Astronomy – General Relativity and Gravitation I
Part 4: Physics – energy, momentum and Einstein's equations

Norman Gray

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Aims You should

1. understand the role of the various forms of the equivalence principle in the develoment of Einstein's theory of gravity;

2. understand how we can derive Newton's theory of gravity as the weak-field limit of Einstein's theory.

Objectives You should be able to demonstrate that you can

1. quote the various forms of the equivalence principle, explaining the significance of the terms in them;

2. equivalently, explain the 'comma-goes-to-semicolon' rule;
3. perform some dynamical calculations, and manipulate indexes confidently.

Proving that nothing ever changes:

> [...] For Aristotle divides theoretical philosophy too, very fittingly, into three primary categories, physics, mathematics and theology. [...] Now the first cause of the first motion of the universe, if one considers it simply, can be thought of as an invisible and motionless deity; the division [of theoretical philosophy] concerned with investigating this [can be called] 'theology', since this kind of activity, somewhere up in the highest reaches of the universe, can only be imagined, and is completely separated from perceptible reality. The division which investigates material and ever-moving nature, and which concerns itself with 'white', 'hot', 'sweet', 'soft' and suchlike qualities one may call 'physics'; such an order of being is situated (for the most part) amongst corruptible bodies and below the lunar sphere. That division which determines the nature involved in forms and motion from place to place, and which serves to investigate shape, number, size, and place, time and suchlike, one may define as 'mathematics'. [...]
>
> From all this we conclude: that the first two divisions of theoretical philosophy should rather be called guesswork than knowledge, theology because of its completely invisible and ungraspable nature, physics because of the unstable and unclear nature of matter; hence there is no hope that philosophers will ever be agreed about them; and that only mathematics can provide sure and unshakeable knowledge to its devotees, provided one approaches it rigorously. For its kind of proof proceeds by indisputable methods, namely arithmetic and geometry. [...] As for physics, mathematics can make a significant contribution. For almost every peculiar attribute of material nature becomes apparent from the peculiarities of its motion from place to place. *Preface to Book 1 of Ptolemy's Almagest, between 150–161* CE *[2]*

Ptolemy is right, here (though some of the details of his cosmology have been adjusted since he wrote this, and what he refers to as ‘theology’ is now more often referred to as ‘Quantum Gravity’): mathematics we can know all about, with certainty; for physics we have to make guesses. He’s dead right about mathematics’ contribution, though, and we’ll discover that our first insights in this section do indeed come from considering the peculiarities of the material world’s motion from place to place.

1 The energy-momentum tensor

The point of this whole course is to describe how gravity, in the form of the curvature of spacetime, is determined by the presence of mass. In newtonian physics, the relationship is straightforward, since the notion of mass is unproblematic. In relativity, however, we know that what matters is not mass alone, but energy-momentum, and so it is not unreasonable that what matters in GR is not mass, but the distribution of energy-momentum, and so we must find a way of describing this in an acceptably geometrical fashion. In this section we are confining ourselves to *special relativity*, and in the next section we discover that this is not, physically, a restriction in fact. This section largely follows Schutz ch. 4.

We start (as we all end) with dust.

1.1 Dust, fluid and flux

A *fluid* in GR and cosmology is, not surprisingly, something which flows; that is, a substance where the forces perpendicular to an imaginary surface (ie, pressure) are much greater than the forces parallel to it (ie, stress, arising from viscosity). The limit of this, a *perfect fluid* is a substance which has pressure but zero stresses. An even simpler substance is termed *dust*, which denotes an idealised form of matter, consisting of a collection of non-interacting particles which are not moving relative to each other, so that the collection has zero pressure. That is to say that there is a frame, called the *momentarily comoving reference frame*, MCRF, with respect to which all the particles in a given volume have zero velocity[1].

We can suppose for the moment that all the dust particles have the same (rest) mass m, but that different parts of the dust cloud may have different number densities n. Just as the particle mass m is the mass in the particle's rest frame, the number density n is always that measured in the MCRF.

If we lorentz-transform to a frame which is moving with velocity $\overline{v}$ with respect to the MCRF, a (stationary) volume element of size $\Delta x \Delta y \Delta z$ will be lorentz-contracted into a (moving) element of size $\Delta x' \Delta y' \Delta z' = (\Delta x/\gamma)\Delta y \Delta z$, where γ is the familiar lorentz factor $\gamma = (1 - v^2)^{-1/2}$, supposing that the frames are chosen such that the relative motion is along the x-axis. That means that the number density of particles, as measured in the frame relative to which the dust is moving, goes *up* to γn. What, then, is the *flux* of particles through an area $\Delta y \Delta z$ in the y'–z' plane? The particles in the volume all pass through the area $\Delta y' \Delta z'$ in a time $\Delta t'$, where $\Delta x' = v \Delta t'$, and so this total number of particles is $(\gamma n)(v \Delta t')\Delta y' \Delta z'$. Thus the total number of particles per unit time and per unit area, which is the flux in the x'-direction, is $\gamma n v$. Writing N^x for this x-directed flux, and v^x for the velocity along the x-axis, v, this is

$$N^x = \gamma n v^x. \tag{4.1}$$

[1] This is also, interchangeably, sometimes called the *Instantaneously* Comoving Reference Frame (ICRF)

We can generalise this, and guess that we can reasonably define a flux vector

$$\overline{N} = n\overline{U}, \tag{4.2}$$

where again n is the dust number density in its MCRF, and $\overline{U}$ is the 4-velocity vector $(\gamma, \gamma v^x, \gamma v^y, \gamma v^z)$. Since the velocity vector has the property $\mathsf{g}(\overline{U}, \overline{U}) = -1$ (remember your SR, and that the 4-velocity vector $\overline{U} = (1, \overline{0})$ in MCRF), we have $\mathsf{g}(\overline{N}, \overline{N}) = N_\alpha N^\alpha = -n^2$. The components of the flux vector $\overline{N}$ in this frame are

$$(\gamma n, \gamma n v^x, \gamma n v^y, \gamma n v^z). \tag{4.3}$$

This flux vector is a *geometrical* object, because $\overline{U}$ is, and so although its components are frame-dependent, the vector as a whole is not.

