Astronomy 2 – Special Relativity – Part 4
Vectors, kinematics and dynamics

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In part 3, we used the axioms of part 2 to obtain the Lorentz Transformation. That allowed us to describe events in two different frames in relative motion. That part was rather mathematical in style. Now we are going to return to the physics, and describe *motion*: velocity, acceleration, momentum, energy and mass.

Aims and objectives for part 4

Aims: You should

1. understand the concept of a 4-vector as a geometrical object, and the distinction between a vector and its components.
2. understand relativistic energy and momentum; the concept of energy-momentum as the norm of the momentum 4-vector; and conservation of the momentum 4-vector.
3. understand the distinction between invariant, conserved and constant quantities.

Objectives: You should be able to

1. quote the expression for the inner-product between two 4-vectors, quote the definition of the norm, and calculate the sum, norm and inner-product of 4-vectors.
2. use the Lorentz Transformation to transform the components of a vector from one frame to another.

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3. categorise 4-vectors as timelike, spacelike, or null.
4. use the velocity, acceleration and frequency 4-vectors, and the Doppler shift formulae of Eqn. (4.18) and Eqn. (4.19) (but you are not obliged to memorise these formulae).
5. use the conservation of 4-momentum, the definition of relativistic energy, Eqn. (4.22), and the relationship between energy and momentum, Eqn. (4.27), to calculate the behaviour of particles in collisions.

1 Kinematics

Part 3 was concerned with static events as observed from moving frames. In this part, we are concerned with particle motion.

Before we can explain motion, we must first be able to describe it. This is the subject of *kinematics*. We will first have to define the vectors of 4-dimensional Minkowski space, and specifically the velocity and acceleration vectors.

The description of 4-vectors in this section draws on the discussion of 4-vectors in chapter 4 of Rindler [1], and the description of dynamics in the following section draws on chapter 5 of the same book; note, however, that Rindler's notational conventions differ from mine in several important respects. Rindler defines the displacement 4-vector as $(x_1, x_2, x_3, x_4) = (\Delta x, \Delta y, \Delta z, \Delta t)$ so that the metric is (implicitly) $\mathrm{diag}(-1, -1, -1, c^2)$. Also, Rindler commits the now-old-fashioned heresy of defining a 'relativistic mass' $m(v) = \gamma(v)m_0$, which doesn't really help things.

The dynamics section below also draws on Chapter 2 of Schutz [2]. This is a textbook on General Relativity (the recommended text for the honours GR course, in fact), but chapter 2 describes Special Relativity using the beautiful and powerful geometrical language which is used extensively in GR. I have toned this down for this section, but the motivation is the same.

The dynamics section also draws on the insightful discussion in Taylor & Wheeler [3].

1.1 Three-vectors

You are familiar with 3-vectors – the vectors of ordinary three-dimensional Euclidean space. To an extent, 3-vectors are merely an ordered triple of numbers, but they are interesting to us as physicists because they represent a more fundamental geometrical object: the three numbers are not just picked at random, but are the vector's *components* – the projections of the vector onto three orthogonal axes (that the axes are orthogonal is not essential to the definition of a vector, but it is almost invariably true in physics, as it makes things so much simpler in all but extreme cases). That is, the components of a vector are functions of both the vector and our choice of axes, and if we change the axes, then the components will change in a systematic way.

For example, consider a prototype displacement vector $(\Delta x, \Delta y, \Delta z)$. These are the components of a vector with respect to the usual axes $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$. If we rotate these axes, say by an angle θ about the z-axis, to obtain axes $\mathbf{e}'_x$, $\mathbf{e}'_y$ and $\mathbf{e}'_z$, we obtain a new set of coordinates $(\Delta x', \Delta y', \Delta z')$, related to the original coordinates by

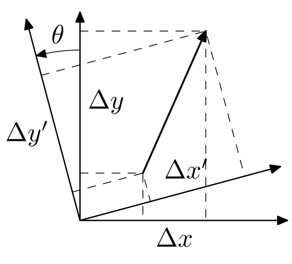

Figure 1

$$\begin{pmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}. \tag{4.1}$$

These new components describe the *same* underlying vector, as shown in Fig. 1. Although any random triple of numbers $(\Delta x, \Delta y, \Delta z)$ describes *some* vector, to some extent what turns the number triple into A Vector is the existence of this underlying object, which implies all the other sets of coordinates $(\Delta x', \Delta y', \Delta z')$, that have a particular functional relation to the original $(\Delta x, \Delta y, \Delta z)$.

Note that this prototype vector is a *displacement vector* – the 'position vector', going from the origin to a point, is *not* a vector in this sense.

Ideally, you should think of the vectors here as being defined as a length plus direction, defined at a point, rather than an arrow spread out ovec a finite quantity of space (you might think of this as some sort of infinitesimal displacement vector).

Swift review of linear algebra (make sure you are comfortable with the following notions): If $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ are vectors, then $\mathbf{a} + \mathbf{b}$ is also a vector, with components $(a_x + b_x, a_y + b_y, a_z + b_z)$. We can define a *scalar-product* $\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$. We can define the *length*, $|\mathbf{a}|$ of a three vector in terms of the scalar product of a vector with itself: $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = a^2 = a_x^2 + a_y^2 + a_z^2$ (we can also see this from Pythagoras' theorem, or indeed from Eqn. (4.1)), and we know that this is an invariant of a rotation – that is, that it takes the same value irrespective of the coordinate system. It is easy to see that $|\mathbf{a}+\mathbf{b}|^2 = a^2+b^2+2\mathbf{a}\cdot\mathbf{b}$ and so, since both a^2 and b^2 are frame-independent, the scalar-product $\mathbf{a} \cdot \mathbf{b}$ must be frame-independent also, even though the individual coordinates a_i and b_i are not. Finally, if the scalar-product of two vectors vanishes, $\mathbf{a} \cdot \mathbf{b} = 0$, we say that the two vectors are *orthogonal*. If a vector is orthogonal to itself ($\mathbf{a} \cdot \mathbf{a} = 0$) then we can deduce that $a_i = 0$. In linear algebra, the scalar-product is more generally termed the *inner-product*, and the length of a vector is termed its *norm*.

1.2 Four-vectors

As we saw in Sect. 3-2.3, we can regard the events of SR taking place in a 4-dimensional space termed spacetime. Here, the prototype displacement 4-vector is $(\Delta t, \Delta x, \Delta y, \Delta z)$, relative to the space axes and wristwatch of a specific observer, and the transformation which takes one 4-vector into another is the familiar LT of Eqn. (3.18), or

$$\begin{pmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \tag{4.2a}$$

for the 'forward transformation' and

$$\begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} \gamma & +\gamma v & 0 & 0 \\ +\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} \tag{4.2b}$$

for the inverse transformation (that the matrices are inverses of each other can be verified by direct multiplication). These give the coordinates of the same displacement as viewed by a second observer whose frame is in standard configuration with respect to the first.

To emphasise the equal standing of the four components of the 4-vector, and to make notation more straightforward, it is usual to write the components of the position vector as (x^0, x^1, x^2, x^3), where $x^0 = t$, $x^1 = x$, $x^2 = y$ and $x^3 = z$ (do note that the superscripts are indexes, not powers), or collectively x^μ, where the greek index μ runs from 0 to 3. We will also occasionally use latin superscripts like i or j: these should be taken to run from 1 to 3.

