Astronomy 3/4 – General Relativity I – Part 1
Introduction

Norman Gray

Martinmas Term, 2004

[Public version 1.0]

Overview

Web pages:

- `http://www.astro.gla.ac.uk/honours/`
- `http://www.astro.gla.ac.uk/users/norman/lectures/AHGR/`
- `http://www.astro.gla.ac.uk/users/norman/lectures/A2SR/`

The course divides into four parts:

Part 1 – Introduction One lecture. Covers the overall motivation for the course – why Special Relativity cannot provide a complete description of gravity, and why gravity is special.

Part 2 – Vectors, tensors and functions Three lectures. Recap of linear algebra, and an introduction to tensors, vectors and one-forms. Basis transforms and components.

Part 3 – Manifolds, vectors and differentiation Four lectures. Introduces differential geometry. Definition of the tangent plane, and differentiation in flat and curved spaces. Introduces geodesics and curvature. Defines Riemann and Ricci tensors, and geodesic deviation.

Part 4 – Physics: energy, momentum and Einstein's equations Two lectures. Back to physics: introduces the energy-momentum tensor. More discussion of the equivalence principle, and a rationale for, and introduction to, Einstein's equations linking the curvature of space-time to the presence of gravitating objects. The Newtonian limit, and classical gravity as the weak-field limit of Einstein's equations.

Aims and objectives for Part 1

The point of Aims and Objectives is twofold. They help me keep on track by reminding me what things it's important I cover; and they help you follow the course, by reminding you of the motivation for the material I'm covering. The distinction between the two, as far as I'm concerned, is simple.

- The *aims* are the point of the course – why you're doing the course, and why I'm teaching it. These are the insights you'll have, and the ideas you'll understand, long after the point where you've forgotten most of the details. Unfortunately, it's easy to claim, but difficult to show, you have this understanding. So....
- The *objectives* are the detailed skills, mastery of which demonstrates that you have in fact achieved the aims of the course. Hint: it is a short step from objectives to exam questions, and I regard the list of objectives as more-or-less coextensive with the set of examinable topics.

Aims You should

1. appreciate why GR is not simply newtonian gravitation plus SR;
2. understand how the equivalence principles lead directly to GR effects such as light deflection and redshift (see also the aims in part 4).

Objectives You should be able to demonstrate that you can

1. quote the strong and weak equivalence principles (see also objectives for part 4), and explain what geodesic deviation is;
2. explain, without detailed calculation, how the equivalence principle leads to light bending in a gravitational field.

1 Three thought experiments on gravitation

What is the problem which General Relativity attempts to solve?

Put another way, why can't we get a theory of gravity by just taking SR and adding some 'relativised' newtonian gravity?

[The following discussion overlaps with the very useful discussion in Schutz §5.1, and with MTW §§7.2–7.3.]

1.1 Gravitational redshift

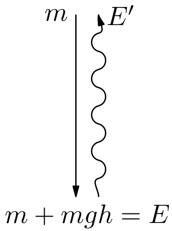

Figure 1

Imagine dropping a particle of mass m through a distance h. The particle starts off with energy m ($E = mc^2$, with $c = 1$, remember), and ends up with energy $E = m + mgh$ (see Fig. 1). Now imagine converting all of this energy into a single photon of energy E, and sending it up towards the original position. It reaches there with energy E', which we convert *back* into a particle. Now, either we have invented a perpetual motion machine, or else $E' = m$:

$$E' = m = \frac{E}{1 + gh},$$

and we discover that a photon loses energy – is redshifted – as a necessary consequence of climbing through a gravitational field, and as a consequence of our demand that energy be conserved.

This phenomenon is termed *gravitational redshift*, and it (or rather, something very like it) has been confirmed experimentally, in the 'Pound-Rebka experiment'. It's also sometimes referred to as 'gravitational doppler shift', but inaccurately, since it is not a consequence of relative motion, and so has nothing to do with the doppler shift you are familiar with.