It is obvious how to recover, from Eq. (4.3), the fluxes N^x across surfaces of constant coordinate (they're just the components in Eq. (4.3)), but we will need to be more general than this. Any function defined over spacetime, $\phi(t, x, y, z)$, defines a surface $\phi =$ constant, and its gradient one-form $\widetilde{\mathrm{d}\phi}$ acts as a normal to this surface (think of the planes in our visualisation of one-forms). The *unit* gradient one-form $\widetilde{n} \equiv \widetilde{\mathrm{d}\phi}/|\widetilde{\mathrm{d}\phi}|$ points in the same direction but has unit magnitude (the notational clash with the number density n is unfortunate but conventional). Consider specifically the coordinate function x: the gradient one-form corresponding to this, $\widetilde{\mathrm{d}x}$, has components $(0, 1, 0, 0)$ (and so is already unit). If we contract this one-form with the flux vector, we find

$$\left\langle \widetilde{\mathrm{d}x}, \overline{N} \right\rangle \equiv \overline{N}(\widetilde{\mathrm{d}x}) = N^x,$$

(where the last expression denotes the x-component of $\overline{N}$, rather than the whole set of components). That is, contracting the flux vector with a gradient one-form produces the flux across the corresponding surface; this is true in general, so that $\overline{N}(\widetilde{n})$ produces the flux across the surface $\phi =$ constant, where $\widetilde{n} = \widetilde{\mathrm{d}\phi}/|\widetilde{\mathrm{d}\phi}|$. The vector $\overline{N} = n\overline{U}$ is manifestly geometrical; it is our ability to recover the flux in this way that justifies our naming this the 'flux vector'.

1.2 The energy-momentum tensor

[We'll switch from (t, x, y, z) to general (x^0, x^1, x^2, x^3), now, but we're still confining ourselves to special relativity.]

We know from our study of special relativity that energy and mass are interconvertible. For our dust particles of mass m, therefore, the *energy density* of the dust, in the MCRF, is mn. In our moving frame, however, as well as the number density rising to γn, the total energy of each particle, as measured in the 'stationary frame', goes up to γm. Thus the energy density of the dust as measured in a moving frame is $\gamma^2 mn$. This double factor of γ cannot result from a lorentz boost of a vector, and is the first indication that to describe the energy-momentum of the dust we will need to use a higher-order tensor.

What geometrical objects do we have to play with? We have the momenta of the dust particles, $\overline{p} = m\overline{U}$, and we have the flux vector $\overline{N} = n\overline{U}$. As mentioned above, we also have the gradient one-forms corresponding to the coordinate functions, $\widetilde{\mathrm{d}}x^\alpha$. By contracting the vectors with these one-forms we can extract the particles' energy $p^0 = \overline{p}(\widetilde{\mathrm{d}}x^0)$ or spatial momenta $p^i = \overline{p}(\widetilde{\mathrm{d}}x^i)$, or the number density $N^0 = \overline{N}(\widetilde{\mathrm{d}}x^0)$ (which we can interpret as the number crossing a surface of constant time, into the future) or number flux $N^i = \overline{N}(\widetilde{\mathrm{d}}x^i)$.

Let us form the $\binom{2}{0}$ tensor

$$\mathsf{T} = \overline{p} \otimes \overline{N} = \rho\overline{U} \otimes \overline{U}, \qquad \text{dust} \tag{4.4}$$

(writing $\rho = mn$ for the *mass density*, and recalling the definition of outer product in Sect. 2.2 of part 2) – this is known as both the *energy-momentum tensor* and the *stress-energy tensor*. We now examine the components of this tensor, obtained by contracting it with the basis one-forms $\widetilde{\omega}^\alpha = \widetilde{\mathrm{d}}x^\alpha$, where the coordinate functions x^i are those corresponding to a frame with respect to which the dust is moving. These components are, of course,

$$T^{\alpha\beta} = \mathsf{T}(\widetilde{\mathrm{d}}x^\alpha, \widetilde{\mathrm{d}}x^\beta) = \overline{p}(\widetilde{\mathrm{d}}x^\alpha) \times \overline{N}(\widetilde{\mathrm{d}}x^\beta).$$

The 0-0 component T^{00} is just $\gamma^2 mn$, which we can recognise as the energy density of the dust, or the flow of the zeroth component of momentum across a surface of constant time.

The 0-i component $T^{0i} = \gamma m \times \gamma n v^i$ (after comparing with Eq. (4.3)). Given that nv has the dimensions of (per-unit-area per-unit-time), and that mass and energy are interconvertible in relativity, this is identifiable as the flux of energy across a (spatial) surface of constant x^i.