This displacement 4-vector $\Delta\mathbf{R} = (\Delta t, \Delta x, \Delta y, \Delta z)$ we can take as the prototype 4-vector, and recognise as a 4-vector anything which transforms in the same way under the coordinate transformation of Eq. (4.2a) (this may seem a rather abstract way of defining vectors, but we will see a concrete example in Sect. 1.5). That is, given an arbitrary vector $\mathbf{A}$, the transformation of its coordinates A^μ in S into its coordinates A'^μ

in S' is

$$\begin{pmatrix} A^{0'} \\ A^{1'} \\ A^{2'} \\ A^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}. \tag{4.3a}$$

Since this is a matrix equation, the inverse transformation is straightforward: it is just the matrix inverse of this:

$$\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^{0'} \\ A^{1'} \\ A^{2'} \\ A^{3'} \end{pmatrix}. \tag{4.3b}$$

Written out explicitly, the expressions in Eq. (4.3a) are

$$\begin{aligned} A^{0'} &= \gamma(A^0 - vA^1) \\ A^{1'} &= \gamma(A^1 - vA^0) \\ A^{2'} &= A^2 \\ A^{3'} &= A^3 \end{aligned}$$

and as you can see, these have the same form as the more familiar expression for the Lorentz transformation of a event's coordinates, as in Eqn. (3.18).

It's no coincidence that the same expression appears in both cases, but it's important that you realise that slightly different things are happening in the two cases. In Eqn. (3.18), the LT relates the *coordinates of a single event* in two reference frames; in Eq. (4.3), the LT relates the *components of a single vector* in two frames. These are not the same thing, because *the coordinates of an event are not the components of a 'position vector'*. The 'position vector' is not a vector in the terms of this section, because the position vector is not frame independent; since it stretches from the frame origin to the position of something, it is obviously different in different frames. This is why it is the displacement 4-vector that we take as the prototype 4-vector above; since the events (ie locations in spacetime) at each end of the displacement vector are frame-independent, the displacement 4-vector is frame-independent also.

Note also that the primed and unprimed frames are perfectly symmetrical. The observers at rest in the unprimed frame S regard the primed frame S' as moving at speed v with respect to them. However, the observers in the primed frame see the unprimed one moving at speed $-v$; they would also naturally swap the assignment of primed and unprimed frame, with their frame, say P, being the unprimed one and the other, P', being the primed one. Now, these two sets of observers are describing the *same* events and frames, but with opposite notation, and so with an opposite sign for v. We can translate between the two sets of notations by simultaneously swapping primed and unprimed quantities and swapping the sign of v. But all this is, is a way of translating between a transformation and its inverse. This is exactly what happens between, for example, Eqn. (3.18) and Eqn. (3.20), or between Eqn. (4.2a) and Eqn. (4.2b).

See example 4.1

I've talked repeatedly of events being frame-independent. Vectors are *also* frame-independent, *even though their components are not* (this is what distinguishes a vector from being just a collection of four numbers).

More linear algebra: Just as with 3-vectors, if **A** and **B** are 4-vectors, so is $\mathbf{A} + \mathbf{B}$, with components $(A^0 + B^0, A^1 + B^1, A^2 + B^2, A^3 + B^3)$. Where the scalar-product of ordinary three vectors is straightforward, the different geometry of spacetime means that the useful inner-product for 4-vectors is defined as

$$\mathbf{A} \cdot \mathbf{B} = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 \tag{4.4a}$$

$$= \sum_{\mu,\nu} \eta_{\mu\nu} A^\mu B^\nu, \tag{4.4b}$$

where the matrix $\eta_{\mu\nu}$ is defined as

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1). \tag{4.5}$$

The inner-product of a vector with itself, $\mathbf{A} \cdot \mathbf{A}$, is its *norm* (or length-squared), and from this definition we can see that the norm of the displacement vector is $\Delta\mathbf{R} \cdot \Delta\mathbf{R} = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = \Delta s^2$.

Just as we discussed in Sect. 3-2.7 for intervals, 4-vectors can be timelike, spacelike or null, depending on whether their norm is positive, negative or zero; note that, since the norm is not positive-definite (ie, it can be negative), even a non-zero vector can be null. Just as in Sect. 1.1, we say that two vectors are *orthogonal* if their inner-product vanishes; it follows that in this geometry, a null vector (with $\mathbf{A} \cdot \mathbf{A} = 0$) is orthogonal to itself! See example 4.2

The norm of a 4-vector is frame-invariant. The norm of $\mathbf{A} + \mathbf{B}$ is $|\mathbf{A} + \mathbf{B}| = |\mathbf{A}|^2 + |\mathbf{B}|^2 + 2\mathbf{A} \cdot \mathbf{B}$, and since $|\mathbf{A} + \mathbf{B}|$, A^2, and B^2 are frame-invariant, so must be the inner-product $\mathbf{A} \cdot \mathbf{B}$.

There's an element of arbitrariness in the choice of the matrix in Eq. (4.1). We could pick any set of numbers here, but only a transformation like this has interesting (which is to say, useful) properties: it preserves the length of the displacement vector, and preserves angles between vectors, and so on. In other words, it preserves euclidean geometry. Similarly, the transformation in Eq. (4.2a) has the crucial property that it preserves the norm of 4-vectors. See example 4.3

Mathematically, we could pick almost *any* set of numbers for the components of $\eta_{\mu\nu}$. However, only the definition in Eqn. (4.5) provides an inner-product which is physically meaningful.

The set of numbers $\eta_{\mu\nu}$ are not just any old matrix, but in particular the components of a *tensor* – a mathematical structure which is a generalisation of a vector, and which is vital for the mathematical study of General Relativity. It is not even just any old tensor: it is the form taken in a non-accelerating frame (ie, in SR) by the *metric tensor* which is a crucial object in General Relativity. It is the metric tensor which creates the notion of the *distance between* two events in spacetime.

1.3 Velocity and acceleration

Since the displacement 4-vector Δx^μ is a vector (in the sense that it transforms properly according to Eqn. (4.2a)), so is the infinitesimal displacement $\mathrm{d}x^\mu$; since the proper time τ (see Sect. 3-3.4) is a Lorentz scalar, we can divide each component of this infinitesimal displacement by the proper time and still have a vector. This latter vector is the *4-velocity*:

$$\mathbf{U} = \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}, \frac{\mathrm{d}x^1}{\mathrm{d}\tau}, \frac{\mathrm{d}x^2}{\mathrm{d}\tau}, \frac{\mathrm{d}x^3}{\mathrm{d}\tau} \right). \tag{4.6}$$

By the same argument, the 4-acceleration

$$\mathbf{A} = \left(\frac{\mathrm{d}^2x^0}{\mathrm{d}\tau^2}, \frac{\mathrm{d}^2x^1}{\mathrm{d}\tau^2}, \frac{\mathrm{d}^2x^2}{\mathrm{d}\tau^2}, \frac{\mathrm{d}^2x^3}{\mathrm{d}\tau^2} \right) \tag{4.7}$$

is a 4-vector, also. We can more naturally write these as $U^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ and $A^\mu = \mathrm{d}U^\mu/\mathrm{d}\tau = \mathrm{d}^2x^\mu/\mathrm{d}\tau^2$.