1.2 Schild's photons

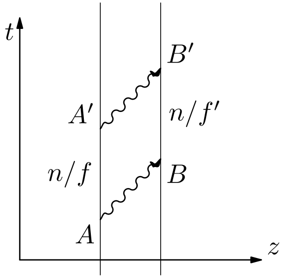

Figure 2

Imagine firing a photon, of frequency f, from a point A to a point B directly above it in a gravitational field (see Fig. 2). As we discovered in Sect. 1.1, the photon will be redshifted to a new frequency f'. After some number of periods, n, we repeat this, and send up another photon (between the points marked A' and B' on the spacetime diagram).

Since nothing will have changed between sending off the two photons, the intervals AB and $A'B'$ will be the same (I've drawn these as straight lines on the diagram, but the argument doesn't depend on that). However the intervals AA' and BB', as measured by local clocks, are *different*. That is, we have not constructed the parallelogram we might have expected, and have therefore discovered that the *geometry* of this spacetime is not the flat geometry we might have expected, and that this is purely as a result of the presence of the gravitational field through which we are sending the photons.

Finding out more about this geometry is what this course is about, and one of the first physical principles we will use is illustrated by a falling lift.

1.3 The falling lift

Recall from Special Relativity that we may define an *inertial frame* to be one in which Newton's laws hold, so that particles which are not acted on by an external force move in straight lines at a constant velocity. In Misner, Thorne and Wheeler's words, inertial frames are defined so that motion looks simple. This is so if we are in a box far away from any gravitational forces, and so we may identify that as a *local inertial frame* (because the previous section suggests that we cannot carelessly make claims about extended frames). Another way of removing gravitational forces, less extreme than going into deep space, is to put ourselves in free fall. Einstein asserted that these two situations are indeed fully equivalent, and defined an inertial frame as one in free fall.

Objects at rest in an inertial frame – in either of the equivalent situations of being far away from gravitating matter or freely falling in a gravitational field – will stay at rest. If we accelerate the box cum inertial frame, perhaps by attaching rockets to its 'floor', then the box will accelerate but its contents won't; they will therefore move towards the floor at an increasing speed, from the point of view of someone in the box. This will happen irrespective of the mass or composition of the objects in the box; they will all appear to increase their speed at the *same* rate.

Note that we are carefully *not* using the word 'accelerate' for the objects' change in speed. We reserve that word for the physical phenomenon measured by an accelerometer, and the result of a real force, and try to avoid using it (not, I fear, always successfully!) to refer to the second derivative of a position – depending on the coordinate system, the one does not always imply the other, as we shall see later.

This is very similar to Galileo's observation that all objects appear to fall under gravity at the same rate, irrespective of their mass or composition, and this has been verified to considerable precision in the Eötvös experiments. Einstein supposed that this was not a coincidence, and that there was a deep equivalence between acceleration and gravity (we shall see later that the force of gravity we feel standing in one place is the result of us being accelerated away from the path we would have if we were in free fall). He raised this to the status of a postulate:

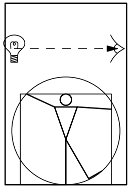

Figure 3

The (weak) Equivalence Principle: Uniform gravitational fields are equivalent to frames that accelerate uniformly relative to inertial frames.

Imagine a box floating freely in space, and imagine shining a torch horizontally across it. Where will the beam end up? Obviously, the beam will end up at a point on the wall directly opposite the torch (Fig. 3). There's nothing exotic about this. The weak equivalence principle tells us that the *same* must happen for a box in free fall. That is, a person inside a falling lift would observe the torch beam to end up level with the point at which it was emitted, in the (inertial) frame of the lift. How would this appear to someone watching the lift fall?

Since the light takes a finite time to cross the lift cabin, the spot on the wall where it strikes will have dropped some finite (though small) distance, and so will be lower than the point of emission, in the frame of someone watching this from a position of safety (Fig. 4). That is, this non-free-fall observer will measure the light's path as being curved in the gravitational field. Even massless light is affected by gravity.