The i-0 component $T^{i0} = p^i \times N^0 = m\gamma v^i \times \gamma n$ is the flux of the i-th component of momentum across a surface of constant time, into the future. By analogy with the energy density, this is known as (the i-th component of) the *momentum density* of the dust. Now, energy flux across a surface is an amount of energy-per-unit-time, per unit area or, since energy and mass are the same thing, mass-per-unit-time, per unit area. However, momentum density is the amount of momentum per unit volume, which is mass-times-speed per unit volume, which is dimensionally the same as energy flux. Another way of getting to the same place (in Schutz's words this time) is that energy flux is the density of mass-energy times the speed it flows at, whereas momentum density is the mass density times the speed the mass flows at, which is the same thing. Thus the identity of T^{i0} and T^{0i} in this case is not coincidental or special to dust, but quite general:

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sity

$$T^{i0} = T^{0i}.$$

Finally the i-j component of the energy-momentum tensor, $T^{ij} = p^i N^j = \gamma m v^i \times \gamma n v^j$, is the flux of i-momentum across a surface of constant x^j. It has the dimensions of momemtum per unit time, per unit area, leading us to identify it as force per unit area, or pressure.

In general, therefore, we can interpret the component $\mathsf{T}(\widetilde{\mathrm{d}x}^\alpha, \widetilde{\mathrm{d}x}^\beta)$ as the flow of the α-th component of momentum across a surface of constant coordinate x^β.

By considering the torques acting on a fluid element we can show (Schutz §4.5) that the tensor T is symmetric in general,

$$T^{\alpha\beta} = T^{\beta\alpha}, \qquad \text{or } \mathsf{T}(\widetilde{p}, \widetilde{q}) = \mathsf{T}(\widetilde{q}, \widetilde{p}), \quad \forall \widetilde{p}, \widetilde{q}. \tag{4.5}$$

In a perfect fluid, there is no preferred direction, so the spatial part of the energy-momentum tensor must be proportional to the spatial part of the metric, which is δ^{ij} in SR. Since there is no viscosity, the only momentum transport possible is perpendicular to the surface of a fluid element, in the form of pressure p (which is force per unit area, remember), giving the constant of proportionality (see Schutz §4.6 for an expanded version of this argument), and so

$$T^{ij} = p\delta^{ij}, \qquad \text{[perfect fluid]}. \tag{4.6}$$

From there it is a short step to show that the energy-momentum tensor for a perfect fluid, as a geometrical object, is

$$\mathsf{T} = (\rho + p)\overline{U} \otimes \overline{U} + p\mathsf{g}, \qquad \text{[perfect fluid]}. \tag{4.7}$$

Dust has no pressure, so its energy-momentum tensor in the MCRF is

$$\mathsf{T} = \mathrm{diag}(\rho, 0, 0, 0), \qquad \text{[dust, MCRF]}. \tag{4.8}$$

The final important property of this tensor is its conservation law. If energy is to be conserved, then the amount of energy-momentum entering an arbitrary four-dimensional box must be the same as the amount leaving it. From this we promptly deduce that

$$\frac{\partial}{\partial x^0}T^{\alpha 0} + \frac{\partial}{\partial x^1}T^{\alpha 1} + \frac{\partial}{\partial x^2}T^{\alpha 2} + \frac{\partial}{\partial x^3}T^{\alpha 3} = 0,$$

or

$$T^{\alpha\beta}{}_{,\beta} = 0. \tag{4.9}$$

By a similar sort of argument, requiring that under any flow of a fluid or of dust the total number of particles is unchanged, we can show that

$$N^{\alpha}{}_{,\alpha} = (nU^{\alpha})_{,\alpha} = 0. \tag{4.10}$$

1.3 Maxwell's equations

For completeness, here are Maxwell's equations in the form appropriate for General Relativity. For fuller details, see exercise 25 in Schutz's §4.10.

Given electromagnetic fields (E^x, E^y, E^z) and (B^x, B^y, B^z), we can define the antisymmetric *Faraday tensor* Farac

$$\mathsf{F} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}. \tag{4.11}$$

We can also define the current vector $\overline{J} = (\rho, j^x, j^y, j^z)$ corresponding to a charge density ρ and current 3-vector $\mathbf{j}$. With these definitions, Maxwell's equations in special relativity become

$$F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu \tag{4.12a}$$

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0. \tag{4.12b}$$

The Faraday tensor F and the energy-momentum tensor T together form the source for the gravitational field. Notwithstanding that, we shall not explicitly include the Faraday tensor in the discussion below.

2 The laws of physics in curved spacetime

So we now have a way to describe the energy-momentum contained within an arbitrary distribution of matter and electromagnetic fields. What we now want to know is how these relate to the curvature of the spacetime they lie within.

2.1 Ricci and Bianchi

First we need to establish useful contractions of the curvature tensor. See Schutz §6.6 for further details of this brief relapse into mathematics.

These contractions are the *Ricci tensor*, obtained by contracting the full curvature tensor over its first and third indexes, Ricci

$$R_{\beta\nu} \equiv g^{\alpha\mu} R_{\alpha\beta\mu\nu} = R^{\mu}{}_{\beta\mu\nu}, \tag{4.13}$$

and the *Ricci scalar* obtained by further contracting the Ricci tensor, Ricci

$$R \equiv g^{\beta\nu} R_{\beta\nu} = g^{\beta\nu} g^{\alpha\mu} R_{\alpha\beta\mu\nu}. \tag{4.14}$$

Note, from Eq. (3.49a), that the Ricci tensor is symmetric: $R_{\alpha\beta} = R_{\beta\alpha}$.

