases as a function of distance from the Sun.



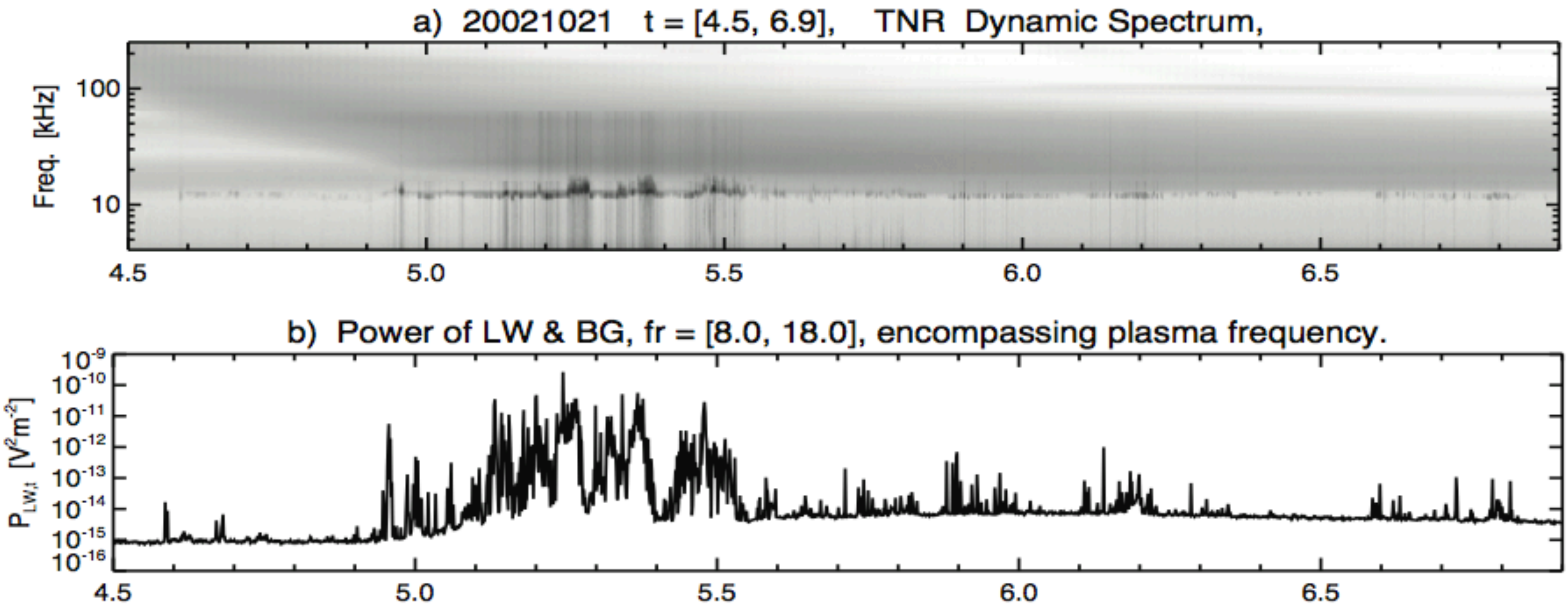
Ratcliffe et al 2014

Density Fluctuations



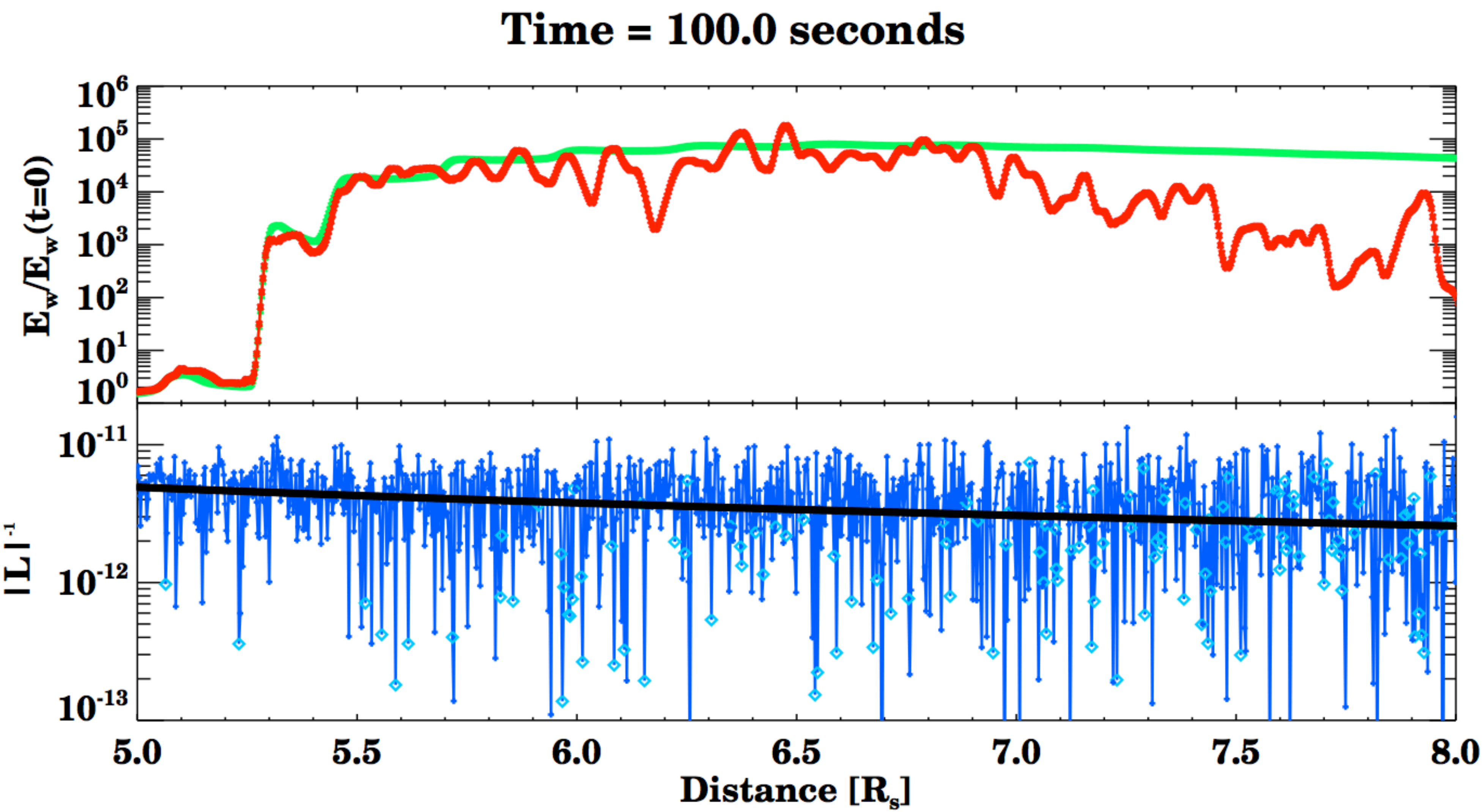