Let us examine these components in more detail. We have

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - |\mathrm{d}\mathbf{r}|^2,$$

(remember that proper time is a 'clock attached to the particle') so that

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = \frac{(\mathrm{d}\tau)^2}{(\mathrm{d}t)^2} = 1 - \frac{|\mathrm{d}\mathbf{r}|^2}{(\mathrm{d}t)^2} = 1 - v^2 = \frac{1}{\gamma^2}.$$

Taking the square root and inverting, we immediately find that

$$dt/d\tau = \gamma, \tag{4.8}$$

and so

$$U^0 = \frac{dx^0}{d\tau} = \frac{dt}{d\tau} = \gamma \tag{4.9a}$$

$$U^i = \frac{dx^i}{d\tau} = \frac{dt}{d\tau}\frac{dx^i}{dt} = \gamma v^i. \tag{4.9b}$$

Thus we can write

$$\mathbf{U} \equiv (U^0, U^1, U^2, U^3) = (\gamma, \gamma v_x, \gamma v_y, \gamma v_z) = \gamma(1, v_x, v_y, v_z). \tag{4.10}$$

We will generally write this, below, as

$$\mathbf{U} = \gamma(1, \mathbf{v}),$$

using $\mathbf{v}$ to represent the three (space) components of the (spatial) velocity vector, but this is perhaps a little ‘slangy’.

In a frame which is co-moving with a particle, the particle’s velocity is $\mathbf{U} = (1, 0, 0, 0)$, so that, from Eqn. (4.4a), $\mathbf{U} \cdot \mathbf{U} = 1$; since the inner-product is frame-invariant, it must have this same value in *all* frames, so that, quite generally, we have

$$\mathbf{U} \cdot \mathbf{U} = 1. \tag{4.11}$$

Here, we defined the 4-velocity by differentiating the displacement 4-vector, and deduced its value in a frame co-moving with a particle. We can now turn this on its head, and *define* the 4-velocity as a vector which has norm 1 and which points along the t-axis of a co-moving frame (this is known as a ‘tangent vector’, and is effectively a vector ‘pointing along’ the worldline). We have thus defined the 4-velocity of a particle as the vector which has components $(1, \mathbf{0})$ in the particle’s rest frame. Note that the norm of the vector is always the same; the particle’s speed relative to a frame S is indicated not by the ‘length’ of the velocity vector – its norm – but by its direction in S. We can then *deduce* the form in Eqn. (4.10) as the Lorentz-transformed version of this. See also the remarks in Sect. 1.4.

Turning now to the acceleration $\mathbf{A}^\mu$, we have

$$A^0 = \frac{d^2x^0}{d\tau^2} = \frac{dU^0}{d\tau} = \frac{d\gamma}{d\tau} = \gamma\frac{d\gamma}{dt} \equiv \gamma\dot{\gamma} \tag{4.12a}$$

$$A^i = \frac{d^2x^i}{d\tau^2} = \frac{dU^i}{d\tau} = \gamma\frac{d}{dt}(\gamma v^i) = \gamma\left(\dot{\gamma}v^i + \gamma a^i\right) \tag{4.12b}$$

(where v^i and a^i are the ordinary velocity and acceleration $v^i = dx^i/dt$, $a^i = d^2x^i/dt^2$, and $\dot{\gamma} = d\gamma/dt$), or

$$\mathbf{A} = \gamma\left(\dot{\gamma}, \dot{\gamma}\mathbf{v} + \gamma\mathbf{a}\right). \tag{4.13}$$

There is no inertial frame in which an accelerating particle is always at rest; however at any instant, such a particle has a definite velocity, and so there is a frame – the *instantaneously co-moving* inertial frame – in which the particle is briefly at rest. In this frame, where $\mathbf{v} = \mathbf{0}$, we have $\mathbf{U} = (1, \mathbf{0})$ and $\mathbf{A} = (\dot{\gamma}, \mathbf{a}) = (0, \mathbf{a})$ (since $\dot{\gamma}$ contains a factor of $\mathbf{v}$), so that

$$\mathbf{U} \cdot \mathbf{A} = 0$$

in this co-moving frame, and therefore in all frames. From the result in this co-moving frame we can deduce the norm of the 4-acceleration

$$\mathbf{A} \cdot \mathbf{A} = -a^2.$$

defining the *proper acceleration* a as the magnitude of the acceleration in the instantaneously co-moving inertial frame. See also Example 4.6.

Finally, given two particles with velocities $\mathbf{U}$ and $\mathbf{V}$, and given that the second has velocity v with respect to the first, then in the first particle's rest frame the velocity vectors have components $\mathbf{U} = (1, \mathbf{0})$ and $\mathbf{V} = \gamma(v)(1, \mathbf{v})$. Thus

$$\mathbf{U} \cdot \mathbf{V} = \gamma(v),$$

and this inner-product is, again, frame-independent.

1.4 Velocities and tangent vectors

[In this (non-examinable) section, I expand on the mention of 'tangent vectors' in the paragraph following Eqn. (4.11). The account here is still rather compressed, and I offer it only to provide a hint of the more abstract, but more beautiful and powerful, way in which these things are handled in General Relativity.]

An alternative route to the velocity vector is to use the idea of a *tangent vector*. This is how vectors are defined in GR, and is a more fundamental approach to vectors than the one described above, which relies on differentiating the mere components of vectors.

Consider a vector $\mathbf{r} = (x, y)$ on the Euclidean plane. If the components x and y are functions of some parameter λ, then the vector function $\mathbf{r}(\lambda)$ will trace out a path $(x(\lambda), y(\lambda))$ on the plane. If we *differentiate* these components with respect to the parameter λ, then we will obtain an object which obviously tells us something about the path. For example, if we follow the path $\mathbf{r}(\theta) = (\cos\theta, \sin\theta)$, parameterised by θ instead of λ, we find it traces out a circle. Differentiating this straightforwardly, we find

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta} = (-\sin\theta, \cos\theta),$$

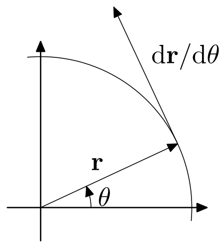

Figure 2

which is a vector which, when plotted at the position $\mathbf{r}(\theta)$, as in Fig. 2, can be clearly seen to be tangent to the path.

Now consider the spacetime vector $\mathbf{R} = (t(\lambda), x(\lambda), y(\lambda), z(\lambda))$, which draws out a path in spacetime. The path this traces out is the *worldline* – the set of events which take place along a moving particle's path through spacetime – and a reasonable parameter to use is the particle's proper time, τ – the time showing on the face of a clock attached to the moving particle. We therefore have a path $\mathbf{R} = \left(x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)\right)$ and, exactly as we did on the Euclidean plane above, can differentiate it to obtain

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}, \frac{\mathrm{d}x^1}{\mathrm{d}\tau}, \frac{\mathrm{d}x^2}{\mathrm{d}\tau}, \frac{\mathrm{d}x^3}{\mathrm{d}\tau}\right),$$

(compare Eqn. (4.6)) as a 4-vector tangent to the worldline, which clearly contains information about the 'speed' of the particle, and which we can *define* to be the velocity 4-vector. By the argument above, we discover that this vector has norm $\mathbf{U} \cdot \mathbf{U} = 1$, and that its direction corresponds precisely to the t-axis of a frame co-moving with the particle.

The point of this approach is that the idea of a *path*, and the idea of a *tangent vector* to that path, are both *geometrical* ideas, existing at a level beneath coordinates (which are more-or-less algebraic things), and so can be defined and discussed without using coordinates, and so without having any dependence on reference frames. They are therefore manifestly frame-independent.

1.5 The frequency vector, and the Doppler shift

In this section, we will examine a particular (non-obvious) 4-vector, and exploit its properties to deduce the relativistic Doppler effect (cf. [1, §4.6]).