By differentiating Eq. (3.48), we can find

$$2R_{\alpha\beta\mu\nu,\lambda} = g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}, \tag{4.15}$$

and noting that partial derivatives commute, deduce

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0. \tag{4.16}$$

Recall that Eq. (3.48) was evaluated in LIF coordinates; however, since in these coordinates $\Gamma^{\mu}_{\alpha\beta} = 0$, partial differentiation and covariant differentiation are equivalent, and Eq. (4.16) can be rewritten

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0, \tag{4.17}$$

which is a tensor equation, known as the *Bianchi identities*. Bian

If we perform the Ricci contraction of Eq. (4.13) on the Bianchi identities, we obtain ties

$$R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu} = 0, \tag{4.18}$$

and if we contract this in turn, we find the *contracted Bianchi identity*

$$G^{\alpha\beta}{}_{;\beta} = 0, \tag{4.19}$$

where the (symmetric) *Einstein tensor* G is defined as

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$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} R. \tag{4.20}$$

From its name, and the alluring property Eq. (4.19), you can guess that this tensor turns out to be particularly important for us.

Anyway, back to the physics.

2.2 The equivalence principle

Back in part 1, we mentioned the equivalence principle (EP). It is now finally time to use this, and to restate it in terms that take advantage of the mathematical work we have done. The material in this section is well-discussed in Schutz §7.1; as well as in Rindler [3], at the end of his chapter 1 and in §§8.9–8.10. It is discussed, one way or another, in essentially every GR textbook, with more or less insight, so you can really take your pick.

One statement of the principle (Einstein's, in fact) is

> *The Equivalence Principle:* All local, freely falling, nonrotating laboratories are fully equivalent for the performance of all physical experiments

(Rindler refers to this as the 'strong' equivalence principle, and discusses it under that title with characteristic care, distinguishing it from the 'semistrong' EP, and the 'weak' EP, which is the statement that inertial and gravitational mass are the same).

The EP gives us a route from the physics we understand to the physics we don't (yet). That is, given that we understand how to do physics in the inertial frames of special relativity, we can import this understanding into the apparently very different world of curved – possibly completely round the twist – spacetimes, since the EP tells us that physics works locally in *exactly* the same way in *any* local inertial frame, free-falling in a curved spacetime.

So that tells us that an electric motor, say, will work as happily as we free-fall into a black hole, as it would work in any less doomed SR frame. It does immediately constrain the general form of physical laws, since it requires that, whatever their form in general, they must reduce to the SR version when expressed in the coordinates of a LIF. For example, whatever form Maxwell's equations take in a curved spacetime, they must reduce to the SR form, Eq. (4.12), when expressed in the coordinates of any LIF. The same goes for conservation laws such as Eq. (4.9) or Eq. (4.10). This form of the EP doesn't, however, rule out the possibility that the curved-spacetime law is (much) more complicated in general, and simply (and even magically) reduces to a simple SR form when in a LIF. Specifically, it doesn't rule out the possibility of *curvature coupling*, where the general form of a conservation law such as Eq. (4.9) has some dependence on the local curvature, which disappears in a LIF.

For that, we need a slightly stronger wording of the EP as quoted above (see Schutz §7.1; Rindler §8.9 quotes this as a 'reformulation' of the EP):

> *The Strong Equivalence Principle:* Any physical law which can be expressed in tensor notation in SR has exactly the same form in a locally inertial frame of a curved spacetime.

The difference here is that this says, in effect, that only geometrical statements count (*this* is why we've been making such a fuss about the primacy of geometrical objects and the relative unimportance of their components, all the way throughout the course). That is, it says that a SR conservation law such as Eq. (4.9), $T^{\mu\nu}{}_{,\nu} = 0$, has the same *form* in a LIF, and as a result, because covariant differentiation reduces to partial differentiation in the LIF, the partial derivative here is *really* just the LIF form of a covariant derivative, and so the general form of this law is

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{4.21}$$

with the comma turning straight into a semicolon, and *no extra curvature terms appearing on the right hand side*. That is why this form of the equivalence principle is sometimes referred to as the 'comma-goes-to-semicolon' rule.

Note that this comma-goes-to-semicolon is emphatically *not* what happened in the step between, for example, Eq. (4.16) and Eq. (4.17), and in numerous other similar moves throughout part three. What was happening there was a *mathematical* step: covariant differentiation of a geometrical object is equivalent to partial differentiation when in a LIF; we have a true statement about partial differentiation in Eq. (4.16), so the same statement must be true of covariant differentiation; such a statement in one frame is true in any frame, hence the generality. The Strong EP comma-goes-to-semicolon rule, on the other hand, is making a *physical* statement, namely that the statement of a physical law in a LIF directly implies a fully covariant law which is *no more complicated.*

It is possibly not obvious, but the Strong EP also tells us how matter is affected by spacetime. In SR, a particle at rest in an inertial frame moves along the time axis of the Minkowski diagram – that is, along the timelike coordinate direction of the local inertial frame, which is a geodesic. The Strong EP tells us that the same must be true in GR, so that this picks out the curves generated by the timelike coordinate of a local inertial frame, which is to say:

> *Space tells matter how to move:* Free-falling particles move on timelike geodesics of the local spacetime. (4.22)

This, like the Strong EP, is a *physical* statement about our universe, rather than a mathematical one.

2.3 Geodesics and the link to 'gravity'

We should say a little more about the rather bald statement (4.22).

This statement describes the motion of a particle in a particular spacetime. If you want to describe or predict the motion of a particle, you do it in two steps. First, you work out which geodesic it will travel along: this involves solving Einstein's equations, and working out from the initial conditions of the motion which of the large number of possible geodesics your particle is actually on. Secondly, you work out how to translate from the simple motion in the inertial coordinates attached to the particle, to the coordinates of interest (presumably attached to you).

