

Astronomy 2 – Special Relativity – Part 4

Vectors, kinematics and dynamics

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In the last part, we used the axioms of part 2 to obtain the Lorentz Transformation. That allowed us to describe events in two different frames in relative motion. That part was rather mathematical in style. Now we are going to return to the physics, and describe *motion*: velocity, acceleration, momentum, energy and mass.

1 Kinematics

- 1.1 Three-vectors
- 1.2 Four-vectors
- 1.3 Velocity and acceleration
- 1.4 Velocities and tangent vectors
- 1.5 The frequency vector, and the Doppler shift

2 Dynamics

- 2.1 Energy and momentum
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Aims and objectives for part 4

Aims: You should

1. understand the concept of a 4-vector as a geometrical object, and the distinction between a vector and its components.
2. understand relativistic energy and momentum; the concept of energy-momentum as the norm of the momentum 4-vector; and conservation of the momentum 4-vector.
3. understand the distinction between invariant, conserved and constant quantities.

Objectives: You should be able to

1. quote the expression for the inner-product between two 4-vectors, quote the definition of the norm, and calculate the sum, norm and inner-product of 4-vectors.

2. use the Lorentz Transformation to transform the components of a vector from one frame to another.
3. categorise 4-vectors as timelike, spacelike, or null.
4. use the velocity, acceleration and frequency 4-vectors, and the Doppler shift formula.
5. use the conservation of 4-momentum, the definition of relativistic energy, Eqn. (4.21), and the relationship between energy and momentum, Eqn. (4.26), to calculate the behaviour of particles in collisions.

1 Kinematics

Part 3 was concerned with static events as observed from moving frames. In this part, we are concerned with particle motion.

Before we can explain motion, we must first be able to describe it. This is the subject of *kinematics*. We will first have to define the vectors of 4-dimensional Minkowski space, and specifically the velocity and acceleration vectors.



The description of 4-vectors in this section draws on the discussion of 4-vectors in chapter 4 of Rindler [1], and the description of dynamics in the following section draws on chapter 5 of the same book; note, however, that Rindler’s notational conventions differ from mine in several important respects. Rindler defines the displacement 4-vector as $(x_1, x_2, x_3, x_4) = (\Delta x, \Delta y, \Delta z, \Delta t)$ so that the metric is (implicitly) $\text{diag}(-1, -1, -1, c^2)$. Also, Rindler commits the now-old-fashioned heresy of defining a ‘relativistic mass’ $m(v) = \gamma(v)m_0$, which doesn’t really help things.



The dynamics section below also draws on Chapter 2 of Schutz [2]. This is a textbook on General Relativity (the recommended text for the honours GR course, in fact), but chapter 2 describes Special Relativity using the beautiful and powerful geometrical language which is used extensively in GR. I have toned this down for this section, but the motivation is the same.

1.1 Three-vectors

You are familiar with 3-vectors – the vectors of ordinary three-dimensional Euclidean space. To an extent, three-vectors are merely an ordered triple of numbers, but they are interesting to us as physicists because they represent a more fundamental geometrical object: the three numbers are not just picked at random, but are the vector’s *components* – the projections of the vector onto three orthogonal axes (that the axes are orthogonal is not essential to the definition of a vector, but it is almost invariably true in physics, as it makes things so much simpler in all but extreme cases). That is, the components of a vector are functions of both the vector and our choice of axes, and if we change the axes, then the components will change in a systematic way.

For example, consider a prototype displacement vector $(\Delta x, \Delta y, \Delta z)$. These are the components of a vector with respect to the usual axes $\mathbf{e}_x, \mathbf{e}_y$ and \mathbf{e}_z . If we rotate these axes, say by an angle θ about the z -axis, to obtain axes $\mathbf{e}'_x, \mathbf{e}'_y$ and \mathbf{e}'_z , we obtain a new set of coordinates $(\Delta x', \Delta y', \Delta z')$, related to the original coordinates by

$$\begin{pmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}. \tag{4.1}$$

These new components describe the *same* underlying vector, as shown in Fig. 1. Although any random triple of numbers $(\Delta x, \Delta y, \Delta z)$ describes *some* vector, to some extent what turns the number triple into A Vector is the existence of this underlying object, which implies all the other sets of coordinates $(\Delta x', \Delta y', \Delta z')$, that have a particular functional relation to the original $(\Delta x, \Delta y, \Delta z)$.

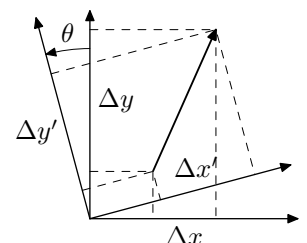


Figure 1

Swift review of linear algebra: If $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ are vectors, then $\mathbf{a} + \mathbf{b}$ is also a vector, with components $(a_x + b_x, a_y + b_y, a_z + b_z)$. We can define a *scalar-product* $\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$. We can define the *length*, $|\mathbf{a}|$ of a three vector in terms of the scalar product of a vector with itself: $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = a^2 = a_x^2 + a_y^2 + a_z^2$ (we can also see this from Pythagoras' theorem, or indeed from Eqn. (4.1)), and we know that this is an invariant of a rotation – that is, that it takes the same value irrespective of the coordinate system. It is easy to see that $|\mathbf{a} + \mathbf{b}|^2 = a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b}$ and so, since both a^2 and b^2 are frame-independent, the scalar-product $\mathbf{a} \cdot \mathbf{b}$ must be frame-independent also, even though the individual coordinates a_i and b_i are not. Finally, if the scalar-product of two vectors vanishes, $\mathbf{a} \cdot \mathbf{b} = 0$, we say that the two vectors are *orthogonal*. If a vector is orthogonal to itself ($\mathbf{a} \cdot \mathbf{a} = 0$) then we can deduce that $a_i = 0$. In linear algebra, the scalar-product is more generally termed the *inner-product*, and the length of a vector is termed its *norm*.