Imagine a series of waves of some type (not necessarily light waves), moving in a direction $\mathbf{n} = (l, m, n)$ at a speed w, and imagine following a point on the crest of one of these waves; the vector $\mathbf{n}$ is a unit vector, so $\mathbf{n} \cdot \mathbf{n} = 1$. The point will have a displacement $(\Delta x, \Delta y, \Delta z)$ in time Δt, and so we will have

$$l\Delta x + m\Delta y + n\Delta z = w\Delta t,$$

for two events on the same wavecrest. Now imagine a whole train of such waves, separated by wavelength λ. Taking now $(\Delta t, \Delta x, \Delta y, \Delta z)$ to be the separation between one event on the crest of a wave and another on a wavecrest N (integer) wavecrests away (that is, a different wavecrest), we can write

$$w\Delta t - l\Delta x - m\Delta y - n\Delta z = N\lambda. \tag{4.14}$$

Defining the frequency $f = w/\lambda$, we can rewrite this as

$$\mathbf{L} \cdot \Delta\mathbf{R} = N, \tag{4.15}$$

where $\Delta\mathbf{R} = (\Delta t, \Delta x, \Delta y, \Delta z)$ and the *frequency 4-vector* is

$$\mathbf{L} = \left(f, \frac{\mathbf{n}}{\lambda}\right) = \left(f, \frac{l}{\lambda}, \frac{m}{\lambda}, \frac{n}{\lambda}\right). \tag{4.16}$$

We have written this as a vector, but what is there to say that this is *really* a vector – that is, that it is the components in one frame of an underlying geometrical object – and isn't merely four numbers in a row? We know that $\Delta\mathbf{R}$ is a vector – it is the prototype 4-vector – so we know that its components transform according to Eqn. (4.2b): $\Delta t = \gamma(\Delta t' + v\Delta x')$, $\Delta x = \gamma(\Delta x' + v\Delta t')$, $\Delta y = \Delta y'$ and $\Delta z = \Delta z'$. We can substitute this into Eqn. (4.15), rearrange to gather terms $\Delta t'$, $\Delta x'$, $\Delta y'$, $\Delta z'$ and, just as we did before Eqn. (4.15), rewrite to find

$$\mathbf{L} \cdot \Delta\mathbf{R} = \left[\gamma\left(f - v\frac{l}{\lambda}\right), \gamma\left(\frac{l}{\lambda} - vf\right), \frac{m}{\lambda}, \frac{n}{\lambda}\right] \cdot \Delta\mathbf{R}'.$$

But the vector in the square brackets is exactly the vector $\mathbf{L}'$ we would obtain if we transformed the frequency vector $\mathbf{L}$ according to the transformation matrix Eqn. (4.3). We have established, therefore, that the object $\mathbf{L}$ defined by Eqn. (4.16) really *is* a 4-vector, since it transforms in the same manner as the prototype 4-vector $\Delta\mathbf{R}$.

Is this not inevitable? Not quite: imagine if we had naïvely defined the frequency 4-vector as a vector whose space components were $\mathbf{n}/\lambda$ and whose time component was defined to be zero. On transformation by either of the routes in the previous paragraph, the vector would acquire a non-zero time component, so that the transformed vector would have a different *form* from the untransformed one. The components of such a 'vector' would *not* transform in the same way as $\Delta\mathbf{R}$, so it would not be a proper 4-vector, so that we would not be able to identify an underlying geometrical object of which these were the components.

See example 4.4

Can we use the frequency 4-vector for anything? Yes. Imagine that the wave-train is moving away from the origin in the S' plane, so that its direction is $\mathbf{n} = (\cos\theta', \sin\theta', 0)$ for some angle θ' in frame S'. In that case we have

$$\mathbf{L}' = \left[f', \frac{\cos\theta'}{\lambda'}, \frac{\sin\theta'}{\lambda'}, 0\right] \tag{4.17a}$$

$$\mathbf{L} = \left[\gamma\left(f' + v\frac{\cos\theta'}{\lambda'}\right), \gamma\left(\frac{\cos\theta'}{\lambda'} + vf'\right), \frac{\sin\theta'}{\lambda'}, 0\right] \tag{4.17b}$$

Now write

$$\mathbf{L} = \left[f, \frac{\cos\theta}{\lambda}, \frac{\sin\theta}{\lambda}, 0 \right], \tag{4.17c}$$

and compare Eqns. (4.17b) and (4.17c) component by component. After a bit of rearrangement, we find

$$f = f'\gamma \left(1 + \frac{v}{w'} \cos\theta' \right) \tag{4.18}$$

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + v w')}. \tag{4.19}$$

Equation (4.18) is the *relativistic Doppler effect*, and describes the change in frequency of a wave, as measured in a frame moving with respect to the frame in which it was emitted. This applies for everything from water waves (for which the effect would be exceedingly small) all the way up to light, for which $w = 1$. Equation (4.19) shows that a wave travelling at an angle θ' in the moving frame S' is measured to be moving at a *different* angle θ in a frame S with respect to which S' is moving with speed v. To calculate the change in the speed of the wave, we could laboriously eliminate variables from Eqns. (4.17b) and (4.17c), but much more directly, we can make use of the fact that the norms of vectors are conserved under Lorentz transformation; thus $\mathbf{L} \cdot \mathbf{L} = \mathbf{L}' \cdot \mathbf{L}'$ or, again using Eqn. (4.17) and $f = w/\lambda$,

See example 4.5
See example 4.6
See example 4.7
See example 4.8

$$f^2 \left(1 - \frac{1}{w^2} \right) = f'^2 \left(1 - \frac{1}{w'^2} \right).$$

We could rewrite this to obtain an expression for w', but simply from this form we can see that if $w = 1$, the fact that neither f nor f' is zero implies that $w' = 1$ also (as the second postulate says).

See example 4.9

Note that my expression Eqn. (4.18) for the Doppler shift is different from the expression in Carroll and Ostlie, in their equation (4.32). That is because their symbols mean different things. In my case, I am talking about a wave moving at an angle θ' to the x-axis, as measured in the moving frame S' (see immediately before Eqn. (4.17)), and Eqn. (4.18) gives the frequency of a wave, as measured in frame S, as a function of frequency f' and angle θ' as measured in S'. Inspection of C&O's equation (4.32), and their figure 4.9, shows that they are describing the observed frequency ν_{obs} (my f) in terms of the frequency in the moving frame ν_{rest} (my f') and the angle θ as measured in the *observer's* frame S (that is, θ rather than θ'); also, the figure shows that C&O's θ would be my $\theta - 180°$. This illustrates the prime importance of being very clear about exactly which frames you're talking about, and being precise about the meanings of the various symbols in an expression.

It's easy to obtain C&O's (4.32) from mine. Rewrite Eqn. (4.18) as $f' = f/\gamma(1 + (v/c)\cos\theta')$. Now swap the roles of the frames S and S', which means making the primed quantities unprimed, and vice versa, and swapping the velocity $v \to -v$. That gives us $f = f'/\gamma(1 - (v/c)\cos\theta)$, which becomes C&O's (4.32) on writing $f = \nu_{\text{obs}}$, $f' = \nu_{\text{rest}}$ and $\theta = \theta + 180°$. Incidentally, I feel this illustrates the greater power of the approach using the frequency 4-vector, which describes the same wave train from two different inertial frames, rather than the rather ad-hoc approach used in C&O, which obscures the symmetry between the 'rest' and 'observer' frames.