The key thing on the way to the important insight here, is to note that if you're moving along a geodesic – if you're in free fall – *you are not being accelerated*, in the very practical sense that if you were carrying an accelerometer, it would register no acceleration. If you stand still and drop a ball from your hand, the ball is showing the path you would have taken, were it not for the floor. That is, it is the force exerted by the floor on your feet that is accelerating you away from your 'natural' free-fall path. If you hold an accelerometer in your hand – for example, a weight on a spring – you can see your acceleration register as the spring extends beyond the length it would have in free-fall.

In other words, GR says we've always been thinking of this situation backwards. We're used to standing-on-the-ground being the normal state and falling being the exceptional one (we're primates, after all, and not falling out of trees has been regarded as a Key Skill). But GR says that we've got that inside out: inertial motion, which in the presence of masses we recognise as free-fall, is the simplest, or normal, state, requiring no explanation, and it's not-falling that has to be explained. The equivalence principle says that the force of gravity doesn't just *feel* like being forced upwards by the floor, it *is* being accelerated upwards by the floor.

2.4 Einstein's equations

We have now worked out how spacetime affects the motion of matter. We now have to work out how matter affects spacetime – where does 'gravity' come from? We can't *deduce* this from anywhere; we can simply make intelligent guesses about this, based on our experience of other parts of physics – see Ptolemy's remarks about this at the beginning of this part – and hope that our (mathematical) deductions from these are corroborated, or not, by experiment. Thus our goal in this section is to *make Einstein's equations plausible.* Schutz does this in his §§8.1–2; Rindler does it very well in his §§8.2 and 8.10.

Newton's theory of gravity can be expressed in terms of a gravitational field ϕ. The gravitational force $\mathbf{f}$ on a test particle of mass m is a three-vector with components $f_i = -m\phi_{,i}$, and the source of the field is mass density ρ, with the field equation connecting the two being

$$\phi^{,i}{}_{,i} = 4\pi G\rho \tag{4.23}$$

(with the sum being taken over the three space indexes, and where $\phi^{,i}{}_{,i} = g^{ij}\phi_{,ij} = g^{ij}\partial^2\phi/\partial x^i\partial x^j$). This is Poisson's equation. In a region which does not contain any matter – for example an area of space which is not inside a star or a planet or a person, the mass density $\rho = 0$, and the *vacuum* field equations are

$$\phi^{,i}{}_{,i} = 0. \tag{4.24}$$

Now cast your mind back to part one, and the expression in the notes there for the acceleration towards each other of two free-falling particles. This expression can be slightly generalised and rewritten here as

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}t^2} = -\phi^{,i}{}_{,j}\xi^j. \tag{4.25}$$

But compare this with Eq. (3.54): they are both equations of geodesic deviation, suggesting that the tensor represented by $R^\mu{}_{\alpha\nu\beta}U^\alpha U^\beta$ is analogous to $\phi^{,i}{}_{,j}$ (we've used the symmetries of the curvature tensor to swap two indexes, note, and used $\overline{U}$ rather than $\overline{X}$ to refer to the free-falling particle velocity). Since the particle velocities are arbitrary, that means, in turn, that the $\phi^{,i}{}_{,i}$ appearing in Poisson's equation is analogous to $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$, and so a good guess at the relativistic analogue of Eq. (4.24) is

$$R_{\mu\nu} = 0. \tag{4.26}$$

This guess turns out to have ample physical support, and Eq. (4.26) are known as Einstein's *vacuum* field equations for GR.

If $R_{\mu\nu} = 0$, then $R = g^{\mu\nu} R_{\mu\nu} = 0$ and therefore

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0.$$

So much for the vacuum equations, but we want to know how spacetime is affected by matter. We can't relate it simply to ρ, since Sect. 1.2 made it clear that this was a frame-dependent quantity; the field is much more likely to be somehow bound to the E-M tensor T instead. Looking back at Eq. (4.23), we might guess

$$R^{\mu\nu} = -\kappa T^{\mu\nu} \tag{4.27}$$

as the field equations in the presence of matter, where κ is some coupling constant, analogous to the newtonian gravitational constant G. This looks plausible, but the conservation law Eq. (4.21) immediately implies that $R^{\mu\nu}{}_{;\nu} = 0$ which, using the Bianchi identity Eq. (4.19), in turn implies that $R_{;\nu} = 0$. But if we use Eq. (4.27) again, this means that $(g_{\alpha\beta} T^{\alpha\beta})_{;\nu} = 0$ also. If we look back to, for example, Eq. (4.8), we see that this field equation, Eq. (4.27), would imply that all matter has a constant density. Which is not the case. So Eq. (4.27) cannot be true.[2]

[2] This argument comes from §8.10 of [3]; Schutz has a more mathematical argument in his §8.1. Which you prefer is a matter of taste, but in keeping with our attempt to talk about physics in this part, we'll prefer the Rindler version for now.

So how about

$$G^{\mu\nu} = -\kappa T^{\mu\nu} \tag{4.28}$$

as an alternative? The Bianchi identity Eq. (4.19) tells us that the conservation equation $T^{\mu\nu}{}_{;\nu} = 0$ is satisfied identically. Additionally – and this is the key part of the argument – numerous experiments tell us that Eq. (4.28) have so-far undisputed physical validity: they have not been shown to be incompatible with our universe. They are known as the *Einstein field equations*, and allow us to complete the other half of the famous slogan Einst equa

> Space tells matter how to move – the statement (4.22) plus equations (3.36) or (3.37);
> matter tells space how to curve – equation (4.28).