1.2 Four-vectors

As we saw in Sect. 3-2.3, we can regard the events of SR taking place in a 4-dimensional space termed spacetime. Here, the prototype displacement 4-vector is $(\Delta t, \Delta x, \Delta y, \Delta z)$, relative to the space axes and wristwatch of a specific observer, and the transformation which takes one 4-vector into another is the familiar LT of Eqn. (3.18), or

$$\begin{pmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \quad (4.2a)$$

for the 'forward transformation' and

$$\begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} \gamma & +\gamma v & 0 & 0 \\ +\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} \quad (4.2b)$$

for the inverse transformation (that the matrices are inverses of each other can be verified by direct multiplication). These give the coordinates of the same displacement as viewed by a second observer whose frame is in standard configuration with respect to the first.

To emphasise the equal standing of the four components of the 4-vector, and to make notation more straightforward, it is usual to write the components of the position vector as (x^0, x^1, x^2, x^3) , where $x^0 = t$, $x^1 = x$, $x^2 = y$ and $x^3 = z$ (do note that the superscripts are indexes, not powers), or collectively x^μ , where the greek index μ runs from 0 to 3. We will also occasionally use latin superscripts like i or j : these should be taken to run from 1 to 3.

This displacement 4-vector $\Delta \mathbf{R} = (\Delta t, \Delta x, \Delta y, \Delta z)$ we can take as the prototype 4-vector, and recognise as a 4-vector anything which transforms in the same way under the coordinate transformation of Eqn. (4.2a) (this may seem a rather abstract way of defining vectors, but we will see a concrete example in Sect. 1.5). That is, the coordinate transformation of an arbitrary vector \mathbf{A} into a vector \mathbf{A}' in S' is

$$\begin{pmatrix} A^{0'} \\ A^{1'} \\ A^{2'} \\ A^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}. \quad (4.3)$$

Since this is a matrix equation, the inverse transformation is straightforward: it is just the matrix inverse of this.

Note also that the primed and unprimed frames are perfectly symmetrical. The observers at rest in the unprimed frame S regard the primed frame S' as moving at speed v with respect to them. However, the observers in the primed frame see the unprimed one moving at speed $-v$; they would also naturally swap the assignment of primed and unprimed frame, with their frame, say P , being the unprimed one and the other, P' , being the primed one. Now, these two sets of observers are describing the *same* events and frames, but with opposite notation, and so with an opposite sign for v . We can translate between the two sets of notations by simultaneously swapping primed and unprimed quantities and swapping the sign of v . But all this is, is a way of translating between a transformation and its inverse. This is exactly what happens between, for example, Eqn. (3.18) and Eqn. (3.20), or between Eqn. (4.2a) and Eqn. (4.2b).

See example 1

I've talked repeatedly of events being frame-independent. Vectors are *also* frame-independent, *even though their components are not* (this is what distinguishes a vector from being just a collection of four numbers).

More linear algebra: Just as with 3-vectors, if \mathbf{A} and \mathbf{B} are 4-vectors, so is $\mathbf{A} + \mathbf{B}$, with components $(A^0 + B^0, A^1 + B^1, A^2 + B^2, A^3 + B^3)$. Where the scalar-product of ordinary three vectors is straightforward, the different geometry of spacetime means that the useful inner-product for 4-vectors is defined as

$$\mathbf{A} \cdot \mathbf{B} = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 \tag{4.4a}$$

$$= \sum_{\mu, \nu} \eta_{\mu\nu} A^\mu B^\nu, \tag{4.4b}$$

where the matrix $\eta_{\mu\nu}$ is defined as

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1). \tag{4.5}$$

The inner-product of a vector with itself, $\mathbf{A} \cdot \mathbf{A}$, is its *norm* (or length-squared), and from this definition we can see that the norm of the displacement vector is $\Delta\mathbf{R} \cdot \Delta\mathbf{R} = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = \Delta s^2$.

Just as we discussed in Sect. 3-2.6 for intervals, 4-vectors can be timelike, spacelike or null, depending on whether their norm is positive, negative or zero; note that, since the norm is not positive-definite (ie, it can be negative), even a non-zero vector can be null. Just as in Sect. 1.1, we say that two vectors are *orthogonal* if their inner-product vanishes; it follows that in this geometry, a null vector (with $\mathbf{A} \cdot \mathbf{A} = 0$) is orthogonal to itself!

See example 2

The norm of a 4-vector is frame-invariant. The norm of $\mathbf{A} + \mathbf{B}$ is $|\mathbf{A} + \mathbf{B}| = |\mathbf{A}|^2 + |\mathbf{B}|^2 + 2\mathbf{A} \cdot \mathbf{B}$, and since $|\mathbf{A} + \mathbf{B}|$, A^2 , and B^2 are frame-invariant, so must be the inner-product $\mathbf{A} \cdot \mathbf{B}$.



Mathematically, we could pick *any* set of numbers for the components of $\eta_{\mu\nu}$. However, only the definition in Eqn. (4.5) provides an inner-product which is physically meaningful.



The set of numbers $\eta_{\mu\nu}$ are not just any old matrix, but in particular the components of a *tensor* – a mathematical structure which is a generalisation of a vector, and which is vital for the mathematical study of General Relativity. It is not even just any old tensor: it is the form taken in a non-accelerating frame (ie, in SR) by the *metric tensor* which is a crucial object in General Relativity. It is the metric tensor which creates the notion of the *distance between* two events in spacetime.

1.3 Velocity and acceleration

Since the displacement 4-vector Δx^μ is a vector (in the sense that it transforms properly according to Eqn. (4.2a)), so is the infinitesimal displacement dx^μ ; since the proper time τ (see Sect. 3-3.4) is a Lorentz scalar, we can divide each component of this

infinitesimal displacement by the proper time and still have a vector. This latter vector is the *4-velocity*:

$$\mathbf{U} = \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau} \right). \quad (4.6)$$

By the same argument, the 4-acceleration

$$\mathbf{A} = \left(\frac{d^2x^0}{d\tau^2}, \frac{d^2x^1}{d\tau^2}, \frac{d^2x^2}{d\tau^2}, \frac{d^2x^3}{d\tau^2} \right) \quad (4.7)$$

is a 4-vector, also. We can more naturally write these as $U^\mu = \frac{dx^\mu}{d\tau}$ and $A^\mu = \frac{dU^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$.