Chen et al 2014

Wave Fluctuations



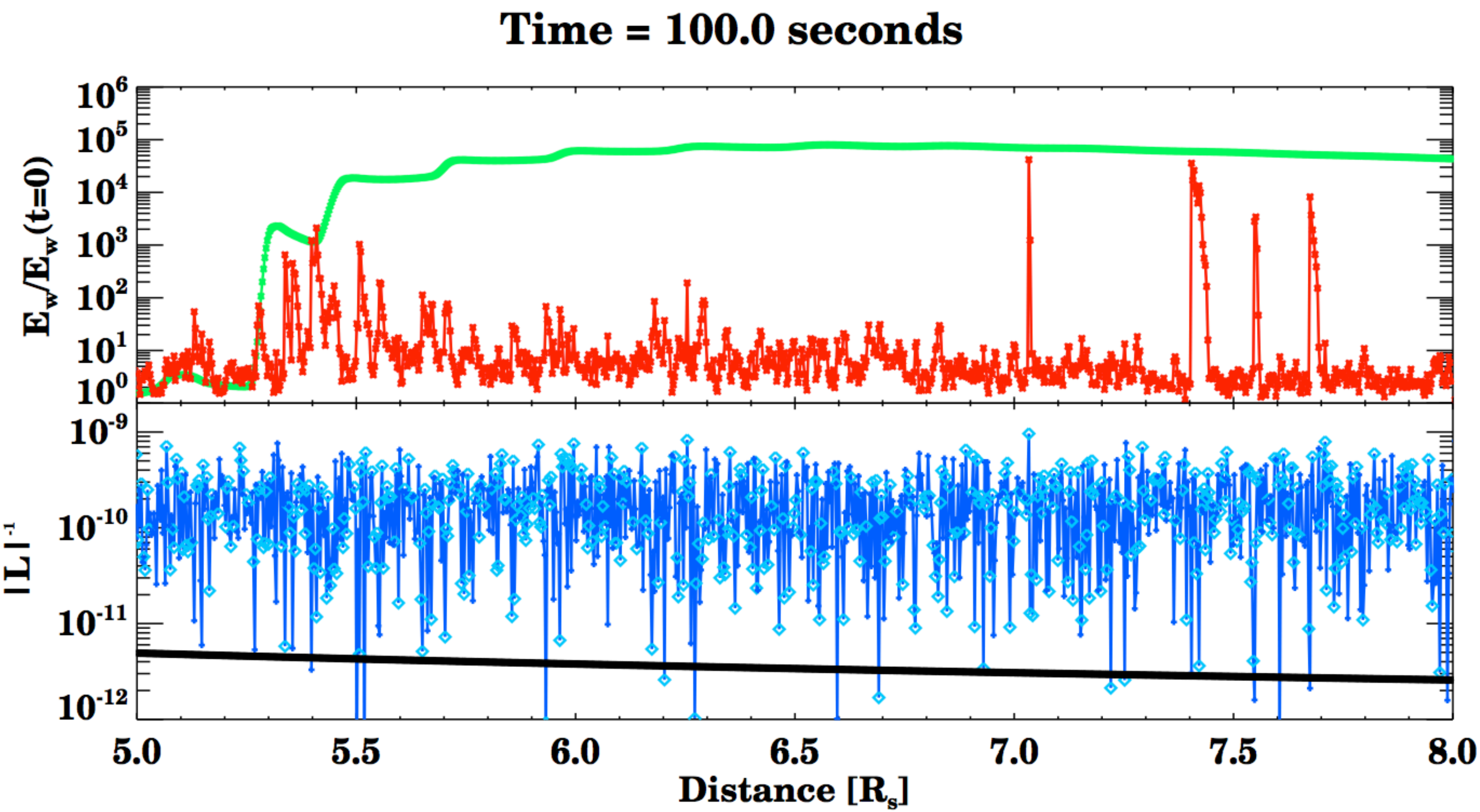
Vidojevic et al 2011

Wave Fluxtuations