2 Dynamics

In the previous section, we have learned how to describe motion; we now want to explain it. In Newtonian mechanics, we do this by defining quantities such as momentum, energy, force and so on. To what extent can we do this in the context of relativity, with our new 4-vector tools?

2.1 Energy and momentum

We can start with momentum. We know that in Newtonian mechanics, momentum is defined as mass times velocity. We have a velocity, so we can try defining a momentum 4-vector as

$$\mathbf{P} = m\mathbf{U} = m\gamma(1, \mathbf{v}). \tag{4.20}$$

Since m is a scalar, and $\mathbf{U}$ is a 4-vector, $\mathbf{P}$ must be a 4-vector also.

In the rest frame of the particle, this becomes $\mathbf{P} = m(1, \mathbf{0})$: it is a 4-vector whose length ($\sqrt{\mathbf{P}\cdot\mathbf{P}}$) is m, and which points along the particle's world-line. That is, it points in the direction of the particle's movement in spacetime. Since this is a vector, its norm and its direction are frame-independent quantities, so a particle's 4-momentum vector *always points in the direction of the particle's world-line*, and the 4-momentum vector's *length is always m*. We'll call this vector the *momentum (4-)vector*, but it's also called the *energy-momentum vector*, and Taylor & Wheeler (coining the word in an excellent chapter on it) call it the *momenergy vector* in order to stress that it is *not* the same thing as the energy or momentum (or mass) that you are used to.

Note that here, and throughout, the symbol m denotes the mass as measured in a particle's rest frame. The reason I mention this is that some treatments of relativity, particularly older ones, introduce the concept of the 'relativistic mass', distinct from the 'rest mass'. The only (dubious) benefit of this is that it makes a factor of γ disappear from a few equations, making them look at little more like their Newtonian counterparts; the cost is that of introducing one more new concept to worry about, which doesn't help much in the long term, and which can obscure aspects of the energy-momentum vector. Rindler introduces the relativistic mass, Taylor & Wheeler and Schutz don't.

Now consider a pair of incoming particles $\mathbf{P}_1$ and $\mathbf{P}_2$ which collide and produce a set of outgoing particles $\mathbf{P}_3$ and $\mathbf{P}_4$ (this can be trivially extended to more than two particles; also, in relativistic collisions, the number of outgoing particles need not equal the number of incoming particles). Suppose that the total momentum is conserved:

$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3 + \mathbf{P}_4. \tag{4.21a}$$

Looking at the time and space coordinates separately, and writing $\mathbf{p} \equiv \gamma m \mathbf{v}$, we have

$$m_1\gamma(v_1) + m_2\gamma(v_2) = m_3\gamma(v_3) + m_4\gamma(v_4) \tag{4.21b}$$

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4. \tag{4.21c}$$

Now recall that, as $v \to 0$, we have $\gamma(v) \to 1$, so that the low-speed limit of the spatial part of the vector $\mathbf{P}$, Eqn. (4.20), is just $m\mathbf{v}$, so that the spatial part of the conservation equation, Eqn. (4.21c), reduces to the statement that $m\mathbf{v}$ is conserved. Both of these prompt us to identify the spatial part of the vector $\mathbf{P}$ as the linear momentum, and to justify giving the 4-vector $\mathbf{P}$ the name 4-momentum.

What, then, of the time component of Eqn. (4.20)? Let us (with, admittedly, a little fore-knowledge) write this as $P^0 = E$, so that

$$E = \gamma m. \tag{4.22}$$

What is the low-v limit of this? Taylor's theorem tells us that

$$\gamma = \left(1 - v^2\right)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}v^2 + O(v^4),$$

so that Eqn. (4.22) becomes

$$E = m + \tfrac{1}{2}mv^2 + O(v^4). \tag{4.23}$$

Now $\frac{1}{2}mv^2$ is the expression for the kinetic energy in Newtonian mechanics, and Eqn. (4.21b), compared with Eqn. (4.22), tells us that this quantity E is conserved in collisions, so we have persuasive support for identifying the quantity E in Eqn. (4.22) as the relativistic *energy* of a particle with mass m and velocity v.

If we rewrite Eqn. (4.22) in physical units, we find

$$E = \gamma m c^2, \tag{4.24}$$

the low-speed limit of which (remember $\gamma(0) = 1$) recovers what has been called the most famous equation of the twentieth century.

The argument presented here after Eqn. (4.21a) has been concerned with giving names to quantities, and, reassuringly for us, linking those newly-named things with quantities we already know about from Newtonian mechanics. This may seem arbitrary, and it is certainly not any sort of proof that the 4-momentum is conserved as Eqn. (4.21a) says it might be. No proof is necessary, however: it turns out *from experiment* that Eqn. (4.21a) is a law of nature, so that we could simply include it as a postulate of relativistic dynamics and proceed to use it without bothering to identify its components with anything we are familiar with.

In case you are worried that we are pulling some sort of fast one, that we never had to do in Newtonian mechanics, note that we *do* have to do a similar thing in Newtonian mechanics. There, we postulate Newton's third law (action equals reaction), and from this we can deduce the conservation of momentum; here, we postulate the conservation of 4-momentum, and this would allow us to deduce a relativistic analogy of Newton's third law (I don't discuss relativistic force here, but it is easy to define). The postulational burden is the same in both cases.

We can see from Eqn. (4.23) that, even when a particle is stationary and $v = 0$, the energy E is non-zero. In other words, a particle of mass m has an energy γm associated with it simply by virtue of its mass. The low-speed limit of Eqn. (4.21b) simply expresses the conservation of mass, but we see from Eqn. (4.22) that it is actually expressing the conservation of energy. In SR there is no real distinction between mass and energy – mass is, like kinetic, thermal and strain energy, merely another form into which energy can be transmuted – albeit a particularly dense store of energy, as can be seen by calculating the energy equivalent, in Joules, of a mass of 1kg. It turns out from GR that it is not mass that gravitates, but energy (most typically, however, in the particularly dense form of mass), so that thermal and electromagnetic energy, for example, and even the energy in the gravitational field itself, all gravitate (it is the non-linearity implicit in the last remark that is part of the explanation for the mathematical difficulty of GR).

See example 4.10

Although the quantities $\mathbf{p} = \gamma m \mathbf{v}$ and E are frame-dependent, and thus not physically meaningful by themselves, the quantity $\mathbf{P}$ defined by Eqn. (4.20) has a physical significance.

Let us now consider the norm of the 4-momentum vector. Like any such norm, it will be frame-invariant, and so will express something fundamental about the vector, analogous to its length. Since this is the momentum vector we are talking about, this norm will be some important invariant of the motion, indicating something like the 'quantity of motion'. From the definition of the momentum, Eqn. (4.20), and its norm, Eqn. (4.11), we have

$$\mathbf{P} \cdot \mathbf{P} = m^2 \mathbf{U} \cdot \mathbf{U} = m^2, \tag{4.25}$$

and we find that this important invariant is the *mass* of the moving particle.

Now using the definition of energy, Eqn. (4.22), we can write $\mathbf{P} = (E, \mathbf{p})$, and find

$$\mathbf{P} \cdot \mathbf{P} = E^2 - \mathbf{p} \cdot \mathbf{p}. \tag{4.26}$$

Writing now $p^2 = \mathbf{p} \cdot \mathbf{p}$, we can combine these to find

$$E^2 = p^2 + m^2. \tag{4.27}$$

Equation (4.27) appears to be simply a handy way of relating E, p and m, but that conceals the more fundamental meaning of Eqn. (4.25). The 4-momentum $\mathbf{P}$ encapsulates the important features of the motion in the energy and spatial momentum. Though they are frame-dependent separately, they combine into a frame-independent quantity.