Let us examine these components in more detail. We have

$$d\tau^2 = dt^2 - |d\mathbf{r}|^2,$$

so that

$$\left(\frac{d\tau}{dt} \right)^2 = \frac{(d\tau)^2}{(dt)^2} = 1 - \frac{|d\mathbf{r}|^2}{(dt)^2} = 1 - v^2 = \frac{1}{\gamma^2}.$$

Thus $dt/d\tau = \gamma$, and

$$U^0 = \frac{dx^0}{d\tau} = \frac{dt}{d\tau} = \gamma \quad (4.8a)$$

$$U^i = \frac{dx^i}{d\tau} = \frac{dt}{d\tau} \frac{dx^i}{dt} = \gamma v^i. \quad (4.8b)$$

Thus we can write, using an obvious notation,

$$\mathbf{U} = \gamma(1, \mathbf{v}). \quad (4.9)$$

In a frame which is co-moving with a particle, the particle's velocity is $\mathbf{U} = (1, \mathbf{0})$, so that, from Eqn. (4.4a), $\mathbf{U} \cdot \mathbf{U} = 1$; since the inner-product is frame-invariant, it must have this same value in *all* frames, so that, quite generally, we have

$$\mathbf{U} \cdot \mathbf{U} = 1. \quad (4.10)$$

Here, we defined the 4-velocity by differentiating the displacement 4-vector, and deduced its value in a frame co-moving with a particle. We can now turn this on its head, and *define* the 4-velocity as a vector which has norm 1 and which points along the t -axis of a co-moving frame (this is known as a 'tangent vector', and is effectively a vector 'pointing along' the worldline). We have thus defined the 4-velocity of a particle as the vector which has components $(1, \mathbf{0})$ in the particle's rest frame. Note that the norm of the vector is always the same; the particle's speed relative to a frame S is indicated not by the 'length' of the velocity vector – its norm – but by its direction in S . We can then *deduce* the form in Eqn. (4.9) as the Lorentz-transformed version of this. See also the remarks in Sect. 1.4.

Turning now to the acceleration \mathbf{A}^μ , we have

$$A^0 = \frac{d^2x^0}{d\tau^2} = \frac{dU^0}{d\tau} = \frac{d\gamma}{d\tau} = \gamma \frac{d\gamma}{dt} \quad (4.11a)$$

$$A^i = \frac{d^2x^i}{d\tau^2} = \frac{dU^i}{d\tau} = \gamma \frac{d}{dt}(\gamma v^i) = \gamma (\dot{\gamma} v^i + \gamma a^i) \quad (4.11b)$$

(where v^i and a^i are the ordinary velocity and acceleration $v^i = dx^i/dt$, $a^i = d^2x^i/dt^2$, and $\dot{\gamma} = d\gamma/dt$), or

$$\mathbf{A} = \gamma (\dot{\gamma}, \dot{\gamma} \mathbf{v} + \gamma \mathbf{a}). \quad (4.12)$$

There is no inertial frame in which an accelerating particle is always at rest; however at any instant, such a particle has a definite velocity, and so there is a frame – the *instantaneously co-moving* inertial frame – in which the particle is briefly at rest. In this frame, where $\mathbf{v} = \mathbf{0}$, we have $\mathbf{U} = (1, \mathbf{0})$ and $\mathbf{A} = (\dot{\gamma}, \mathbf{a}) = (0, \mathbf{a})$ (since $\dot{\gamma}$ contains a factor of \mathbf{v}), so that

$$\mathbf{U} \cdot \mathbf{A} = 0$$

in this co-moving frame, and therefore in all frames. From the result in this co-moving frame we can deduce the norm of the 4-acceleration

$$\mathbf{A} \cdot \mathbf{A} = -a^2.$$

defining the *proper acceleration* a as the magnitude of the acceleration in the instantaneously co-moving inertial frame. See also Example 5.

Finally, given two particles with velocities \mathbf{U} and \mathbf{V} , and given that the second has velocity v with respect to the first, then in the first particle’s rest frame the velocity vectors have components $\mathbf{U} = (1, \mathbf{0})$ and $\mathbf{V} = \gamma(v)(1, \mathbf{v})$. Thus

$$\mathbf{U} \cdot \mathbf{V} = \gamma(v),$$

and this inner-product is, again, frame-independent.

1.4 Velocities and tangent vectors



[In this (non-examinable) section, I expand on the mention of ‘tangent vectors’ in the paragraph following Eqn. (4.10). The account here is still rather compressed, and I offer it only to provide a hint of the more abstract, but more beautiful and powerful, way in which these things are handled in General Relativity.]

An alternative route to the velocity vector is to use the idea of a *tangent vector*. This is how vectors are defined in GR, and is a more fundamental approach to vectors than the one described above, which relies on differentiating the mere components of vectors.

Consider a vector $\mathbf{r} = (x, y)$ on the Euclidean plane. If the components x and y are functions of some parameter λ , then the vector function $\mathbf{r}(\lambda)$ will trace out a path $(x(\lambda), y(\lambda))$ on the plane. If we *differentiate* these components with respect to the parameter λ , then we will obtain an object which obviously tells us something about the path. For example, if we follow the path $\mathbf{r}(\theta) = (\cos \theta, \sin \theta)$, parameterised by θ instead of λ , we find it traces out a circle. Differentiating this straightforwardly, we find

$$\frac{d\mathbf{r}}{d\theta} = (-\sin \theta, \cos \theta),$$

which is a vector which, when plotted at the position $\mathbf{r}(\theta)$, as in Fig. 2, can be clearly seen to be tangent to the path.

Now consider the spacetime vector $\mathbf{R} = (t(\lambda), x(\lambda), y(\lambda), z(\lambda))$, which draws out a path in spacetime. The path this traces out is the *worldline* – the set of events which take place along a moving particle’s path through spacetime – and a reasonable parameter to use is the particle’s proper time, τ – the time showing on the face of a clock attached to the moving particle. We therefore have a path $\mathbf{R} = (x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau))$ and, exactly as we did on the Euclidean plane above, can differentiate it to obtain

$$\frac{d\mathbf{R}}{d\tau} = \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau} \right),$$

(compare Eqn. (4.6)) as a 4-vector tangent to the worldline, which clearly contains information about the ‘speed’ of the particle, and which we can *define* to be the velocity

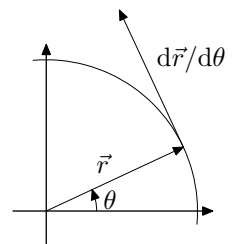


Figure 2

4-vector. By the argument above, we discover that this vector has norm $\mathbf{U} \cdot \mathbf{U} = 1$, and that its direction corresponds precisely to the t -axis of a frame co-moving with the particle.

