Astronomy – General Relativity and Gravitation I
Notation

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Context matters. . . .

Parts 2 and 3 introduce a great deal of sometimes confusing notation. The best way to get used to this is: to get used to it, by working through examples, but while you're slogging there, this crib might be useful.

Tensors

A $\binom{M}{N}$ tensor is a linear function of M one-forms and N vectors, which turns them into a number. A $\binom{1}{0}$ tensor is called a vector, and a $\binom{0}{1}$ tensor is a one-form. Vectors are written with a bar over them, $\overline{V}$, and one-forms with a tilde $\widetilde{p}$ (§2-2.1). In my (informal) notation in the lectures, $\mathsf{T}(\widetilde{\cdot}, \overline{\cdot})$ is a $\binom{1}{1}$ tensor – a machine with a single one-form-shaped slot and a single vector-shaped slot. Note that this is a different beast from $\mathsf{T}(\overline{\cdot}, \widetilde{\cdot})$, which is also a $\binom{1}{1}$ tensor, but with the slots differently arranged.

Coordinates and components

In a space of dimension n, a set of n linearly independent vectors $\overline{e}_i$ $(i = 1, \ldots, n)$ forms a basis for all vectors in the space; a set of n linearly independent one-forms $\widetilde{\omega}^i$ forms a basis for all one-forms in the space.

$\overline{e}_i(\widetilde{\omega}^j) = \delta_i{}^j \Leftrightarrow \widetilde{\omega}^j(\overline{e}_i) = \delta^j{}_i.$	*Choose* basis vectors and one-forms to be *dual* (remember that $\overline{e}_i$ and $\widetilde{\omega}^i$ are functions)	(2.3a)
$\overline{V} = V^0\overline{e}_0 + V^1\overline{e}_1 + \cdots$	Vectors have components, written with raised indexes	§2-2.5
$\widetilde{p} = p_0\widetilde{\omega}^0 + p_1\widetilde{\omega}^1 + \cdots$	. . . so do one-forms, but written with lowered indexes	
$V^i = \overline{V}(\widetilde{\omega}^i), \quad p_i = \widetilde{p}(\overline{e}^i)$	Components of vectors and one-forms (a consequence of the above)	(2.4)
$T^i{}_j = \mathsf{T}(\widetilde{\omega}^i, \overline{e}_j)$	Tensors have components, too	(2.5)
$T_i{}^j = \mathsf{T}(\overline{e}_i, \widetilde{\omega}^j)$	A different beast (note the arrangement of indexes)	

The object $T^i{}_j$ is a *number* – a component of a tensor in a particular basis. However we also (loosely, wickedly) use this same notation to refer to the corresponding *matrix* of numbers, and even to the corresponding $\binom{1}{1}$ tensor T.

The vector space in which these objects live is the tangent plane to the manifold M at the point P, $T_P(M)$ (§3-1.1). In this space, the basis vectors are $\overline{e}_i = \partial/\partial x^i$ (§3-1.2), and the basis one-forms $\widetilde{\mathrm{d}x}^i$, where x^i is the i-th coordinate (more precisely, coordinate function; note that x^i is *not* a component of any vector, though the notation makes it look a bit like one). These bases are dual: $\overline{e}_i(\widetilde{\omega}^i) = \partial/\partial x^i(\widetilde{\mathrm{d}x}^j) = \delta_i{}^j$ (cf, (3.6)).

Contractions

A contraction is a tensor with some or all of its arguments filled in.

$\overline{V}(\widetilde{p}) = \widetilde{p}(\overline{V})$	by choice	§2-2.5
$\widetilde{p}(\overline{V}) \equiv \langle \widetilde{p}, \overline{V} \rangle$	special notation for vectors contracted with one-forms	(2.1)
$\widetilde{p}(\overline{V}) = p_i V^i$	basis independent	§2-2.5
$\mathsf{T}(\widetilde{p}, \widetilde{\cdot}, \overline{V})^j = p_i V^k T^{ij}{}_k$	partially contracted $\binom{2}{1}$ tensor (a vector)	§2-2.5
$\overline{p} = \mathsf{g}(\widetilde{p}, \widetilde{\cdot})$	a vector, with components…	§3-2.3
$\mathsf{g}(\widetilde{p}, \widetilde{\omega}^j) = p_i \mathsf{g}(\widetilde{\omega}^i, \widetilde{\omega}^j) \equiv p^j$	definition of vector $\overline{p}$, with raised indexes, dual to one-form $\widetilde{p}$, written with lowered indexes	
$g_{ij} = \mathsf{g}(\overline{e}_i, \overline{e}_j)$,	components of the metric	(2.8)
$g^{ij} = \mathsf{g}(\widetilde{\omega}^i, \widetilde{\omega}^j)$	…up and down	
$g_{ij} g^{kl} T^j{}_l = T_i{}^k$	the metric raises and lowers indexes; $\mathsf{T}(\widetilde{\cdot}, \overline{\cdot})$ and $\mathsf{T}(\overline{\cdot}, \widetilde{\cdot})$ are distinct but related	
$g_{ij}, \quad g^i{}_j = \delta^i{}_j, \quad g^{ij}$	different tensors in principle, but all referred to as ‘the metric’	(2.10)