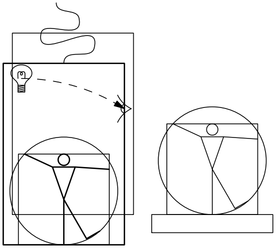

Figure 4

2 Relativity and gravitation

2.1 Tides and geodesic deviation

Consider two particles, A and B, falling towards the earth (Fig. 5). They start off level with each other, at a height $z(t)$ from the centre of the earth, and separated by a horizontal distance $\xi(t)$.

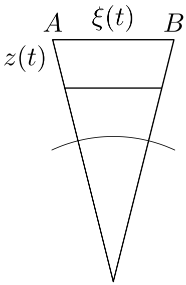

Figure 5

From the diagram, the separation $\xi(t)$ is proportional to $z(t)$, so that $\xi(t) = kz(t)$, for some constant k. The gravitational force on a particle of mass m, at altitude z is $F = GMm/z^2$, thus

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = k\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -k\frac{F}{m} = -k\frac{GM}{z^2} = -\xi\frac{GM}{z^3}.$$

This tells us that the inertial frames attached to these freely falling particles approach each other at an increasing speed (that is, they 'accelerate' towards each other in the sense that the second derivative of their separation is non-zero, but since they are in free fall, there is no physical acceleration). Ex.1

2.2 There is no universal inertial frame

A lot of Special Relativity depended on inertial frames having infinite extent: if I am an inertial observer, then any other inertial observer must be moving at a constant velocity with respect to me.

Consider observers plummeting down liftshafts, in free-fall, on opposite sides of the earth. These are inertial observers, but the second derivative of their spatial separation is not zero – they are accelerating with respect to one another. This means that, if I am one of these inertial observers, then (presuming I do not have more pressing things to worry about) I cannot use SR to calculate what the *other* inertial observer would measure in their frame, nor calculate what I would measure if I observed a bit of physics that I understand, which was happening in the other inertial observer's frame.

But this is precisely what I do want to do, supposing that the bit of physics in question is happening in free fall in the accretion disk surrounding a black hole, and I want to interpret what I am seeing through my telescope. Gravitational redshift of spectral lines is just the beginning of it!

It is General Relativity which tells us how we must patch together such disparate inertial frames.

2.3 What does GR plan to do about it?

Newton's second law is

$$\bar{F} = m\bar{a}.$$

That makes the geometrical statement that when you apply to an object a force acting in a certain direction, that object accelerates in the *same direction* as the force, with an acceleration which is proportional to it. If we want to give numerical values to this statement, then we need a coordinate system – where is the origin, what scale are the axes, and so on – but the physical law is true irrespective of which system we pick, and it remains true if we change our mind.

This is not just a peculiar property of Newton's laws. We (and Einstein) can elevate this to another principle:

> *The principle of general covariance:* All physical laws must be invariant under all coordinate transformations.

That is, only geometrical objects matter – to be a physical law, an equation must be expressible in a form which is purely geometrical, and thus independent of any coordinate system used to represent it.

That is what GR does: it describes the physics of gravity in a purely geometrical way, avoiding giving fundamental importance to any particular set of coordinates. It describes gravity, not as the rather mysterious, instantly-acting, force which Newton described in his 'law of universal gravitation', but instead as the inevitable consequence of our movement through a curved spacetime.

The problem is, that doing geometry on a curved space is tricky….

3 Natural units

In Special Relativity, we normally use *natural units*, in which we use the same units, metres, to measure both distance and time, with the result that we measure distance in these two directions in spacetime using the same units (because of the high speed of light, metres and seconds are otherwise absurdly mismatched). We extend this in General Relativity, but now measuring mass in metres also. First, a recap of natural units in SR.

It is straightforward to measure distances in seconds, and we do this naturally when we talk of the Earth being 8 light-minutes from the sun, or the nearest star being a little more than 4 light-years away, or Edinburgh being 50 minutes from Glasgow (ScotRail permitting). In fact, since 1981 or so, the International Standard definition of the metre is that it is the distance light travels in 1/299792458 seconds; that is, the speed of light is $299\,792\,458\,\mathrm{m\,s^{-1}}$ *by definition*, and so c is therefore demoted to being merely a conversion factor between two different units of time. In the same sense, the inch is defined as $2.54\,\mathrm{cm}$, and this figure of 2.54 is merely a conversion factor between two different, and only historically distinct, units of length. We write this as $1in = 2.54cm$, or $1 = 2.54\,\mathrm{cm\,in^{-1}}$.