The point of this approach is that the idea of a *path*, and the idea of a *tangent vector* to that path, are both *geometrical* ideas, existing at a level beneath coordinates (which are more-or-less algebraic things), and so can be defined and discussed without using coordinates, and so without having any dependence on reference frames. They are therefore manifestly frame-independent.

1.5 The frequency vector, and the Doppler shift

In this section, we will examine a particular (non-obvious) 4-vector, and exploit its properties to deduce the relativistic Doppler effect (cf. [1, §4.6]).

Imagine a series of waves of some type (not necessarily light waves), moving in a direction $\mathbf{n} = (l, m, n)$ at a speed w , and imagine following a point on the crest of one of these waves. The point will have a displacement $(\Delta x, \Delta y, \Delta z)$ in time Δt , and so we will have

$$l\Delta x + m\Delta y + n\Delta z = w\Delta t.$$

Now imagine a whole train of such waves, separated by wavelength λ . Taking now $(\Delta t, \Delta x, \Delta y, \Delta z)$ to be the displacement between one event on the crest of a wave and another on a wavecrest N (integer) wavecrests away, we can write

$$w\Delta t - l\Delta x - m\Delta y - n\Delta z = N\lambda. \quad (4.13)$$

Defining the frequency $f = w/\lambda$, we can rewrite this as

$$\mathbf{L} \cdot \Delta\mathbf{R} = N, \quad (4.14)$$

where $\Delta\mathbf{R} = (\Delta t, \Delta x, \Delta y, \Delta z)$ and the *frequency 4-vector* is

$$\mathbf{L} = \left(f, \frac{\mathbf{n}}{\lambda} \right). \quad (4.15)$$

We have written this as a vector, but what is there to say that this is *really* a vector – that is, that it is the components in one frame of an underlying geometrical object – and isn't merely four numbers in a row? We know that $\Delta\mathbf{R}$ is a vector – it is the prototype 4-vector – so we know that its components transform according to Eqn. (4.2b): $\Delta t = \gamma(\Delta t' + v\Delta x')$, $\Delta x = \gamma(\Delta x' + v\Delta t')$, $\Delta y = \Delta y'$ and $\Delta z = \Delta z'$. We can substitute this into Eqn. (4.14), rearrange to gather terms $\Delta t'$, $\Delta x'$, $\Delta y'$, $\Delta z'$ and, just as we did before Eqn. (4.14), rewrite to find

$$\mathbf{L} \cdot \Delta\mathbf{R} = \left[\gamma \left(f - v \frac{l}{\lambda} \right), \gamma \left(\frac{l}{\lambda} - vf \right), \frac{m}{\lambda}, \frac{n}{\lambda} \right] \cdot \Delta\mathbf{R}'.$$

But the vector in the square brackets is exactly the vector \mathbf{L}' we would obtain if we transformed the frequency vector \mathbf{L} according to the transformation matrix Eqn. (4.3). We have established, therefore, that the object \mathbf{L} defined by Eqn. (4.15) really *is* a 4-vector, since it transforms in the same manner as the prototype 4-vector $\Delta\mathbf{R}$.

Is this not inevitable? Not quite: imagine if we had naïvely defined the frequency 4-vector as a vector whose space components were \mathbf{n}/λ and whose time component was defined to be zero. On transformation by either of the routes in the previous paragraph, the vector would acquire a non-zero time component, so that the transformed vector would have a different *form* from the untransformed one. The components of such a 'vector' would *not* transform in the same way as $\Delta\mathbf{R}$, so it would not be a proper 4-vector, so that we would not be able to identify an underlying geometrical object of which these were the components.

See example 3

Can we use the frequency 4-vector for anything? Yes. Imagine that the wave-train is moving away from the origin in the S' plane, so that its direction is $\mathbf{n} = (\cos \alpha', \sin \alpha', 0)$ for some angle α' in frame S' . In that case we have

$$\mathbf{L}' = \left[f', \frac{\cos \alpha'}{\lambda'}, \frac{\sin \alpha'}{\lambda'}, 0 \right] \quad (4.16a)$$

$$\mathbf{L} = \left[\gamma \left(f' + v \frac{\cos \alpha'}{\lambda'} \right), \gamma \left(\frac{\cos \alpha'}{\lambda'} + v f' \right), \frac{\sin \alpha'}{\lambda'}, 0 \right] \quad (4.16b)$$

Now write

$$\mathbf{L} = \left[f, \frac{\cos \alpha}{\lambda}, \frac{\sin \alpha}{\lambda}, 0 \right], \quad (4.16c)$$

and compare Eqns. (4.16b) and (4.16c) component by component. After a bit of rearrangement, we find

$$f = f' \gamma \left(1 + \frac{v}{w'} \cos \alpha' \right) \quad (4.17)$$

$$\tan \alpha = \frac{\sin \alpha'}{\gamma (\cos \alpha' + v w')} \quad (4.18)$$

Equation (4.17) is the *relativistic Doppler effect*, and describes the change in frequency of a wave, as measured in a frame moving with respect to the frame in which it was emitted. This applies for everything from water waves (for which the effect would be exceedingly small) all the way up to light, for which $w = 1$. Equation (4.18) shows that a wave travelling at an angle α' in the moving frame S' is measured to be moving at a *different* angle α in a frame S with respect to which S' is moving with speed v . To calculate the change in the speed of the wave, we could laboriously eliminate variables from Eqns. (4.16b) and (4.16c), but much more directly, we can make use of the fact that the norms of vectors are conserved under Lorentz transformation; thus $\mathbf{L} \cdot \mathbf{L} = \mathbf{L}' \cdot \mathbf{L}'$ or, again using Eqn. (4.16) and $f = w/\lambda$,

$$f^2 \left(1 - \frac{1}{w^2} \right) = f'^2 \left(1 - \frac{1}{w'^2} \right).$$

We could rewrite this to obtain an expression for w' , but simply from this form we can see that if $w = 1$, the fact that neither f nor f' is zero implies that $w' = 1$ also (as the second postulate says).