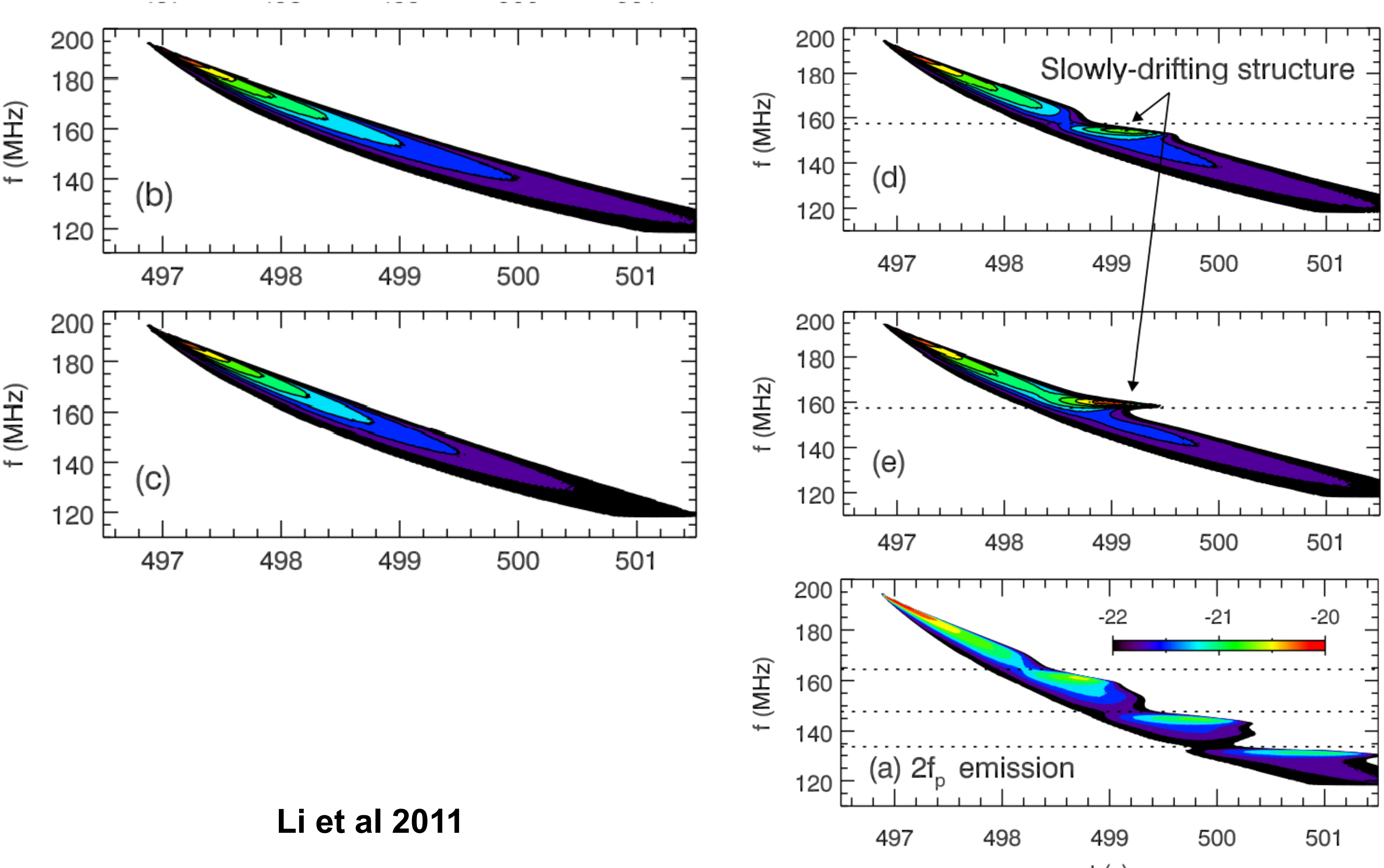
Reid & Kontar 2010

Wave Fluctuations



Reid et al 2010

Temperature Fluctuations

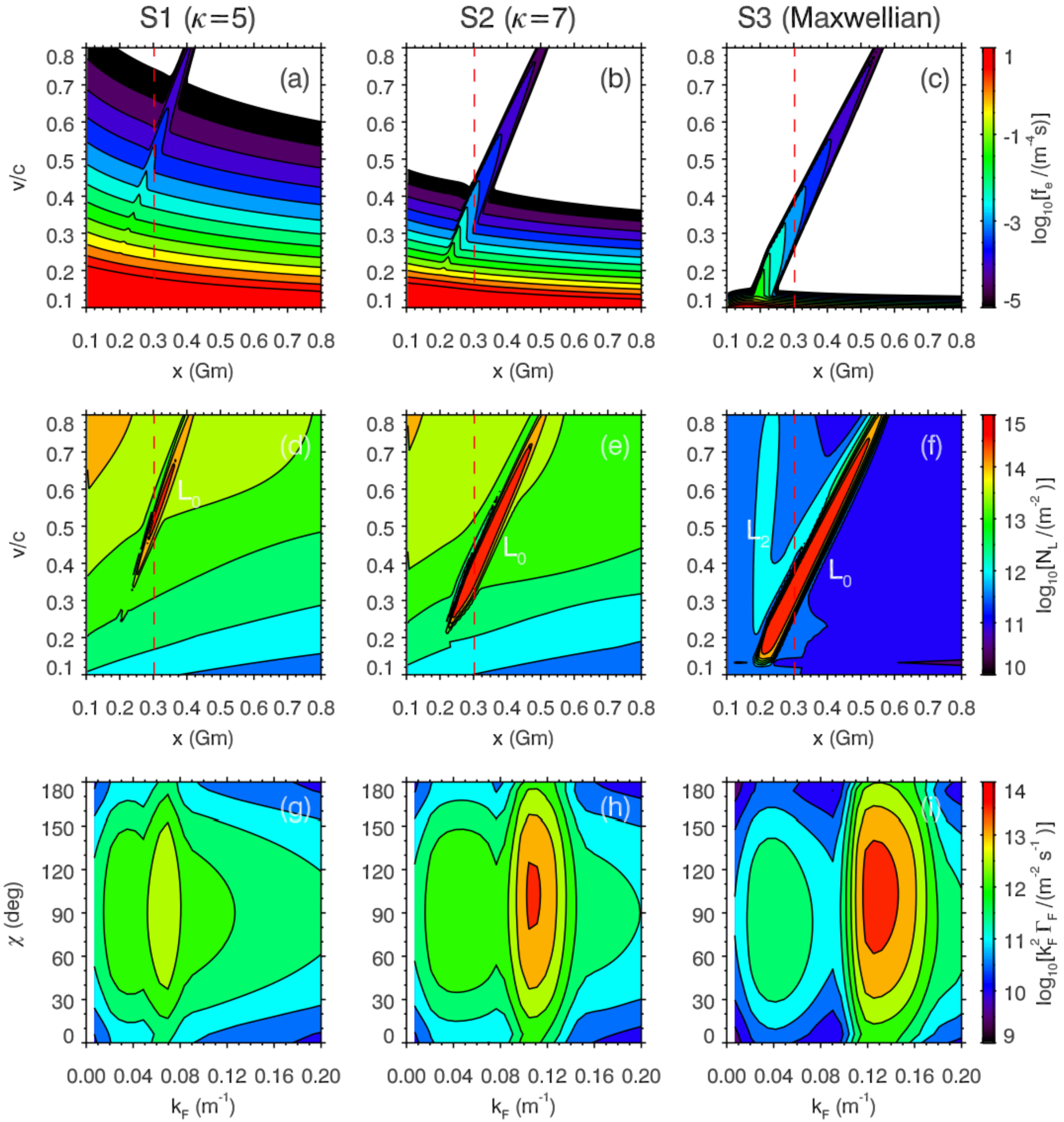