There are several advantages to this. (i) In relativity, space and time are not really distinct, but having different units for the two ‘directions’ can obscure this. (ii) If we measure time in metres, then we no longer need the conversion factor c in our equations, which are consequently simpler. (iii) In these units, light travels a distance of one metre in a time of one metre, giving the speed of light as an easy-to-remember, and dimensionless, $c = 1$. We also quote other speeds in these units of metres per metre, so that all speeds are dimensionless and less than one. That means that $1 = c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$ so that, just as with the expression $1 = 2.54\,\mathrm{cm\,in^{-1}}$ above, we are using the figure 3×10^8 as a conversion factor between the alternative length units of metres and seconds.

In GR we must additionally deal with the masses of objects, and we measure masses in metres also, with the conversion factor between kilogrammes and metres fixed by the demand that the gravitational constant have the easy-to-remember value $G = 1$. That means that the expression $1 = G = 6.673 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ becomes a conversion factor between kilogrammes and the other units.

It is easy, once you have a little practice, to convert values and equations between the different systems of units. Throughout the rest of this course, I will quote equations in units where $c = 1$, and, when we come to that, $G = 1$, so that the factors c and G disappears from the equations.

XXX
natu

For example, to convert $10\,\mathrm{J} = 10\,\mathrm{kg\,m^2\,s^{-2}}$ to natural units, we could proceed in two ways. Since $c = 1$, we have $1\,\mathrm{s} = 3 \times 10^8\,\mathrm{m}$ (this looks very bizarre, but compare the closely analogous statement $1\,\mathrm{in} = 2.54\,\mathrm{cm}$), and so $1\,\mathrm{s^{-2}} = (9 \times 10^{16})^{-1}\,\mathrm{m^{-2}}$. So $10\,\mathrm{kg\,m^2\,s^{-2}} = 10\,\mathrm{kg\,m^2} \times (9 \times 10^{16})^{-1}\,\mathrm{m^{-2}} = 1.1 \times 10^{-16}\,\mathrm{kg}$.

Alternatively, we can write $1 = 3 \times 10^8\,\mathrm{m\,s^{-1}}$ (compare $1 = 2.54\,\mathrm{cm\,in^{-1}}$), or $1 = (3 \times 10^8)^{-1}\,\mathrm{s\,m^{-1}}$. Thus

$$\begin{aligned} 10\,\mathrm{J} &= 10\,\mathrm{kg\,m^2\,s^{-2}} \times (1)^2 \\ &= 10\,\mathrm{kg\,m^2\,s^{-2}} \times (3 \times 10^8)^{-2}\,\mathrm{s^2\,m^{-2}} \\ &= 1.1 \times 10^{-16}\,\mathrm{kg}. \end{aligned}$$

Ex.1

In General Relativity, people tend to work in units where mass has the same units as distance and time, and the gravitational constant $G = c = 1$. In relativistic quantum mechanics, likewise, units are chosen so that $\hbar = c = 1$.

4 Further reading

When learning relativity, even more than with other subjects, you benefit from hearing or reading things multiple times, from different authors, and from different points of view. I mention a couple of good introductions below, but there is really no substitute for going to section ‘Physics C25’ in the library, looking through the books there, and finding one which makes sense to *you*.

The course book is [1] (herafter simply ‘Schutz’). Though possession of this book is not compulsory, it is very heavily recommended: of the books below, it is the one closest in style to these lectures; also, I will occasionally direct you to particular sections of it.

Other textbooks you might want to look at are:

- Rindler [2] always explains the physics clearly, particularly the differences between the strong and weak equivalence principles, and the motivation for GR. It’s now rather old-fashioned in many respects, in particular its treatment of differential geometry.