It seems odd to think of a particle's momentum as being always non-zero, irrespective of how rapidly it's moving; this is the same oddness that has the particle's velocity always being of length 1. One way of thinking about this is that it shows that the 4-momemtum vector (or energy-momentum or momenergy vector) is a 'better' thing than the ordinary 3-momentum: it's frame-independent, and so has a better claim to being something intrinsic to the particle. Another way of thinking about it is to conceive of the 4-velocity as showing the particle's movement through spacetime. In a frame in which the particle is 'at rest', the particle is in fact moving at speed c into the future. If you are looking at this particle from another frame, you'll see the particle move in space (it has non-zero space components to its 4-velocity in your frame), and it will consequently (in order that $\mathbf{U} \cdot \mathbf{U} = 1$) have a *larger* time component in your frame than it has in its rest frame. In other words, the particle moves through more time in your frame than it does in its rest frame – another manifestation of time dilation. Multiply this velocity by the particle's mass, and you can imagine the particle moving into the future with a certain momentum in its rest frame; observe this particle from a moving frame and its spatial momentum becomes non-zero, and the time component of its momentum (its energy) has to become bigger – the particle packs more punch – as a consequence of the length of the momentum vector being invariant.

2.2 Photons

For a photon, the interval represented by $\mathrm{d}\mathbf{R} \cdot \mathrm{d}\mathbf{R}$ is always zero ($\mathrm{d}\mathbf{R} \cdot \mathrm{d}\mathbf{R} = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = 0$ for photons. But this means that the proper time $\mathrm{d}\tau^2$ is also zero for photons. This means, in turn, that we cannot define a 4-velocity vector for a photon by the same route that led us to Eqn. (4.6), and therefore cannot define a 4-momentum as in Eqn. (4.20).

We can do so, however, by a different route. Recall that we defined (in the paragraph below Eqn. (4.11)) the 4-velocity as a vector pointing along the worldline, which resulted in the 4-momentum being in the same direction. From the discussion of the momentum of massive particles above, we see that the P^0 component is related to the energy, so we can use this to define a 4-momentum for a *massless* particle, and again write

$$\mathbf{P}_\gamma = (E, \mathbf{p}_\gamma).$$

Since the photon's velocity 4-vector is null, the photon's 4-momentum must be also (since it is defined above to be pointing in the same direction). Thus we must have $\mathbf{P}_\gamma \cdot \mathbf{P}_\gamma = 0$, thus $\mathbf{p}_\gamma \cdot \mathbf{p}_\gamma = E^2$, recovering the $m = 0$ version of Eqn. (4.27),

$$E^2 = p^2 \qquad \text{(massless particle)}, \tag{4.28}$$

so that even massless particles have a non-zero momentum.

In quantum mechanics, we learn that the energy associated with a quantum of light – a photon – is $E = hf$, where h is Planck's constant, $h = 6.626 \times 10^{-34}$ Js (or 2.199×10^{-42} kg m in natural units), so that

$$\mathbf{P} = (hf, hf, 0, 0) \qquad \text{(photon)}. \tag{4.29}$$

2.3 Relativistic collisions and the centre-of-momentum frame

Consider two particles, of momenta $\mathbf{P}_1$ and $\mathbf{P}_2$, which collide and produce a single particle of momentum $\mathbf{P}_3$ – you can think of this as describing either as two balls of relativistic putty, or an elementary particle collision which produces a single new outgoing particle (this example is adapted from [3, §8.3]); note also that the subscripts denote the momenta of different particles, but the superscripts denote the different momentum components.

The particles have momenta

$$\mathbf{P}_i = \gamma_i m_i (1, \mathbf{v}), \tag{4.30}$$

where the three particles have velocities $\mathbf{v}$, and $\gamma_i \equiv \gamma(v_i)$. From momentum conservation, we also know that

$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3. \tag{4.31}$$

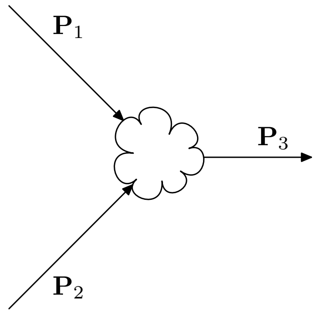

Figure 3

Of course, this also indicates that *each component* of the vectors is separately conserved.

Let's simplify this, and imagine the collision taking place in one dimension, with an incoming particle moving along the x-axis to strike a stationary second particle. The resulting particle will also move along the x-axis. In this subsection we will, to avoid clutter, write only the x-component of the spatial part of vectors, missing out the y and z components, which are zero in this one-dimensional setup. Thus we'll write (t, x) rather than (t, x, y, z) or $(t, \mathbf{r})$, and (P^0, P^1) rather than (P^0, P^1, P^2, P^3).

Let the 'incoming' particles have masses $m_1 = m_2 = 8$ units, the first be travelling with speed $v_1 = 15/17$ along the x-axis in the 'lab frame' (so that $\gamma_1 = \gamma(15/17) = 17/8$), and the second be stationary, $v_2 = 0$ (so $\gamma_2 = 1$). As discussed above, the appropriate measure of the 'length' of the $\mathbf{P}$ vectors is the quantity m of Eqn. (4.25): $m = \sqrt{\mathbf{P} \cdot \mathbf{P}}$. Recall that $\mathbf{P}_i = (E_i, p_i) = (\gamma_i m_i, \gamma_i m_i v_i)$. Thus we can make a table of the various kinematical parameters:

particle	$\mathbf{P}_i = (E_i, p_i)$	v_i	m_i
1	$(17, 15)$	$15/17$	$\sqrt{17^2 - 15^2} = 8$
2	$(8, 0)$	0	8
3	$(17 + 8, 15 + 0) = (25, 15)$	$3/5$	$\sqrt{25^2 - 15^2} = 20$

Does this hang together? The speed of a particle is $v = (P^1/P^0)$ (cf Eqn. (4.30)), giving $v_3 = P_3^1/P_3^0 = 15/25 = 3/5$ and therefore $\gamma_3 = 5/4$. Thus $m_3 = E_3 = P_3^0 = \gamma_3 m_3 = 25$, and so $m_3 = 20$, as above.

Notice the following points.

(i) The two incoming particles have the same energy-momentum, $m_1 = m_2 = 8$. This may seem surprising, since one is moving much faster than the other, but recall that the inner-product $\mathbf{P}_1 \cdot \mathbf{P}_1$ is an *invariant*, so it has the same value in the frame in which particle 1 is moving at speed $v_1 = 15/17$ as it has in the frame in which particle 1 is stationary; in the latter frame, $\mathbf{P}'_1 = (\gamma(0) m_1, v_1) = (8, 0)$, showing the same values as $\mathbf{P}_2$ in the lab frame.

(ii) Notice that the mass in the 'outgoing' particle 3 is *not* merely the sum of the two incoming masses: it is the components of $\mathbf{P}$, including total energy $P_1^0 + P_2^0 = P_3^0 = \gamma m_3$ that are the quantities conserved in collisions, not mass (in the sense of the norm of $\mathbf{P}$).

(iii) The quantity $m^2 = \mathbf{P} \cdot \mathbf{P}$ is frame-invariant, but not conserved; P^μ is conserved, but not frame-invariant. Taylor & Wheeler [3, Box 7-3] usefully contrast the terms 'conserved', invariant' and 'constant':

Conserved refers to a quantity which is not changed (over time?) by some process – momentum, for example, is conserved in a collision. This refers to one frame at a time, and a conserved quantity will typically have different numerical values in different frames.