Li et al 2011

Landau Damping

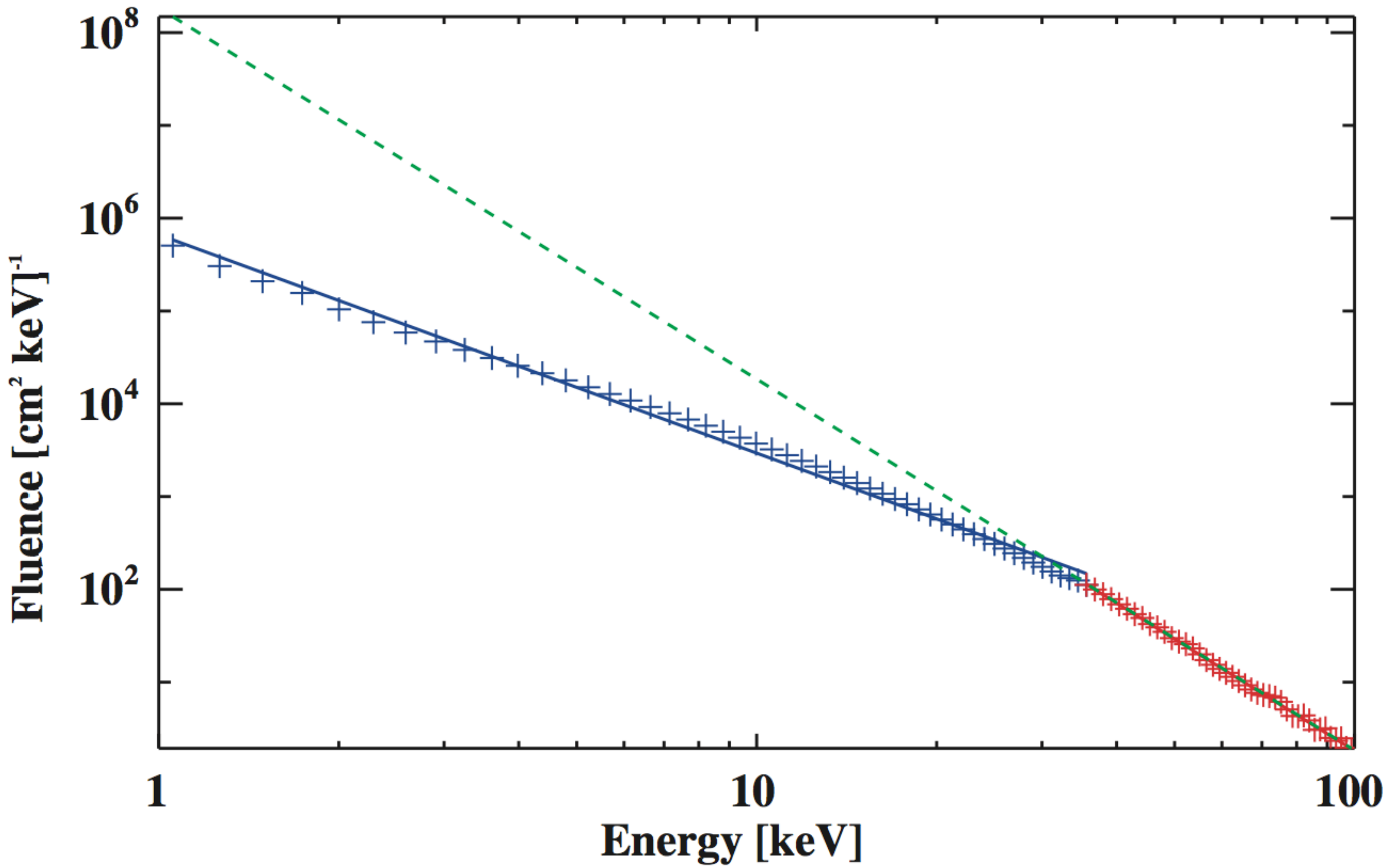
- The background plasma damps the Langmuir waves that have phase velocity close to the thermal velocity.
- In the solar wind the background plasma is NOT a Maxwellian but a kappa distribution.
- The kappa tail causes more damping of the plasma.