- Wald [3] is comprehensive and well thought-of.

- Misner, Thorne and Wheeler [4] is a glorious, comprehensive, doorstop of a book. Its distinctive prose style and typographical oddities have fans and enemies in roughly equal numbers. If you liked Taylor & Wheeler’s *Spacetime Physics*, there’s a good chance you’ll like this one. There’s much, much, more in here than you need for the course. Chapter 1 in particular is worth reading for an overview of the subject.

- Longair’s book [5] is worth looking at. The section on GR (only a smallish part of the book) is concerned with motivating the subject, and is in a seat-of-the-pants style that might be to your taste.

There are also many more advanced texts. The following are graduate-level texts, and so well beyond the level of this course. They are typically mathematically very sophisticated.

If, however, your tastes run that way, then the introductory chapters of these books might be instructive, and give you a taste of the vast wonderland of beautiful maths that can be found in this subject.

- Chapter 1 of [6] covers more than the content of this course in just 60 pages.
- [7] is by the same author as [1] above – from their covers, they're usually referred to as 'blue Schutz' and 'green Schutz' respectively. It's a lovely book, which explains the differential geometry clearly and sparsely, including applications beyond relativity and cosmology. However, it appeals only to those with a strong mathematical background, and horrifies everyone else.
- Hawking and Ellis [8], chapter 2, covers more than all the differential geometry of this course.

Notation conventions There are a number of different *sign conventions* in use in relativity books. The sign conventions used in this course match those in Schutz, [4] and [8]. The book [6] has the opposite signs for g, R and G; and [2] has opposite signs for g and G.

sign
tions

References

[1] Bernard F Schutz. *A First Course in General Relativity*. Cambridge University Press, 1985.

[2] Wolfgang Rindler. *Essential Relativity: Special, General and Cosmological*. Springer-Verlag, 2nd edition, 1977.

[3] Robert M Wald. *General Relativity*. University of Chicago Press, 1984.

[4] Charles W Misner, Kip S Thorne, and John Archibald Wheeler. *Gravitation*. Freeman, 1973.

[5] M S Longair. *Theoretical Concepts in Physics*. Cambridge University Press, 1984.

[6] John Stewart. *Advanced General Relativity*. Cambridge, 1991.

[7] Bernard F Schutz. *Geometrical Methods of Mathematical Physics*. Cambridge University Press, 1980.

[8] Stephen W Hawking and G F R Ellis. *The Large Scale Structure of Space-Time*. Cambridge, 1973.

Examples

Example 1.1 (section 2.1)

If two 1kg balls, 1m apart, fall down a liftshaft near the surface of the earth, how much is their tidal acceleration towards each other? How much is their acceleration towards each other as a result of their mutual gravitational attraction?

Example 1.2 (section 3)

Convert the following to units in which $c = 1$: (a) $10\,\mathrm{J}$; (b) lightbulb power, $100\,\mathrm{W}$; (c) Planck's constant, $\hbar = 1.05 \times 10^{-34}\,\mathrm{J\,s}$; (d) velocity of a car, $v = 30\,\mathrm{m\,s^{-1}}$; (e) momentum of a car, $3 \times 10^4\,\mathrm{kg\,m\,s^{-1}}$; (f) pressure of 1 atmosphere, $10^5\,\mathrm{N\,m^{-2}}$; (g) density of water, $10^3\,\mathrm{kg\,m^{-3}}$; (h) luminosity flux, $10^6\,\mathrm{J\,s^{-1}\,cm^{-2}}$.

Convert the following to physical units (SI): (i) velocity, $v = 10^{-2}$; (j) pressure $10^{19}\,\mathrm{kg\,m^{-3}}$; (k) time $10^{18}\,\mathrm{m}$; (l) energy density $u = 1\,\mathrm{kg\,m^{-3}}$; (m) acceleration $10\,\mathrm{m^{-1}}$; (n) Eqn. (**??**), $t' = \gamma(t - vx)$; (o) the 'mass-shell' equation $E^2 = p^2 + m^2$. (Example slightly adapted from Schutz [1, ch.1])