See example 4

See example 5

See example 6

See example 7

2 Dynamics

In the previous section, we have learned how to describe motion; we now want to explain it. In Newtonian mechanics, we do this by defining quantities such as momentum, energy, force and so on. To what extent can we do this in the context of relativity, with our new 4-vector tools?

2.1 Energy and momentum

We can start with momentum. We know that in Newtonian mechanics, momentum is defined as mass times velocity. We have a velocity, so we can try defining a momentum 4-vector as

$$\mathbf{P} = m\mathbf{U} = m\gamma(1, \mathbf{v}). \quad (4.19)$$

Since m is a scalar, and \mathbf{U} is a 4-vector, \mathbf{P} must be a 4-vector also.



Note that here, and throughout, the symbol m denotes the mass as measured in a particle's rest frame. The reason I mention this is that some treatments of relativity,

particularly older ones, introduce the concept of the ‘relativistic mass’, distinct from the ‘rest mass’. The only (dubious) benefit of this is that it makes a factor of γ disappear from a few equations, making them look at little more like their Newtonian counterparts; the cost is that of introducing one more new concept to worry about, which doesn’t help much in the long term, and can obscure aspects of the energy-momentum vector. Rindler introduces the relativistic mass, Taylor & Wheeler and Schutz don’t.

Now consider a pair of incoming particles \mathbf{P}_1 and \mathbf{P}_2 which collide and produce a set of outgoing particles \mathbf{P}_3 and \mathbf{P}_4 (this can be trivially extended to more than two particles; also, in relativistic collisions, the number of outgoing particles need not equal the number of incoming particles). Suppose that the total momentum is conserved:

$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3 + \mathbf{P}_4. \quad (4.20a)$$

Looking at the time and space coordinates separately, and writing $\mathbf{p} \equiv \gamma m \mathbf{v}$, we have

$$m_1 \gamma(v_1) + m_2 \gamma(v_2) = m_3 \gamma(v_3) + m_4 \gamma(v_4) \quad (4.20b)$$

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4. \quad (4.20c)$$

Now recall that, as $v \rightarrow 0$, we have $\gamma(v) \rightarrow 1$, so that the low-speed limit of the spatial part of the vector \mathbf{P} , Eqn. (4.19), is just $m\mathbf{v}$, so that the spatial part of the conservation equation, Eqn. (4.20c), reduces to the statement that $m\mathbf{v}$ is conserved. Both of these prompt us to identify the spatial part of the vector \mathbf{P} as the linear momentum, and to justify giving the 4-vector \mathbf{P} the name 4-momentum.

What, then, of the time component of Eqn. (4.19)? Let us (with, admittedly, a little fore-knowledge) write this as $P^0 = E$, so that

$$E = \gamma m. \quad (4.21)$$

What is the low- v limit of this? Taylor’s theorem tells us that

$$\gamma = (1 - v^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}v^2 + O(v^4),$$

so that Eqn. (4.21) becomes

$$E = m + \frac{1}{2}mv^2 + O(v^4). \quad (4.22)$$

Now $\frac{1}{2}mv^2$ is the expression for the kinetic energy in Newtonian mechanics, and Eqn. (4.20b), compared with Eqn. (4.21), tells us that this quantity E is conserved in collisions, so we have persuasive support for identifying the quantity E in Eqn. (4.21) as the relativistic *energy* of a particle with mass m and velocity v .

If we rewrite Eqn. (4.21) in physical units, we find

$$E = \gamma mc^2, \quad (4.23)$$

the low-speed limit of which (remember $\gamma(0) = 1$) recovers what has been called the most famous equation of the twentieth century.

The argument presented here after Eqn. (4.20a) has been concerned with giving names to quantities, and, reassuringly for us, linking those newly-named things with quantities we already know about from Newtonian mechanics. This may seem arbitrary, and it is certainly not any sort of proof that the 4-momentum is conserved as Eqn. (4.20a) says it might be. No proof is necessary, however: it turns out *from experiment* that Eqn. (4.20a) is a law of nature, so that we could simply include it as a postulate of relativistic dynamics and proceed to use it without bothering to identify its components with anything we are familiar with.



In case you are worried that we are pulling some sort of fast one, that we never had to do in Newtonian mechanics, note that we *do* have to do a similar thing in Newtonian mechanics. There, we postulate Newton’s third law (action equals reaction), and

from this we can deduce the conservation of momentum; here, we postulate the conservation of 4-momentum, and this would allow us to deduce a relativistic analogy of Newton’s third law (I don’t discuss relativistic force here, but it is easy to define). The postulational burden is the same in both cases.

We can see from Eqn. (4.22) that, even when a particle is stationary and $v = 0$, the energy E is non-zero. In other words, a particle of mass m has an energy γm associated with it simply by virtue of its mass. The low-speed limit of Eqn. (4.20b) simply expresses the conservation of mass, but we see from Eqn. (4.21) that it is actually expressing the conservation of energy. In SR there is no real distinction between mass and energy – mass is, like kinetic, thermal and strain energy, merely another form into which energy can be transmuted – albeit a particularly dense store of energy, as can be seen by calculating the energy equivalent, in Joules, of a mass of 1kg. It turns out from GR that it is not mass that gravitates, but energy (most typically, however, in the particularly dense form of mass), so that thermal and electromagnetic energy, for example, and even the energy in the gravitational field itself, all gravitate.

See example 8

Although the quantities $\mathbf{p} = \gamma m \mathbf{v}$ and E are frame-dependent, and thus not physically meaningful by themselves, the quantity \mathbf{P} defined by Eqn. (4.19) has a physical significance.

Let us now consider the norm of the 4-momentum vector. Like any such norm, it will be frame-invariant, and so will express something fundamental about the vector, analogous to its length. Since this is the momentum vector we are talking about, this norm will be some important invariant of the motion, indicating something like the ‘quantity of motion’. From the definition of the momentum, Eqn. (4.19), and its norm, Eqn. (4.10), we have

$$\mathbf{P} \cdot \mathbf{P} = m^2 \mathbf{U} \cdot \mathbf{U} = m^2, \quad (4.24)$$

and we find that this important invariant is the *mass* of the moving particle. We could give this invariant a special name such as ‘energy-momentum’ – Taylor & Wheeler coin the term ‘momenergy’ – but we discover that we might as well stick to the familiar term ‘mass’, or perhaps ‘invariant mass’, even though it is not necessarily in the form of hard lumps of matter. In other words, whenever we use the term ‘mass’ in relativity, we are not necessarily referring to lumps of stuff, but directly to the norm of the appropriate \mathbf{P} vector.

