Numerical Astronomy I – Part 1, Introduction

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Inverse problems are common throughout physics, but ubiquitous in astronomy, where we only occasionally have direct, experimental, access to the objects we study.

The most general (one dimensional, linear) inverse problem (IP) has the form

$$g(x) = \alpha(x)u(x) + \lambda \int_{a(x)}^{b(x)} K(x,y)u(y)\,\mathrm{d}y,$$

where $u(x)$ is the underlying function we wish to uncover, $g(x)$ is some data, and $K(x,y)$ is a (known) *kernel function*, which expresses the effect of the physical processes hiding this underlying function from us.

This is distinguished from the 'forward problem' or 'direct problem', of going from some *known* underlying function $u(x)$ to a prediction of the observation $g(x)$ which would result from it.

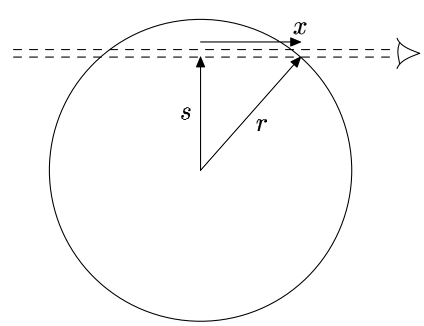
1 Types of inverse problems, and examples

A rough typology of inverse problem types is as follows (this typology is Glasko's – others exist). The classification is not complete, but it illustrates the range of inverse problems, and the broad areas of common technique.

1.1 The 'interpretation problem'

This is perhaps the most typical IP, and consists of the attempt to understand or measure an underlying process, after it has been distorted by the effects of some other physical process.

As an example, recall that, in observations, three spatial dimensions are projected into two. We may wish to measure a number density of stars $n(r)$ in a globular cluster, but what we can observe is the number of stars projected into a line of sight, $N(s)$. From the figure,



$$\begin{aligned}
N(s) &= \int_{-\infty}^{\infty} n(s,x)\,\mathrm{d}x && (1.1)\\
&= 2\int_s^{\infty} n(r)\left|\frac{\mathrm{d}x}{\mathrm{d}r}\right|\,\mathrm{d}r && (1.2)\\
&= 2\int_s^{\infty} \frac{rn(r)\,\mathrm{d}r}{\sqrt{r^2-s^2}} = \int_{\sqrt{\eta}}^{\infty} \frac{n(\sqrt{\xi})\,\mathrm{d}\xi}{\sqrt{\xi-\eta}} && (1.3)
\end{aligned}$$

(in terms of $r = \sqrt{\xi}$ and $s = \sqrt{\eta}$ – this is Abel's equation). Thus we need to solve an inverse problem to recover $n(r)$ from observed $N(s)$.

1.2 The 'instrument problem'

This is to some extent a subset of the interpretation problem. In this case, the underlying process has been distorted by the instrument we use to measure it, and this means that we can study the distortion process in detail, and manipulate it, possibly by improving the instrument.

The most typical example of such a problem is that of a telescope's point spread function, or a photomultiplier's response function, both of which have the form of a *convolution* (that is, having a kernel which is a function of $(x - y)$ only):

$$g(x) = \int_{-\infty}^{\infty} K(x-y)u(y)\,\mathrm{d}y.$$

1.3 The 'synthesis problem'

If we wish to design an instrument with specific properties, then we have to solve an inverse problem to establish those parameters for a putative instrument which give it properties closest to those we want. The distinction between this problem and the previous one is that there we wish to find $u(x)$ given $K(x, y)$, but in this case we (loosely) wish to find $K(x, y)$, given (a suitably restricted range of) $u(x)$.

1.4 The 'control problem'

This is like the synthesis problem in its aims, but mathematically like the interpretation problem in that we, again, wish to find $u(x)$ given $g(x)$ and $K(x, y)$. The difference is in the sort of questions we ask of the solution, in terms of error behaviour and the like.

As an example of this, consider the (simplified) heat conduction equation,

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2},$$

in appropriate units. By fourier-analyzing the problem, we can find (see Craig and Brown, pp 22–3) that

$$T(z,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} T(\zeta, 0) \exp -\frac{(z-\zeta)^2}{4t}\,\mathrm{d}\zeta,$$

a gaussian convolution. We can solve this to obtain the initial temperature distribution $T(z, 0)$ which produces a required distribution $T(z, t)$ at time t.

2 Background and reading

You might like to refresh your memory of linear algebra. Although the lectures will be concerned with numerical methods of solution, I won't be getting involved in detailed numeric examples. However, if you're following the Numerical Methods course, there will be a little cross-over.

2.1 Further reading

Though there is an active research literature on inverse problems, there is a lack of accessible books. The first two I mention are both on short loan in the GUL.

I J D Craig and J C Brown, *Inverse Problems in Astronomy*. Adam Hilger (1986). This book is accessible and, as the title suggests, strongly focused on astronomical applications.

V B Glasko, *Inverse Problems of Mathematical Physics*. American Institute of Physics Translation Series (1986). This is much more mathematically abstract than Craig and Brown – it contains useful insights, but definitely counts as supplementary.

W H Press *et al.*, *Numerical Recipes in C*, 2nd edition. Cambridge University Press (1992). *Numerical Recipes* concentrates on numerical algorithms, but includes lucid and practical, though *not* detailed, introductions to inverse problem theory. There are also editions in Fortran and Fortran90; the second edition corrects errors in the first.