Einstein's equation constitutes ten second-order non-linear differential equations (ten since there are only ten independent components in the einstein tensor), which reduce to six independent equations when we take account of the four differential identities Eq. (4.19). Between them, these determine six of the ten independent components of the metric $g_{\mu\nu}$, with the remaining four functional degrees of freedom in the metric corresponding to coordinate transformations which do not change the geometry. The non-linearity (meaning that adding together two solutions to the equation does *not* produce another solution) is what allows spacetime to couple to itself without the presence of any curvature terms in the energy-momentum tensor (which acts as the source of the field); it is also what makes Eq. (4.28) devilishly difficult to solve, and a good deal of the second part of this course, GRG II, is devoted to examining some of the solutions which have been derived over the years.

The identities Eq. (3.49) together reduce the number of independent components of the curvature tensor from 256 (4^4) to 20. That corresponds to there being 20 independent second derivatives of the metric $g_{\alpha\beta,\mu\nu}$ rather than 100 (100 since both the metric and partial differentiation are symmetric). See the end of Schutz §6.2. Ex.4

There are two further points to make, both relating to the arbitrariness which is evident in our justification of Eq. (4.28).

The first is to acknowledge that, although we were forced to go from Eq. (4.27) to Eq. (4.28) by the observation that the universe is in fact lumpy, there is nothing other than Occam's razor which forces us to stop adding complication when we do arrive at Einstein's equations, which do so far appear to match the universe we find ourselves in. There have been various attempts to play with more elaborate theories of gravity, but almost none so far which have acquired experimental support. Chandrasekhar's words on this, quoted in Schutz §8.1, are good:

> The element of controversy and doubt, that have continued to shroud the general theory of relativity to this day, derives precisely from this fact, namely that in the formulation of his theory Einstein incorporates aesthetic criteria; and every critic feels that he is entitled to his own differing aesthetic and philosophic criteria. Let me simply say that I do not share these doubts; and I shall leave it at that.

The one variation of Einstein's equation which is now being taken seriously is one that Einstein himself reluctantly suggested. Since $g_{\alpha\beta;\mu} = 0$ identically, we can add any constant multiple of the metric to the einstein tensor without disturbing the right-hand side of Eq. (4.28). Specifically, we can write

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = -\kappa T^{\mu\nu}. \tag{4.29}$$

The extra term is referred to as the *cosmological constant*. Einstein introduced it in order to permit a static solution to the field equations, but the experimental evidence for the big bang showed that this was not in fact a requirement, and the parameter Λ was determined to be vanishingly small. Much more recently, however, studies of dark matter and the cosmic energy budget have shown that the large-scale structure of the universe is not completely determined by its matter content, and so Λ, in the form of 'dark energy', is now again the subject of feverish study. co

The results of NASA's WMAP mission (see `http://map.gsfc.nasa.gov/`) showed in 2003 that such a cosmological term, related to a dark energy field, is a necessary addition to Einstein's equations of Eq. (4.28) in order to match the universe we find ourselves in.

2.5 The field equations from a variational principle

The account above, of how we obtain Einstein's equations, is pragmatic, and broadly follows Einstein's own approach to obtaining them. As well, it conveniently introduces the idea of the energy-momentum tensor, and lets us develop some intuitions about it. It is not the only way to obtain the equations, however.

In Sect. 4.1 of part 3, we saw, in passing, how we could obtain the geodesic equation by extremising the integrated length of proper distance between two points, $\mathrm{d}s = |g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu|^{1/2}$. We can do something very similar with the *Einstein-Hilbert action*,

Einst
Hilbe

$$S = \frac{1}{16\pi}\int_\Omega R(-g)^{1/2}\mathrm{d}^4x. \tag{4.30}$$

Here, R is the Ricci curvature scalar of Eq. (4.14), g is the determinant of the metric, and the volume of integration, Ω is the region interior to some boundary where we can take the variation to be zero. The Ricci scalar is a simple object – a scalar field on spacetime – which characterises the local curvature at each point. Under a change of basis, the volume element d^4x is scaled by a factor of the jacobian $|\partial x^{\bar{\mu}}/\partial x^\mu|$, and the determinant g by a factor of $(\text{jacobian})^{-2}$ (these are 'tensor densities'), so that the quantity $\sqrt{-g}\,\mathrm{d}^4x$ is a scalar.

We assume that this action, S, is *extremised* by the variation $\delta g^{\mu\nu}$ in the metric. *This is a physical statement*, and it is startling that such a simple statement – almost the simplest non-trivial statement we can make with these raw materials – combined with the very profound ideas of the calculus of variations, can lead us to the einstein equations.

Calculating the variation, δS, resulting from a variation $\delta g^{\mu\nu}$, we find

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu} \tag{4.31}$$

(the calculation is not long, but is somewhat tricky, and is described in Carroll [4, §4.3], and in MTW [5, box 17.2 & ch. 21]). You will recognise the term in square brackets from Eq. (4.20);

requiring that $\delta S = 0$ for all variations $g_{\mu\nu}$ therefore implies that

$$G_{\mu\nu} = 0,$$

recovering Einstein's vacuum field equations.

We can add a second term S_M to the action, which depends on the energy-momentum content of the spacetime volume, then perform the same calculation, and discover the field equations in the presence of matter. Choosing what that term S_M should be is of course an intricate matter, but if we obtain from it the tensor

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}},$$

then we can recover the einstein equations of Eq. (4.28).

3 The newtonian limit

We cannot finish this course without using at least one physical metric, and the one we shall briefly examine is that in the *weak field limit*, where spacetime is curved only slightly, such as round a small object like the earth.