Now using the definition of energy, Eqn. (4.21), we can write $\mathbf{P} = (E, \mathbf{p})$, and find

$$\mathbf{P} \cdot \mathbf{P} = E^2 - \mathbf{p} \cdot \mathbf{p}. \quad (4.25)$$

Writing now $p^2 = \mathbf{p} \cdot \mathbf{p}$, we can combine these to find

$$E^2 = p^2 + m^2. \quad (4.26)$$

Equation (4.26) appears to be simply a handy way of relating E , p and m , but that conceals the more fundamental meaning of Eqn. (4.24). The 4-momentum \mathbf{P} encapsulates the important features of the motion in the energy and spatial momentum. Though they are frame-dependent separately, they combine into a frame-independent quantity.

2.2 Photons

For a photon, the interval represented by $d\mathbf{R} \cdot d\mathbf{R}$ is always zero ($d\mathbf{R} \cdot d\mathbf{R} = dt^2 - dx^2 - dy^2 - dz^2 = 0$ for photons). But this means that the proper time $d\tau^2$ is also zero for photons. This means, in turn, that we cannot define a 4-velocity vector for a photon by the same route that led us to Eqn. (4.6), and therefore cannot define a 4-momentum as in Eqn. (4.19).

We can do so, however, by a different route. Recall that we defined (in the paragraph below Eqn. (4.10)) the 4-velocity as a vector pointing along the worldline, which

resulted in the 4-momentum being in the same direction. From the discussion of the momentum of massive particles above, we see that the P^0 component is related to the energy, so we can use this to define a 4-momentum for a *massless* particle, and again write

$$\mathbf{P}_\gamma = (E, \mathbf{p}_\gamma).$$

Since the photon's velocity 4-vector is null, the photon's 4-momentum must be also (since it is defined above to be pointing in the same direction). Thus we must have $\mathbf{P}_\gamma \cdot \mathbf{P}_\gamma = 0$, thus $\mathbf{p}_\gamma \cdot \mathbf{p}_\gamma = E^2$, recovering the $m = 0$ version of Eqn. (4.26),

$$E^2 = p^2 \quad (\text{massless particle}), \quad (4.27)$$

so that even massless particles have a non-zero momentum.

In quantum mechanics, we learn that the energy associated with a quantum of light – a photon – is $E = hf$, where h is Planck's constant, $h = 6.626 \times 10^{-34}$ Js (or 2.199×10^{-42} kg m in natural units), so that

$$\mathbf{P} = (hf, hf, 0, 0) \quad (\text{photon}). \quad (4.28)$$

2.3 Relativistic collisions and the centre-of-momentum frame

In this subsection we will, to avoid clutter, write only the x -component of the spatial part of vectors, writing (t, x) rather than (t, x, y, z) and (P^0, P^1) rather than (P^0, P^1, P^2, P^3) .

Consider two particles, of momenta \mathbf{P}_1 and \mathbf{P}_2 , which collide and produce a single particle of momentum \mathbf{P}_3 – you can think of this as describing either as two balls of relativistic putty, or an elementary particle collision which produces a single new outgoing particle (this example is adapted from [3, §8.3]).

The particles have momenta

$$\mathbf{P}_i = \gamma_i m_i (1, \mathbf{v}_i), \quad (4.29)$$

where the three particles have velocities \mathbf{v}_i , and $\gamma_i \equiv \gamma(\mathbf{v}_i)$. From momentum conservation, we also know that

$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3. \quad (4.30)$$

Note that this also indicates that *each component* of the vectors is separately conserved. Let the 'incoming' particles have masses $m_1 = m_2 = 8$ units, the first be travelling with speed $v_1 = 15/17$ along the x -axis in the 'lab frame' (so that components v_1^2 and v_1^3 are both zero, and $\gamma_1 = \gamma(15/17) = 17/8$), and the second be stationary, $v_2 = 0$. There's nothing special about this rather peculiar value for v_1 , except that it makes the arithmetic below work out neatly. As discussed above, the appropriate measure of the 'length' of the \mathbf{P} vectors is the quantity m of Eqn. (4.24): $m = \sqrt{\mathbf{P} \cdot \mathbf{P}}$. Thus

$$\mathbf{P}_1 = (\gamma_1 m_1, \gamma_1 m_1 v_1) = (17, 15) \Rightarrow m_1 = \sqrt{17^2 - 15^2} = 8 \quad (4.31a)$$

$$\mathbf{P}_2 = (8, 0) \Rightarrow m_2 = 8 \quad (4.31b)$$

$$\mathbf{P}_3 = \mathbf{P}_1 + \mathbf{P}_2 = (25, 15) \Rightarrow m_3 = 20 \quad (4.31c)$$

As a pleasing consistency check, we can calculate the speed of particle 3 using the relation $v = P^1/P^0$ (cf. Eqn. (4.29)), giving $v_3 = P_3^1/P_3^0 = 15/25 = 3/5$. We thus have $P_3^0 = E_3 = \gamma(3/5)m_3 = 25$ giving $m_3 = 20$ as in Eqn. (4.31c).

Notice the following points.

(i) The two incoming particles have the same energy-momentum, $m_1 = m_2 = 8$. This may seem surprising, since one is moving much faster than the other, but recall that the inner-product $\mathbf{P}_1 \cdot \mathbf{P}_1$ is an *invariant*, so it has the same value in the frame

in which particle 1 is moving at speed $v_1 = 15/17$ as it has in the frame in which particle 1 is stationary; in the latter frame, $\mathbf{P}'_1 = (\gamma(0)m_1, v_1) = (8, 0)$, showing the same values as \mathbf{P}_2 in the lab frame.

(ii) From Eqn. (4.31a), notice that the mass in the ‘outgoing’ particle 3 is *not* merely the sum of the two incoming masses: it is the components of \mathbf{P} , including total energy $P_1^0 + P_2^0 = P_3^0 = \gamma m_3$ that are the quantities conserved in collisions, not mass (in the sense of the norm of \mathbf{P}).