Kappa Background



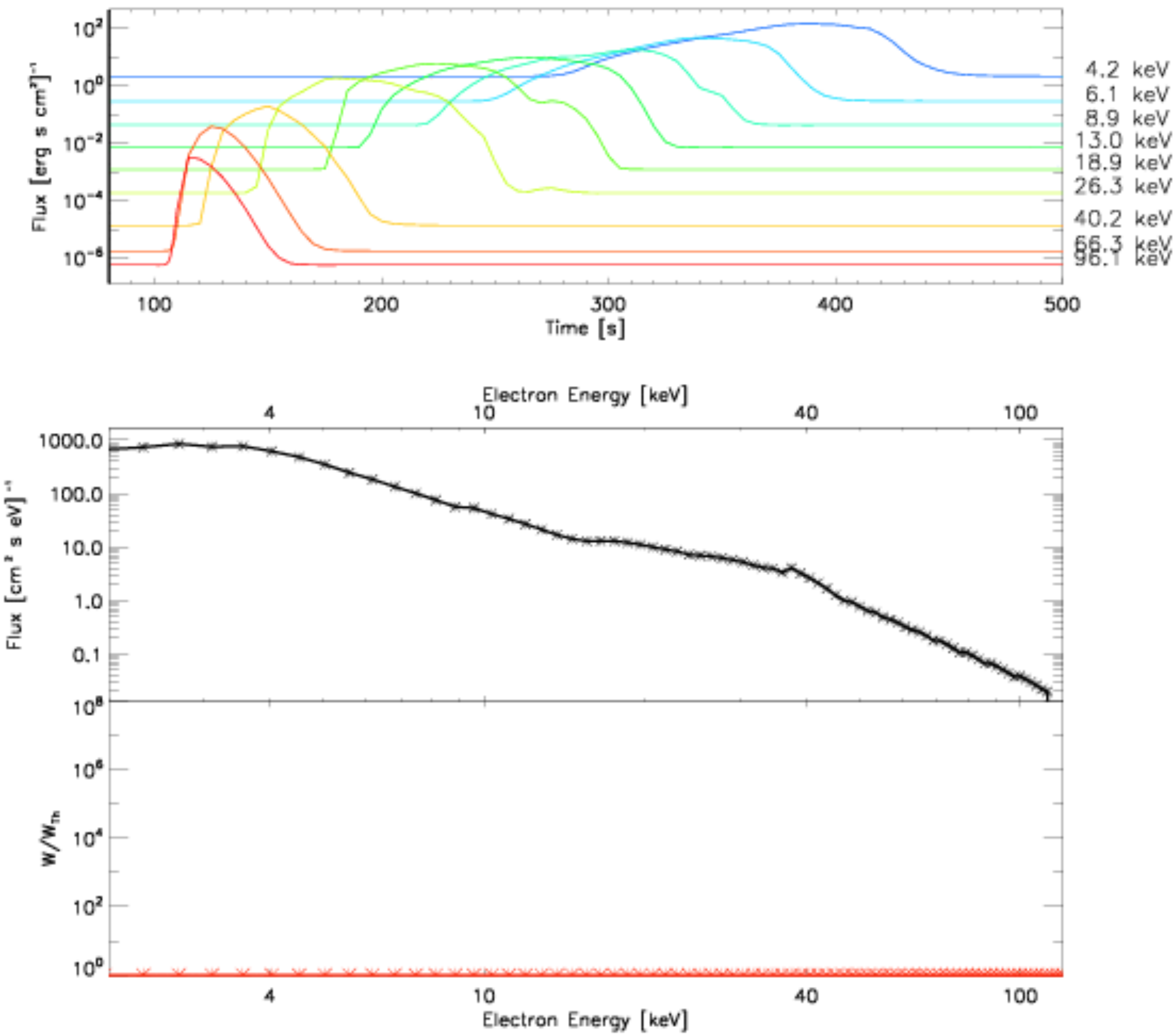
Li et al 2014

Electron Spectra

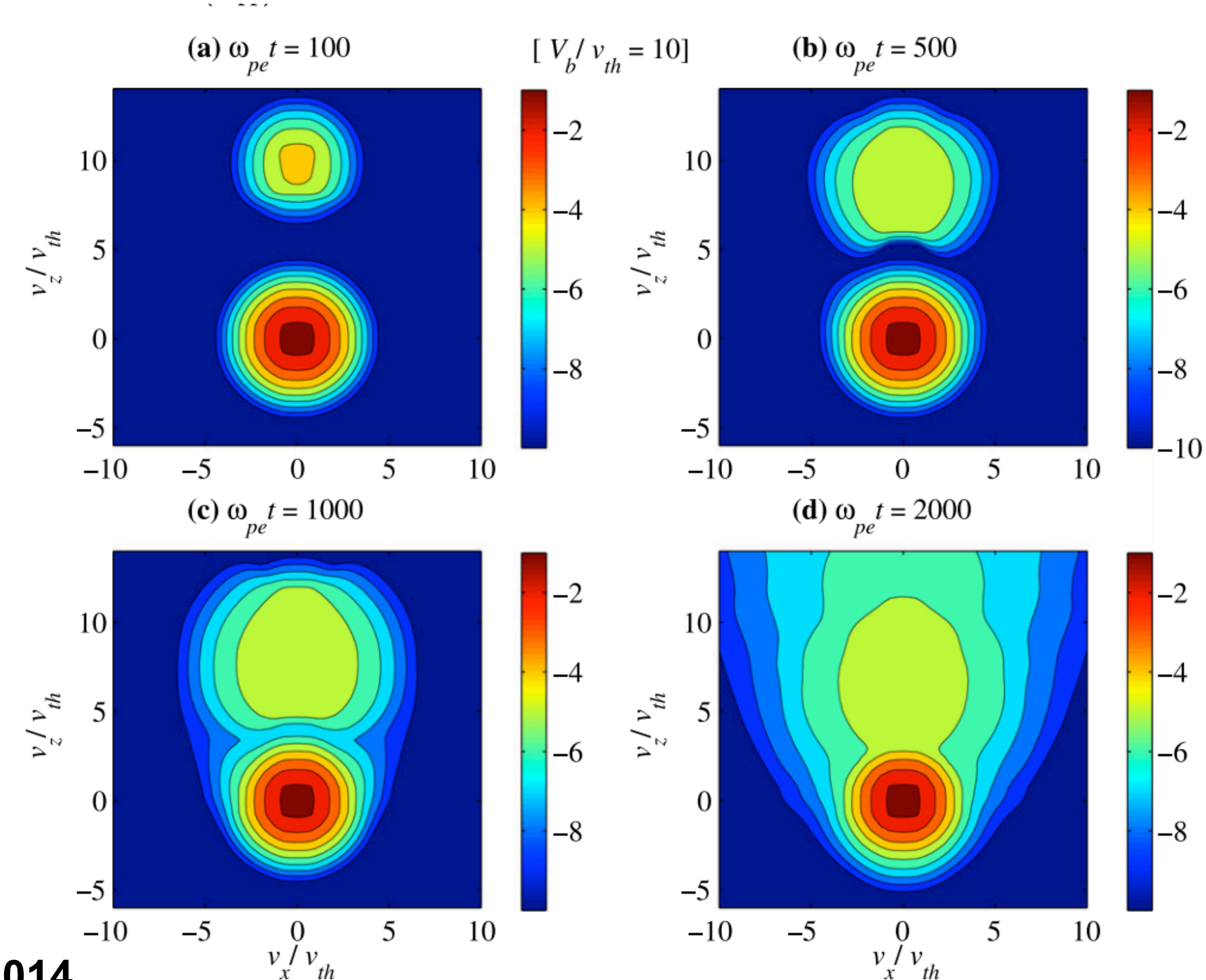


Kontar & Reid 2009

Electron Spectra

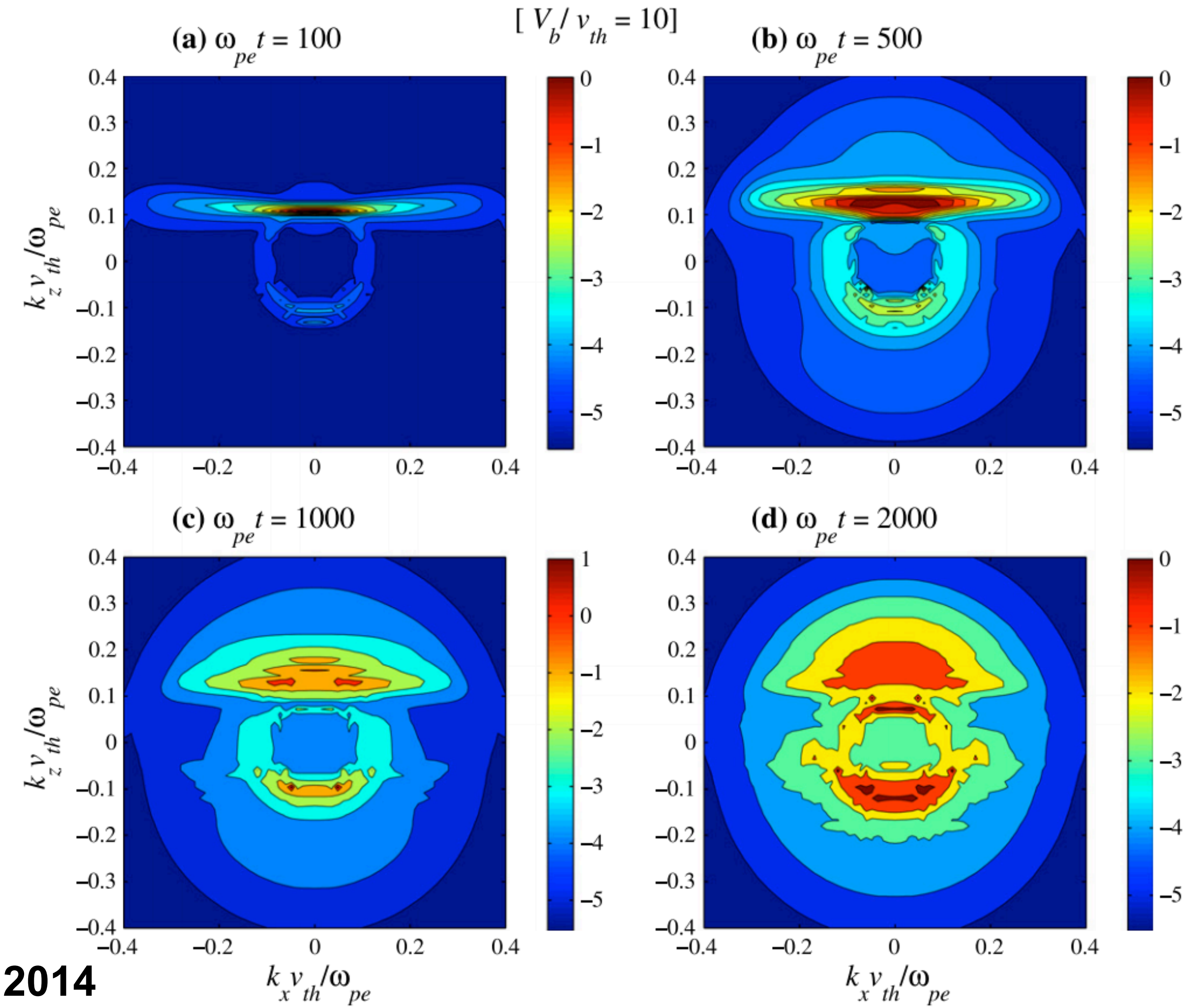


2D Electron Distribution



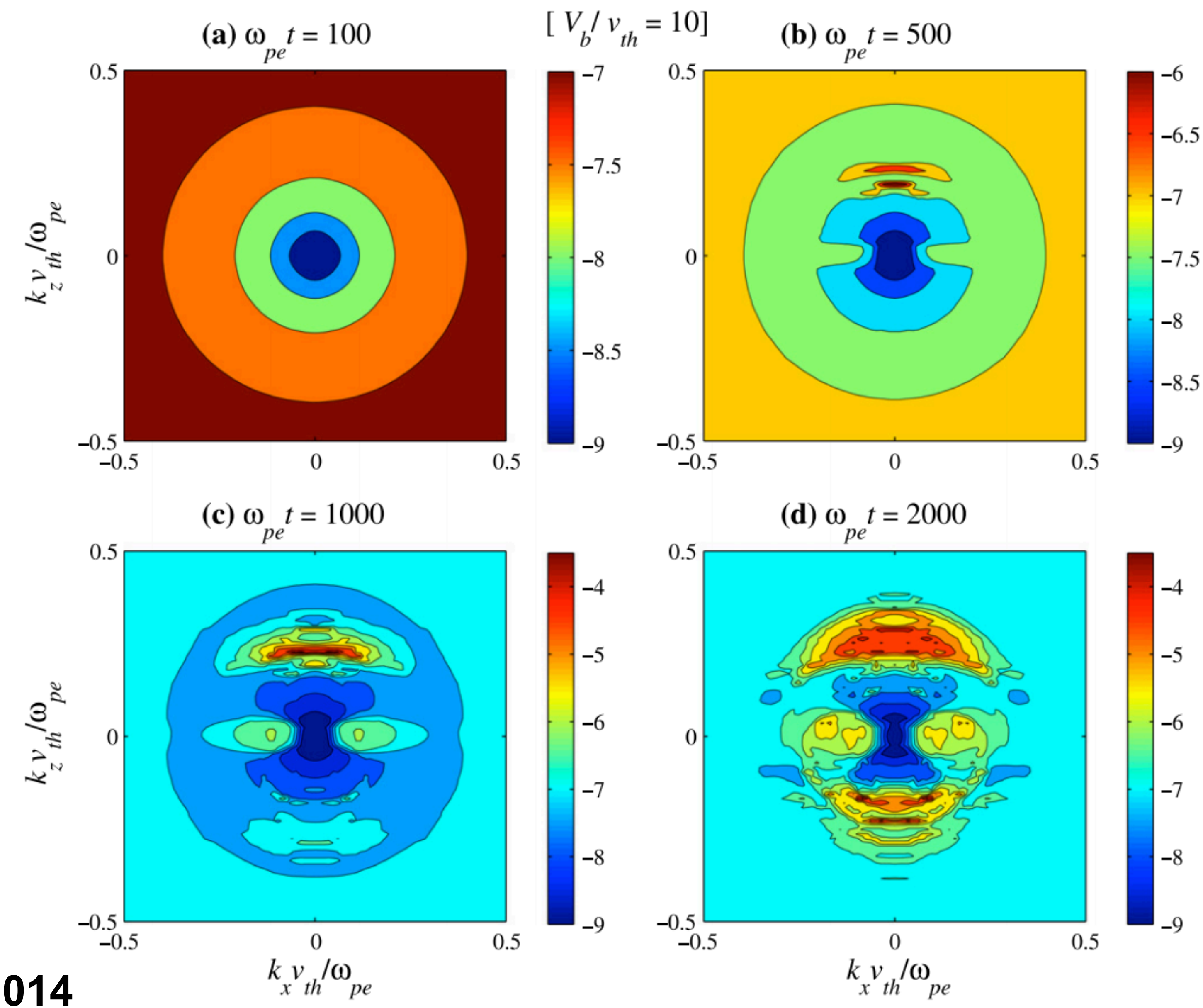
Ziebell et al 2014

2D Langmuir Waves



Ziebell et al 2014

2D Ion-Sound Waves



Ziebell et al 2014

Multi-dimensions

- What about when we have more than one dimension in space? What will happen?

- NOTE: 1D – n points. 3D – n^3 points.
- 1D – 10 points. 3D – 1000 points.
- 1D – 1000 points. 3D – 1000000000 points

- Best to start small and slowly increase number of points.

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Conclusions

- Numerical simulations are a powerful tool to understand the world around us.

- Large effort for numerical simulations to understand solar accelerated electrons and their radio signatures.

- Still much more to be understood. We need you!