Differentiation

$V^i{}_{,j} \equiv \partial V^i/\partial x^j, \quad p_{i,j} \equiv \partial p_i/\partial x^j$	(non-covariant) derivative of a component (no surprises)	
$\nabla \overline{V}$	covariant derivative of $\overline{V}$ (a $\binom{1}{1}$ tensor)	§3-2.2
$\nabla_{\overline{U}} \overline{V}$	covariant derivative of $\overline{V}$ in the direction $\overline{U}$ (a vector)	cf. (3.33)
$\nabla_{\overline{e}_i} \overline{V} \equiv \nabla_i \overline{V}$	shorthand	
$(\nabla \overline{V})^i{}_j = V^i{}_{;j}$	components of the $\binom{1}{1}$ tensor $\nabla \overline{V}$	(3.18)
$(\nabla \widetilde{p})_{ij} = p_{i;j}$	components of the $\binom{0}{2}$ tensor $\nabla \widetilde{p}$	
$V^{i;j} = g^{jk} V^i{}_{;k}$	it’s a tensor, so you can raise its indexes, too	
$\nabla_{\overline{U}} \overline{V} = U^\alpha V^\beta{}_{;\alpha} \overline{e}_\beta$	…putting all that together	

Changing bases

We might change from a basis $\overline{e}_i$ (for example $\overline{e}_0, \dots, \overline{e}_3$) to a basis $\overline{e}_{\bar{\imath}}$ ($\overline{e}_{\bar{0}}, \dots, \overline{e}_{\bar{3}}$), noting that the bar goes over the *index* and not, as might be more intuitive, the vector (that is, we don't write $\overline{\overline{e}}_i$ or $\overline{e}'_i$). The transformation is described using a matrix (it's not a tensor) $\Lambda^{i}_{\bar{\jmath}}$ (§2-2.7).

$\overline{e}_{\bar{\imath}} = \Lambda^{i}_{\bar{\imath}} \overline{e}_i, \quad \widetilde{\omega}^{\bar{\imath}} = \Lambda^{\bar{\imath}}_{i} \widetilde{\omega}^{i}$	transformation of basis vectors and one-forms	§3-1.4
$V^{\bar{\imath}} = \Lambda^{\bar{\imath}}_{i} V^i, \quad p_{\bar{\imath}} = \Lambda^{i}_{\bar{\imath}} p_i$	transformation of components	(2.12)
$\overline{e}_i = \Lambda^{\bar{\jmath}}_{i} e_{\bar{\jmath}} = \Lambda^{\bar{\jmath}}_{i} \Lambda^{j}_{\bar{\jmath}} \overline{e}_j$	the transformation matrix goes in both directions	
$\Rightarrow \Lambda^{\bar{\jmath}}_{i} \Lambda^{j}_{\bar{\jmath}} = \delta^{j}_{i}$	matrix inverses	(2.18)

Notes: (1) these look complicated to remember, but as long as each Λ has one barred and one unbarred index, you'll find there's only one place to put each index, consistent with the summation convention. (2) This shows why it's useful to have the bars on the indexes. (3) Some books use hats for the alternate bases: $\overline{e}_{\hat{\imath}}$.

Einstein's summation convention

See §2-2.5.

$$p_i V^i \equiv \sum_i p_i V^i.$$

The convention works only for *two* repeated indexes, one up, one down. This is one of the reasons why one-form components are written with lowered indexes and vectors with raised ones; the other is to distinguish the components p_i of the one-form $\widetilde{p}$ from the components $p^i = g^{ij} p_j$ of the related vector $\overline{p} = \mathsf{g}(\widetilde{p}, \widetilde{\cdot})$. Points to watch:

- A term should have at most two duplicate indexes, one up, one down; if you find something like $U^i V^i$ or $U^i T_{ii}$, you've made a mistake.
- All the terms in an expression should have the same unmatched index(es): $A^i = B^j T^i{}_j + C^i$ is all right, $A^i = B^j T^k{}_j$ is a mistake (typo or thinko).
- You can change or swap repeated indexes: $A^{ij} T_{ij}$, $A^{ik} T_{ik}$ and $A^{ji} T_{ji}$ all mean *exactly* the same thing, but all are different from $A^{ij} T_{ji}$ (unless T happens to be symmetric).

However, sometimes we will refer to particular components of a tensor or matrix, such as referring to the diagonal elements of the metric as g^{ii} – there's no summation convention here, so the proscriptions above aren't relevant. Context again – sorry.

Miscellaneous

$\delta^i{}_j = \delta^{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$	Kronecker delta symbol	§2-1.1
$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$	the metric (tensor) of Minkowski space	
$[\overline{A}, \overline{B}] = \overline{AB} - \overline{BA}$	the 'commutator'	before (3.51)

Note: $\Lambda^i_{\ j}$ and $\Gamma^i_{\ jk}$ are matrices, not the components of tensors, so the indexes don't correspond to arguments, and so don't have to be staggered (though I sometimes do, in the notes, for no particularly good reason).

Different authors adopt different conventions for the 'signature' of the metric (the sum of the signs on the diagonal), and adopt different conventions for the Riemann tensor (3.43). This '+2' convention matches Schutz.

In general, component indexes are roman letters, i, j, and so on. When discussing specifically spacetime, it is traditional but not universal to use greek letters such as μ, ν, α, β, and so on, for component indexes ranging over $\{0, 1, 2, 3\}$, and roman letters for indexes ranging over the spacelike components $\{1, 2, 3\}$.

Components are usually standing in for numbers, however we'll sometimes replace them with letters when a particular coordinate system suggests them. For example $\overline{e}_x$ rather than $\overline{e}_1$ in the context of cartesian coordinates, or write $\Gamma^\theta_{\phi\phi}$ rather than, say, Γ^1_{22} when writing the Christoffel symbols for coordinates (θ, ϕ). There shouldn't be confusion (context, again), because x, y, θ and ϕ are never used as (variable) component indexes; see eg. (3.13).