Invariant refers to a quantity which is not changed by some transformation – the radius r is an invariant of the rotation in Fig. 3-5, and we have just discovered that m^2 is an invariant of the Lorentz transformation. The term necessarily refers to more than one reference frame; there is no reason to expect that an invariant quantity will be conserved in a particular process.

Constant refers to a quantity which does not change in time, such as the mass of the University.

The speed of light is one of the very few things which has the full-house of being conserved, invariant, *and* constant.

From Eqn. (4.30) and Eqn. (4.31) we can write

$$\mathbf{P}_3 = (\gamma_1 m_1 + \gamma_2 m_2, \gamma_1 m_1 v_1 + \gamma_2 m_2 v_2).$$

Now use the LT, Eqn. (4.3), to transform this to a frame which is moving with speed V with respect to the lab frame. We therefore have

$$P_3'^1 = \gamma(V)(P_3^1 - VP_3^0) = \gamma(V)(\gamma_1 m_1 v_1 + \gamma_2 m_2 v_2 - V(\gamma_1 m_1 + \gamma_2 m_2)).$$

We can choose the speed V to be such that this spatial momentum is zero, $P_3'^1 = 0$, giving

$$V = \frac{\gamma_1 v_1 + \gamma_2 v_2}{\gamma_1 + \gamma_2} = \frac{3}{5}$$

(since here $m_1 = m_2$). In this frame we have (transforming the other momentum 4-vectors, and using $\gamma(V = 3/5) = 5/4$ and, for example $P_i'^0 = \gamma(P_i^0 - VP_i^1)$):

particle	$\mathbf{P}_i' = (E_i', p_i')$	v_i	m_i
1	$(10, 6)$	$3/5$	$\sqrt{10^2 - 6^2} = 8$
2	$(10, -6)$	$-3/5$	8
3	$(20, 0)$	0	20

Notice: (i) In this frame also, $\mathbf{P}_1' + \mathbf{P}_2' = \mathbf{P}_3'$: the conservation of a vector quantity means that it is conserved in all frames. (ii) Also $m_1' = (\mathbf{P}_1' \cdot \mathbf{P}_1')^{1/2} = 8$, $m_2' = 8$ and $m_3' = 20$: these are frame-invariant quantities. (iii) Both the momenta *and* energy of the vectors in this frame are different from the lab frame, even though their norms are the same.

This frame, in which the total spatial momentum is zero, so that the incoming particles have equal and opposite spatial momenta, is known as the *centre of momentum* (CM) frame, and the energy available for particle production in this frame (P^0_{CM}) is known as the *centre of mass energy*.

2.4 But where's this mass coming from?

What's going on here? How much mass is actually present here? Put in physical terms, is this telling us that there would be more *gravity* after than collision than before? Excellent questions.

Imagine putting a box round the collision in Fig. 3, and asking the only apparently vague question "How much dynamics is there in this box?' – how much 'oomph' is here?" Saying "there are two particles of mass m" doesn't seem an adequate answer, because it doesn't account for the fact that the particles are moving towards each other. Different observers will disagree about which particle is moving, and how quickly, but there can be no doubt that the particles are going to hit each other, quite hard. As well as the two particles, there's a lot of kinetic energy in this box.

If we imagine that these balls of relativistic putty are each a mere eight grammes in mass, and reverting to physical units for the moment, the incoming particle in the first case, above, has an energy of $E = \gamma mc^2 = 17/8 \times (8 \times 10^{-3}\,\mathrm{kg}) \times (3 \times 10^8\,\mathrm{m\,s^{-1}})^2 =$

1.53 PJ (about a third of a megaton of TNT), only 720 TJ of which is accounted for by the energy equivalent of the putty's 8 g of mass. That's a lot of energy, and it may not be surprising that that energy of motion makes as important a 'dent' in spacetime as the energy represented by the particles' masses.

Thus, as we have seen above, it is not mass that is conserved in collisions, but 4-momentum (or energy-momentum). In General Relativity, it is not mass that is the source of gravitation, but 4-momentum.

Taylor & Wheeler's chapter 7 is a good discussion of the puzzles here.

2.5 An example: Compton scattering

As a further example of a relativistic collision, we can examine the collision between a photon – a quantum of light energy – and an electron. This is *Compton scattering*, and is not the same as the classical Thomson scattering of light by electrons: that is a purely electromagnetic effect whereas this is an inherently relativistic and quantum-mechanical effect, where we are treating both the electron and the incoming light as relativistic particles. See also French [4, pp.194–6].

The collision is as shown in Fig. 4. An incoming photon strikes a stationary electron and both recoil. The incoming photon has energy $Q_1 = hf_1 = h/\lambda_1$ and the outgoing one $Q_2 = hf_2 = h/\lambda_2$; the outgoing electron has energy E, spatial momentum $\mathbf{p}$, and mass m. The four momentum 4-vectors are therefore

$$\begin{aligned} \mathbf{P}_{1e} &= (m, 0, 0, 0) & \mathbf{P}_{2e} &= (E, p\cos\theta, p\sin\theta, 0) \\ \mathbf{P}_{1\gamma} &= (Q_1, Q_1, 0, 0) & \mathbf{P}_{2\gamma} &= (Q_2, Q_2\cos\phi, Q_2\sin\phi, 0)\,, \end{aligned}$$

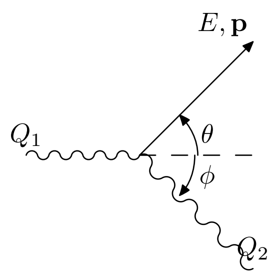

Figure 4

where subscripts 1 and 2 denote momenta before and after the collision, respectively.

Momentum conservation implies $\mathbf{P}_{1e}+\mathbf{P}_{1\gamma} = \mathbf{P}_{2e}+\mathbf{P}_{2\gamma}$. Comparing components, we find

$$Q_1 + m = Q_2 + E \tag{4.32a}$$
$$Q_1 = p\cos\theta + Q_2\cos\phi \tag{4.32b}$$
$$0 = p\sin\theta + Q_2\sin\phi. \tag{4.32c}$$

Squaring and adding Eqn. (4.32b) and Eqn. (4.32c), we have

$$(Q_1 - Q_2\cos\phi)^2 + (-Q_2\sin\phi)^2 = p^2. \tag{4.33}$$

Using the relation $E^2 = p^2 + m^2$ in this equation, then substituting for E from Eqn. (4.32a), and rearranging, we find

$$Q_1 Q_2(1-\cos\phi) = (Q_1 - Q_2)m \tag{4.34}$$

which, on dividing through by $Q_1 Q_2 m$, gives

$$\frac{1}{Q_2} - \frac{1}{Q_1} = \frac{1-\cos\phi}{m} \tag{4.35}$$

or, in terms of wavelength,

$$\lambda_2 - \lambda_1 = \frac{h}{m}(1-\cos\phi). \tag{4.36}$$

We can see that the scattered photon will have a longer wavelength (smaller energy) than the incident one. Note also that this is not any type of Doppler effect: the photon has a different energy not because we are observing it in any sort of moving frame, but has a lower energy because it has had to give some up to the electron.

3 Conclusion

That is all we have time for. Over the past few weeks, we have examined the axioms on which SR is based, derived the immediate consequences of these axioms in the form of the Lorentz Transformation equations, and finally examined how our understanding of dynamics has to be transformed within the context of relativity.