Before we do that we need to get units straight. In special relativity we chose our unit of time to be the metre, and we followed that convention in this course. That meant that the speed of light c was dimensionless:

$$1 = c = 299\,792\,458\,\mathrm{m\,s^{-1}}.$$

In gravitational physics, we use *geometrical units*, for much the same reason. In SI units, Newton's gravitational constant has the dimensions $[G] = \mathrm{kg^{-1}m^3s^{-2}}$, but it is convenient in GR to have G dimensionless, and to this end we choose our unit of *mass* to be the metre, with the conversion factor between this and the other mass unit, kg, obtained by: geom units

$$1 = \frac{G}{c^2} = 7.425 \times 10^{-28}\,\mathrm{m\,kg^{-1}}.$$

See Schutz's §8.1 for a table of physical values in these units. Measuring masses in metres turns out unexpectedly intuitive: when you learn about (Schwartzschild) black holes you discover that the radius of the event horizon of an object has the same value as the object's mass expressed in metres. Also, within the solar system, the mass of the sun is less well-known than the value of the 'heliocentric gravitational constant', $GM_\odot$, which has units of $\mathrm{m^3\,s^{-2}}$ in SI units, and thus units of metres in natural units.[3]

In the weak-field approximation, we take the spacetime round a small object to be nearly minkowskian, with

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \tag{4.32}$$

[3] $GM_\odot$ is known to one part in 10^{10}, but since G is known only to one part in 10^4 or so, the uncertainty in $M_\odot$ has the same value.

where $|h_{\alpha\beta}| \ll 1$, and the matrix $\eta_{\alpha\beta}$ is the matrix of components of the metric in minkowski space. Note that Eq. (4.32), defining $h_{\alpha\beta}$, is a *matrix* equation, not a tensor one: we are choosing coordinates in which the matrix of components $g_{\alpha\beta}$ of the metric tensor g is approximately equal to $\eta_{\alpha\beta}$. If we lorentz-transform Eq. (4.32) – using the $\Lambda^{\alpha}_{\bar{\alpha}}$ of SR, for which $\eta_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}_{\bar{\alpha}} \Lambda^{\beta}{}_{\bar{\beta}} \eta_{\alpha\beta}$ – we get an equation just like Eq. (4.32), but in the new coordinates; that is, the components $h_{\alpha\beta}$ transform as if they were the components of a tensor in SR. This allows us to express $R^{\alpha}{}_{\beta\mu\nu}$, $R_{\alpha\beta}$ and $G^{\alpha\beta}$, and thus Einstein's equation itself, in terms of $h_{\alpha\beta}$ plus corrections of order $|h_{\alpha\beta}|^2$. The picture here is that $g_{\alpha\beta}$ is the result of a *perturbation* on flat (minkowski) spacetime, and that h (which encodes that perturbation) is a tensor in minkowski space: expressing Einstein's equations in terms of h (accurate to first order in $h_{\alpha\beta}$) gives us a mathematically tractable problem to solve.

The next step is to observe that in the newtonian limit, which is the limit where Newton's gravity works, the gravitational potential $|\phi| \ll 1$ and speeds $|\mathbf{v}| \ll 1$. This implies that $|T^{00}| \gg |T^{0i}| \gg |T^{ij}|$ (because $T^{00} \propto m$, $T^{0i} \propto v^i$ and $T^{ij} \propto v^i v^j$, with $v_{\text{earth}} \approx 10^{-4}$). We then identify $T^{00} = \rho + O(\rho v^2)$. By matching the resulting form of Einstein's equation with Newton's equation for gravity, we fix the constant κ in Eq. (4.28), so that, in geometrical units,

$$G^{\mu\nu} = 8\pi T^{\mu\nu}. \tag{4.33}$$

The solution to this equation, in this approximation, is

$$h^{00} = h^{11} = h^{22} = h^{33} = -2\phi, \tag{4.34}$$

which translates into a metric for newtonian spacetime

$$\mathsf{g} \to \operatorname{diag}(-(1+2\phi), 1-2\phi, 1-2\phi, 1-2\phi), \tag{4.35a}$$

which can be alternatively written as the interval

$$\mathrm{d}s^2 = -(1+2\phi)\mathrm{d}t^2 + (1-2\phi)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{4.35b}$$

See Schutz §§8.3–4 for the slightly intricate details of this derivation to Eq. (4.35), and see his §7.2 for the derivation of the newtonian geodesics below. Carroll [4] gives an overlapping account of the same material in §4.1 and (with more technical background) §7.1. Also, Prof. Hendry goes into this calculation in considerable detail in Chapter 5 of GRG2, as part of the discussion of gravitational radiation.

The decomposition of Eq. (4.32) can be viewed either in terms of tensors in a background (flat) spacetime (as above), or as exploitation of a gauge freedom in GR. Because of the coordinate invariance of GR, we are free to choose coordinates (ie choose a gauge) in which the matrix $h_{\alpha\beta}$ has desirable (ie simplifying) properties. The details omitted here are to do with identifying what the desirable simplifications are, and proving that a suitable choice of coordinates is indeed always possible. The solution to Eq. (4.33) in terms of h can then be fairly directly shown to be Eq. (4.34).