(iii) The quantity $m^2 = \mathbf{P} \cdot \mathbf{P}$ is frame-invariant, but not conserved; P^μ is conserved, but not frame-invariant. Taylor & Wheeler [3, Box 7-3] usefully contrast the terms ‘conserved’, ‘invariant’ and ‘constant’:

Conserved refers to a quantity which is not changed (over time?) by some process – momentum, for example, is conserved in a collision. This refers to one frame at a time, and a conserved quantity will typically have different numerical values in different frames.

Invariant refers to a quantity which is not changed by some transformation – the radius r is an invariant of the rotation in Fig. 3-5, and we have just discovered that m^2 is an invariant of the Lorentz transformation. The term necessarily refers to more than one reference frame; there is no reason to expect that an invariant quantity will be conserved in a particular process.

Constant refers to a quantity which does not change in time, such as the mass of the University.

The speed of light is one of the very few things which has the full-house of being conserved, invariant, *and* constant.

From Eqn. (4.29) and Eqn. (4.30) we can write

$$\mathbf{P}_3 = (\gamma_1 m_1 + \gamma_2 m_2, \gamma_1 m_1 v_1 + \gamma_2 m_2 v_2).$$

Now use the LT, Eqn. (4.3), to transform this to a frame which is moving with speed V with respect to the lab frame. We therefore have

$$P_3^{1'} = \gamma(V)(P_3^1 - V P_3^0) = \gamma(V)(\gamma_1 m_1 v_1 + \gamma_2 m_2 v_2 - V(\gamma_1 m_1 + \gamma_2 m_2)).$$

We can choose the speed V to be such that this spatial momentum is zero, $P_3^{1'} = 0$, giving

$$V = \frac{\gamma_1 v_1 + \gamma_2 v_2}{\gamma_1 + \gamma_2} = \frac{3}{5}$$

(since here $m_1 = m_2$). In this frame we have (transforming the other momentum 4-vectors, and using $\gamma(V = 3/5) = 5/4$)

$$\mathbf{P}'_1 = (\gamma(17 - V \cdot 15), \gamma(15 - V \cdot 17)) = (10, 6) \quad (4.32a)$$

$$\mathbf{P}'_2 = (10, -6) \quad (4.32b)$$

$$\mathbf{P}'_3 = (20, 0). \quad (4.32c)$$

Notice: (i) In this frame also, $\mathbf{P}'_1 + \mathbf{P}'_2 = \mathbf{P}'_3$: the conservation of a vector quantity means that it is conserved in all frames. (ii) Also $m'_1 = (\mathbf{P}'_1 \cdot \mathbf{P}'_1)^{1/2} = 8$, $m'_2 = 8$ and $m'_3 = 20$: these are frame-invariant quantities. (iii) Both the momenta *and* energy of the vectors in this frame are different from the lab frame, even though their norms are the same.

This frame, in which the total spatial momentum is zero, so that the incoming particles have equal and opposite spatial momenta, is known as the *centre of momentum* (CM) frame, and the energy available for particle production in this frame (P_{CM}^0) is known as the *centre of mass energy*.

2.4 Compton scattering

As a further example of a relativistic collision, we can examine the collision between a photon – a quantum of light energy – and an electron. This is not the same as the classical Thomson scattering of light by electrons: that is a purely electromagnetic effect whereas this is an inherently relativistic and quantum-mechanical effect, where we are treating both the electron and the incoming light as relativistic particles. See also French [4, pp.194–6].

The collision is as shown in Fig. 3. An incoming photon strikes a stationary electron and both recoil. The incoming photon has energy $Q_1 = hf_1 = h/\lambda_1$ and the outgoing one $Q_2 = hf_2 = h/\lambda_2$; the outgoing electron has energy E spatial momentum \mathbf{p} , and mass m . The four momentum 4-vectors are therefore

$$\begin{aligned} \mathbf{P}_{1e} &= (m, 0, 0, 0) & \mathbf{P}_{2e} &= (E, p \cos \theta, p \sin \theta, 0) \\ \mathbf{P}_{1\gamma} &= (Q_1, Q_1, 0, 0) & \mathbf{P}_{2\gamma} &= (Q_2, Q_2 \cos \phi, Q_2 \sin \phi, 0), \end{aligned}$$

where subscripts 1 and 2 denote momenta before and after the collision, respectively.

Momentum conservation implies $\mathbf{P}_{1e} + \mathbf{P}_{1\gamma} = \mathbf{P}_{2e} + \mathbf{P}_{2\gamma}$. Comparing components, we find

$$Q_1 + m = Q_2 + E \quad (4.33a)$$

$$Q_1 = p \cos \theta + Q_2 \cos \phi \quad (4.33b)$$

$$0 = p \sin \theta + Q_2 \sin \phi. \quad (4.33c)$$

Squaring and adding Eqn. (4.33b) and Eqn. (4.33c), we have

$$(Q_1 - Q_2 \cos \phi)^2 + (-Q_2 \sin \phi)^2 = p^2. \quad (4.34)$$

Using the relation $E^2 = p^2 + m^2$ in this equation, then substituting for E from Eqn. (4.33a), and rearranging, we find

$$Q_1 Q_2 (1 - \cos \phi) = (Q_1 - Q_2) m \quad (4.35)$$

which, on dividing through by $Q_1 Q_2 m$, gives

$$\frac{1}{Q_2} - \frac{1}{Q_1} = \frac{1 - \cos \phi}{m} \quad (4.36)$$

or, in terms of wavelength,

$$\lambda_2 - \lambda_1 = \frac{h}{m} (1 - \cos \phi). \quad (4.37)$$

We can see that the scattered photon will have a longer wavelength (smaller energy) than the incident one. Note also that this is not any type of Doppler effect: the photon has a different energy not because we are observing it in any sort of moving frame, but has a lower energy because it has had to give some up to the electron.

3 Conclusion

That is all we have time for. Over the past few weeks, we have examined the axioms on which SR is based, derived the immediate consequences of these axioms in the form of the Lorentz Transformation equations, and finally examined how our understanding of dynamics has to be transformed within the context of relativity.