Has it been worth it? While you will only *very* rarely have to calculate the speed of clocks on passing spaceships, and rather less often on passing trains, the physical world picture you have learned in these lectures is what underlies areas of physics as diverse as relativistic quantum mechanics, particle physics and general relativity, which are simply unintelligible otherwise. Even if you do not go on to study those areas, you have come to have a more fundamental view of the universe than you had from your intuitive understanding of Newtonian physics, and gained access to one of the 20th century's most radical intellectual adventures.

Enjoy!

Questions

In Sect. 1.2, the vectors $A = (1, 2)$ and $B = (2, 1)$ are orthogonal, but if you draw them on the Minkowski diagram, they're clearly not at right-angles. Why? The term 'orthogonal' means nothing more nor less than $\mathbf{A} \cdot \mathbf{B} = 0$. In the special case of Euclidean space (that is, the space we're used to), we give this the traditional name of 'right-angled'. That's all that 'right-angled' is: the traditional name for 'orthogonal', when used in Euclidean space. Another way of thinking about it is that, just as maps of the Earth are distorted when they're put on a flat sheet, Minkowski space naturally gets distorted when it's forced onto the flat Euclidean page.

References

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[2] Bernard F Schutz. *A First Course in General Relativity*. Cambridge University Press, second edition, 2009.

[3] Edwin F Taylor and John Archibald Wheeler. *Spacetime Physics*. W H Freeman, 2nd edition, 1992.

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The notation a-b refers to part a page b.

Examples

Example 4.1 (section 1.2)

Calculate the matrix inverse of Eqn. (4.3), giving the transformation $\mathbf{A}' \mapsto \mathbf{A}$. Then obtain the inverse of Eqn. (4.3) by swapping primed with unprimed quantities and changing the sign of v. Verify that the two ways of obtaining the inverse match.

Example 4.2 (section 1.2)

Consider the vectors $\mathbf{A} = (5, 3, 4, 0)$, $\mathbf{B} = (1, 1, -1, -1)$ and $\mathbf{C} = (3, 2, -1, 0)$. Calculate $\mathbf{A}\cdot\mathbf{B}$, $\mathbf{A}\cdot\mathbf{C}$ and $\mathbf{B}\cdot\mathbf{C}$: which pair is orthogonal? Calculate also the norms of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, and indicate which vectors are null, timelike and spacelike. (Objective 1)

Example 4.3 (section 1.2)

Consider, instead of Eq. (4.2a), the transformation

$$\begin{pmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \tag{i}$$

which is the simplest transformation which ‘mixes’ the x- and t-coordinates. By *requiring* that $\Delta s'^2 = \Delta s^2 = \Delta t^2 - \Delta x^2$ after this transformation, find constraints on the parameters a, b, c, d (you can freely add the constraint $b = c$; why?), and by setting b proportional to a deduce the matrix Eq. (4.2a).

Example 4.4 (section 1.5)

One might naïvely consider defining a 4-vector representing the electric field as $(0, E_x, E_y, E_z)$, where the E_i are the components of the electric field, and the time component is defined to be identically zero. Explain briefly why this *cannot* be a 4-vector [adapted from degree exam, 2001].

Example 4.5 (section 1.5)

Obtain the inverse to these equations by applying Eqn. (4.3) to Eqn. (4.17c) and comparing with Eqn. (4.17). Verify that the resulting expression matches that which you obtain through the procedure described in Sect. 1.2: swapping primed with unprimed quantities and changing the sign of v.

Example 4.6 (section 1.5)

A rocket starts at rest at a space station, and accelerates along the x-axis at a constant rate α in its own frame S'. The rocket’s (proper) acceleration 4-vector is therefore $A' = (0, \alpha, 0, 0)$.

(a) Using the (inverse) Lorentz transformation equations, to transform from an inertial frame instantaneously co-moving with the rocket into the spacestation frame, calculate the components of the rocket’s acceleration vector in the spacestation’s frame, and confirm that $\mathbf{A} \cdot \mathbf{A} = -\alpha^2$ in this frame also. Write down the rocket’s velocity 4-vector in this frame, and confirm that $\mathbf{U} \cdot \mathbf{A} = 0$. Differentiate this with respect to the proper time, to obtain

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\tau} = \gamma \left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}, \frac{\mathrm{d}}{\mathrm{d}t}(\gamma v), 0, 0 \right).$$

Hence deduce that $\gamma v = \alpha t$, and thus that

$$\frac{1}{v^2} = \frac{1}{\alpha^2 t^2} + 1. \tag{i}$$

(b) If, at time t, the rocket sets off a flashbulb which has frequency f' in its frame, use the Doppler formula to show that the light is observed, at the spacestation, to have frequency

$$f = f'(\sqrt{1 + \alpha^2 t^2} - \alpha t).$$

What is this factor if the flashbulb is set off at time $t = 3/(4\alpha)$?

(c) Rearrange Eqn. (i) to produce an expression for v, write $v = \mathrm{d}x/\mathrm{d}t$, and integrate to obtain the constant-acceleration equation for SR,

$$(x + 1/\alpha)^2 - t^2 = 1/\alpha^2$$

(you may or may not find the substitution $\alpha t = \sinh\theta$ useful or obvious here). This is the equation for a hyperbola. By sketching this on a Minkowski diagram and considering the point at which the asymptote intersects the t-axis, demonstrate that it is impossible for the space station to signal to the retreating rocket after a time $1/\alpha$.

Example 4.7 (section 1.5)

Consider an ambulance passing you with its siren sounding. At the instant the ambulance is level with you, its siren has its 'rest' frequency. If the ambulance were moving at a relativistic speed, what would be the measured frequency of the ambulance's blue light, as a function of its frequency in its rest frame, f'? If we put $\alpha' = \pi/2$ into Eq. (4.18) (examining the light which is emitted perpendicular to the ambulance's motion), then we obtain $f = \gamma f'$. But this suggests that the frequency in our frame is measured to be *higher* than the frequency in the rest frame, and appears to directly contradict the calculations in Example 3-**??**. What is wrong with this argument?

Example 4.8 (section 1.5)

The traditional traffic-lights example: You are driving towards some traffic lights showing red (take wavelength $\lambda = 600\,\mathrm{nm}$). How fast do you have to be driving so that they are Doppler-shifted enough that they appear green (wavelength $\lambda = 500\,\mathrm{nm}$)? [Hint: rewrite Eqn. (4.18) in terms of $\rho = f'/f$, set $w = c$ and $\theta = 0$, and rearrange to make v/c the subject.] Does this sound like a reasonable defence when you're had up in court on a charge of dangerous driving? (Objective 4)

Example 4.9 (section 1.5)

By squaring Eqn. (4.19) and setting $w = 1$ (and using trigonometric identities to reexpress $\sin^2\theta'$), show that

$$\cos\theta' = \frac{\cos\theta - v}{v\cos\theta}. \tag{i}$$

A source in the moving frame S emits light isotropically. Deduce that light emitted into the forward half-sphere is observed, in the moving frame, to be emitted into a cone with half-angle θ'_+, where $\cos\theta'_+ = v$. This is known as the *headlight effect.* (Objective 4)

Example 4.10 (section 2.1)

The luminosity of the sun is $L_\odot = 3.86 \times 10^{26}\,\mathrm{W}$; this is powered by nuclear fusion. How much mass does it consume, in fusion reactions, per second? (Objective 5)