What are the geodesics in this spacetime? The geodesic equation is $\nabla_{\overline{U}}\overline{U} = 0$. This geodesic curve has affine parameter τ, but by rescaling this parameter though an affine transformation ($\tau \to \tau/m$), we can express this in terms of the momentum $\overline{p} = m\overline{U}$. This has the advantage that the resulting geodesic equation

$$\nabla_{\overline{p}}\overline{p} = 0 \tag{4.36}$$

is also valid for photons, which have a well-defined momentum even though they have no mass m. We shall now *solve* this equation, to find the path of a free-falling particle through this spacetime. Ex.4

The component form of Eq. (4.36) is

$$p^\alpha p^\mu{}_{,\alpha} + \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = 0.$$

If we restrict ourselves to the motion of a non-relativistic particle through this spacetime, we have $|p^0| \gg |p^i|$, and we reduce this equation to

$$m\frac{\mathrm{d}}{\mathrm{d}\tau} p^\mu + \Gamma^\mu_{00}(p^0)^2 = 0. \tag{4.37}$$

The 0–0 Christoffel symbols for this metric, in this approximation, are

$$\Gamma^0_{00} = \phi_{,0} + O(\phi^2) \tag{4.38}$$

$$\Gamma^i_{00} = -\tfrac{1}{2}(-2\phi)_{,j}\,\delta^{ij}. \tag{4.39}$$

The 0-th component of Eq. (4.37) then tells us that

$$\frac{\mathrm{d}p^0}{\mathrm{d}\tau} = -m\frac{\partial\phi}{\partial\tau},$$

so that the energy of the particle in this frame is conserved in a non-time-dependent field (the particle picks up kinetic energy as it falls, and loses gravitational potential energy). The space component is

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = -m\phi^{,i}$$

which is simply Newton's law of gravitation, $\mathbf{f} = -m\nabla\phi$.

Thus we have come a long way in this course, from Special Relativity back, through Ptolemy and Newton, to well before the place we started. We have discovered that the universe is simple (the strong EP and Eq. (4.28)), and that we are now well-placed to look upward and outward, towards the physical applications of GR II.

> I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me. *Isaac Newton, as quoted in Brewster, Memoirs of the life, writings, and discoveries of Sir Isaac Newton*

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These notes have benefitted from very thoughtful comments, criticism and error-checking, received from both colleagues and students, over the years this course has been presented. Without downplaying other contributions, Craig Stark (in 2004) and Liam Moore (2011) were helpfully relentless in finding ambiguities and errors. If you find more, please do let me know.

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[2] Claudius Ptolemaeus. *Ptolemy's Almagest*. Duckworth, 1984. Trans. G. J. Toomer.

[3] Wolfgang Rindler. *Essential Relativity: Special, General and Cosmological*. Springer-Verlag, 2nd edition, 1977.

[4] Sean M Carroll. *Spacetime and Geometry*. Pearson Education, 2004. See also `http://spacetimeandgeometry.net`.

[5] Charles W Misner, Kip S Thorne, and John Archibald Wheeler. *Gravitation*. Freeman, 1973.

Examples

Example 4.1 (section 1.2)

Deduce Eq. (4.7), given that you have only the tensors $\overline{U} \otimes \overline{U}$ and $\mathsf{g} = \eta$ to work with, that the result must be proportional to both ρ and p, and that it must be consistent with both Eq. (4.6), and Eq. (4.4) in the limit $p = 0$. Thus write down the general expression $\mathsf{T} = (a\rho + bp)\overline{U} \otimes \overline{U} + (c\rho + dp)\mathsf{g}$ and apply the various constraints. Recall that $\overline{U} = (1, \overline{0})$ in the MCRF.

Example 4.2 (section 2.2)

(a) Quote, and describe the significance of, the Equivalence Principle and the Strong Equivalence Principle. [4]

(b) Explain briefly, and without calculation, how the Equivalence Principle implies that light will be observed to bend when in a gravitational field. [4]

(c) What would happen to an electric motor in free fall across the event horizon of a black hole (ignore any tidal effects)? [2]

(d) Explain what is meant by the 'comma-goes-to-semicolon' rule in general relativity, and why it follows from the principles of covariance and equivalence. [Objective 1, Objective 2; adapted from Degree exam 2003]

Example 4.3 (section 2.4)

Prove that the curvature tensor has only 20 independent components for a 4-dimensional manifold, when you take equations Eq. (3.49a) and Eq. (3.49b) into account.

Example 4.4 (section 3)

The geodesic equation, in terms of the momentum $\overline{p}$, is $\nabla_{\overline{p}}\overline{p} = 0$ or, in components,

$$p^\alpha p_{\beta;\alpha} = 0.$$

By expanding this, taking advantage of the symmetry of the resulting expression under index swaps, and using the relation $p^\alpha \mathrm{d}/\mathrm{d}x^\alpha = m\mathrm{d}/\mathrm{d}\tau$, show that

$$m\frac{\mathrm{d}p_\alpha}{\mathrm{d}\tau} = \frac{1}{2} g_{\beta\gamma,\alpha} p^\beta p^\gamma. \tag{i}$$

You may need the relations

$$\begin{aligned} p_{\alpha;\beta} &= p_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta} p_\gamma \\ \Gamma^\gamma_{\alpha\beta} &= \frac{1}{2} g^{\gamma\lambda}(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}) \end{aligned}$$

[Objective 3, and other objectives from parts 2 and 3; from degree exam 2009, 10 marks]

Example 4.5 (section 3)

The Schwartzschild metric is

$$g_{tt} = -(1-a) \qquad g_{rr} = (1-a)^{-1}$$
$$g_{\theta\theta} = r^2 \qquad g_{\phi\phi} = r^2 \sin^2\theta,$$

where $a = 2M/r$, M is a constant, and all other metric components are zero.

Calculate the five non-zero derivatives of the metric.

Using Eq. (i), and by considering the relevant components of $\mathrm{d}p_\alpha/\mathrm{d}\tau$, demonstrate that:

1. if a particle is initially moving in the equatorial plane (that is, with $\theta = \pi/2$ and $p_\theta = 0$), then it remains in that plane;
2. if a particle is released from rest in these coordinates (that is, with $p_r = p_\theta = p_\phi = 0$, and $p_t \neq 0$), it initially moves radially inwards.

[Objective 3, and other objectives from parts 2 and 3; from degree exam 2009, 15 marks]

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