Has it been worth it? While you will only very rarely have to calculate the speed of clocks on passing spaceships, and rather less often on passing trains, the physical world picture you have learned in these lectures is what underlies areas of physics as diverse as relativistic quantum mechanics, particle physics and general relativity,

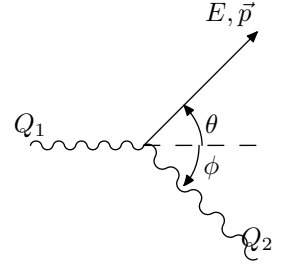


Figure 3

which are simply unintelligible otherwise. Even if you do not go on to study those areas, you have come to have a more fundamental view of the universe than you had from your intuitive understanding of Newtonian physics, and gained access to one of the 20th century's most radical intellectual adventures.

Enjoy!

Questions

In Sect. 1.2, the vectors $A = (1, 2)$ and $B = (2, 1)$ are orthogonal, but if you draw them on the Minkowski diagram, they're clearly not at right-angles. Why? The term 'orthogonal' means nothing more nor less than $\mathbf{A} \cdot \mathbf{B} = 0$. In the special case of Euclidean space (that is, the space we're used to), we give this the traditional name of 'right-angled'. That's all that 'right-angled' is: the traditional name for 'orthogonal', when used in Euclidean space. Another way of thinking about it is that, just as maps of the Earth are distorted when they're put on a flat sheet, Minkowski space naturally gets distorted when it's forced onto the flat Euclidean page.

References

- [1] Wolfgang Rindler. *Essential Relativity: Special, General and Cosmological*. Springer-Verlag, 2nd edition, 1977.
- [2] Bernard F Schutz. *A First Course in General Relativity*. Cambridge University Press, 1985.
- [3] Edwin F Taylor and John Archibald Wheeler. *Spacetime Physics*. W H Freeman, 2nd edition, 1992.
- [4] A P French. *Special Relativity*. Chapman and Hall, 1971.

Examples

Example 1 (section 1.2)

Calculate the matrix inverse of Eqn. (4.3), giving the transformation $\mathbf{A}' \mapsto \mathbf{A}$. Then obtain the inverse of Eqn. (4.3) by swapping primed with unprimed quantities and changing the sign of v . Verify that the two ways of obtaining the inverse match.

Example 2 (section 1.2)

Consider the vectors $\mathbf{A} = (5, 3, 4, 0)$, $\mathbf{B} = (1, 1, -1, -1)$ and $\mathbf{C} = (3, 2, -1, 0)$. Calculate $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{A} \cdot \mathbf{C}$ and $\mathbf{B} \cdot \mathbf{C}$: which pair is orthogonal? Calculate also the norms of \mathbf{A} , \mathbf{B} and \mathbf{C} , and indicate which vectors are null, timelike and spacelike. (Objective 1)

Example 3 (section 1.5)

One might naïvely consider defining a 4-vector representing the electric field as $(0, E_x, E_y, E_z)$, where the E_i are the components of the electric field, and the time component is defined to be identically zero. Explain briefly why this *cannot* be a 4-vector [adapted from degree exam, 2001].

Example 4 (section 1.5)

Obtain the inverse to these equations by applying Eqn. (4.3) to Eqn. (4.16c) and comparing with Eqn. (4.16). Verify that the resulting expression matches that which you obtain through the procedure described in Sect. 1.2: swapping primed with unprimed quantities and changing the sign of v .

Example 5 (section 1.5)

A rocket starts at rest at a space station, and accelerates along the x -axis at a constant rate α in its own frame S' . The rocket's acceleration 4-vector is therefore $A' = (0, \alpha, 0, 0)$.

(a) Using the (inverse) Lorentz transformation equations, calculate the components of the rocket's acceleration vector in the space station's frame, and confirm that $\mathbf{A} \cdot \mathbf{A} = -\alpha^2$ in this frame also. Write down the rocket's velocity 4-vector in this frame, and confirm that $\mathbf{U} \cdot \mathbf{A} = 0$. Differentiate this with respect to the proper time, to obtain

$$\frac{d\mathbf{U}}{d\tau} = \gamma \left(\frac{d\gamma}{dt}, \frac{d}{dt}(\gamma v), 0, 0 \right).$$

Hence deduce that $\gamma v = \alpha t$, and thus that

$$\frac{1}{v^2} = \frac{1}{\alpha^2 t^2} + 1. \quad (\text{i})$$

(b) If, at time t , the rocket sets off a flashbulb which has frequency f' in its frame, use the Doppler formula to show that the light is observed, at the spacestation, to have frequency

$$f = f'(\sqrt{1 + \alpha^2 t^2} - \alpha t).$$

What is this factor if the flashbulb is set off at time $t = 3/(4\alpha)$?

(c) Rearrange Eqn. (i) to produce an expression for v , write $v = dx/dt$, and integrate to obtain the constant-acceleration equation for SR,

$$(x + 1/\alpha)^2 - t^2 = 1/\alpha^2$$

(you may or may not find the substitution $\alpha t = \sinh \theta$ useful or obvious here). This is the equation for a hyperbola. By sketching this on a Minkowski diagram and considering the point at which the asymptote intersects the t -axis, demonstrate that it is impossible for the space station to signal to the retreating rocket after a time $1/\alpha$.

Example 6 (section 1.5)

The traditional traffic-lights example: You are driving towards some traffic lights showing red (take wavelength $\lambda = 600$ nm). How fast do you have to be driving so that they are Doppler-shifted enough that they appear green (wavelength $\lambda = 500$ nm)? [Hint: rewrite Eqn. (4.17) in terms of $\rho = f'/f$, set $w = c$ and $\alpha = 0$, and rearrange to make v/c the subject.] Does this sound like a reasonable defence when you're had up in court on a charge of dangerous driving? (Objective 4)

Example 7 (section 1.5)

By squaring Eqn. (4.18) and setting $w = 1$ (and using trigonometric identities to reexpress $\sin^2 \alpha'$), show that

$$\cos \alpha' = \frac{\cos \alpha - v}{v \cos \alpha}. \quad (\text{i})$$

A source in the moving frame S emits light isotropically. Deduce that light emitted into the forward half-sphere is observed, in the moving frame, to be emitted into a cone with half-angle α'_+ , where $\cos \alpha'_+ = v$. This is known as the *headlight effect*. (Objective 4)

Example 8 (section 2.1)

The luminosity of the sun is $L_{\odot} = 3.86 \times 10^{26}$ W. How much mass does it lose per second? (Objective 5)