Astronomy 2 – Special Relativity – Part 3
Spacetime and the Lorentz Transformation

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In this part, we are going to investigate the consequences of the two axioms we introduced last time.

Aims and objectives for part 3

Aims: You should

1. understand why the concept of simultaneity is necessarily relative to the speed of the observer.
2. follow the derivation, and recognise the significance, of the Lorentz Transformation.
3. appreciate the role of geometry in understanding Spacetime, specifically the importance of the invariant interval and the Minkowski diagram.
4. appreciate the importance of frame-invariance as a property of a quantity.

Objectives: You should be able to

1. provide short definitions of the terms 'spacetime', 'worldline', 'proper time' and 'invariant interval'.
2. be able to quote the length contraction and time dilation formulae, and use them in solving problems.
3. translate a problem into a set of events in frames in standard configuration, and solve it using the Lorentz Transformation, and/or the velocity transformation (but you are not obliged to memorise these).
4. define (and calculate) the invariant interval separating events; provide a definition of the terms timelike, spacelike and null, as they relate to separations between events; categorise an interval between events as timelike, spacelike, or null; and draw appropriate conclusions about causal influence between events.
5. place events and worldlines on a Minkowski diagram, and draw conclusions from it.

1 Length contraction and time dilation

1.1 Simultaneity

Imagine standing in the centre of a train carriage, with suitably agile friends at either end: Fred (at the Front) and Barbara (at the Back). At a prearranged time, say time ‘0’, on your carefully synchronised watches, you fire off a flashbulb and your friends note down the time showing on their watches when the flash reaches them. Since you are standing in the middle of the carriage, Fred’s and Barbara’s times must be the same as each other. Comparing notes afterwards, you all find that it took some time for the flash to travel from the middle of the carriage to the end, and that your friends have noted down the same arrival time on their watches, time ‘3’, say. In other words, Fred’s and Barbara’s watches both reading ‘3’, are *simultaneous* events in the frame of the carriage.

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These watches are obviously not calibrated in seconds, but these could be sensible values if the watches are telling time in the natural relativistic time unit of metres! See Sect. 2.1 and Example 3.13 below.

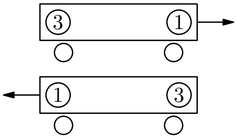

Figure 1: Passing trains

But if this train is moving through a station as all this goes on, and you look from the platform into the carriage, what would you see from this point of view? You would see the light from the flash move both forward towards Fred and backwards towards Barbara, but remember that you would *not* see the light moving forwards faster than the speed of light – its speed would not be enhanced by the motion of the train – nor would you see the light moving backwards at less than c. Since the back of the train is rushing towards where the light was emitted, the flash would naturally get to Barbara first. If, standing on the platform, you were to take a photograph at this point, you would get something like the upper part of Fig. 1. Barbara's watch *must* read '3', since the flash meeting her and her watch reading '3' are simultaneous at the same point in space, and so must be simultaneous for observers in any frame. But at this point, the light moving towards Fred cannot yet have caught up with him: since the light reaches Fred when *his* watch reads '3', his watch must still be reading something less than that, '1', say. In other words, Barbara's watch reading '3' and Fred's watch reading '1' are simultaneous events in *your* inertial frame on the platform.

What is going on here? Are these events simultaneous or not? What this tells us is that our notion of simultaneity is rather naïve, and that we have to be very careful exactly what we mean when we talk of events as being simultaneous. The only case where two events are quite unambiguously simultaneous is if they take place at exactly the same point in space. That's why we could say without hesitation that the light reached Barbara when her watch read '3', because that's what she had seen and noted down.

1.2 Length contraction and time dilation, qualitatively

We're not finished with the trains, yet. Imagine now we're standing on the platform and see now two trains go past. We've cunningly arranged the speeds, timetable and flashbulbs so that we can get the photograph in Fig. 1, where the light has reached both rear observers and neither front one. Now pause a moment, and take another photograph when the two rear observers are beside each other, this time getting Fig. 2.

Note, by the way, that there is nothing fundamental about the presence of this third party, taking 'photographs'. They are there simply as proxy for a suitable set of observers lined up along the platform edge. In the same way, these 'photographs' are not what you would see if you used a real camera. A real such photo would include optical effects such as aberration and the Doppler effect, which would hopelessly confuse the issue here.

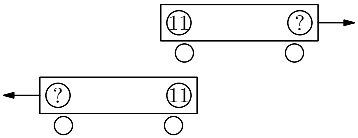

Figure 2: More trains

After this, the various observers calm down, amble together, and compare notes. Barbara (standing at the back of the top carriage) could remark "I saw the front of the other carriage pass us when my clock was reading '3' " (this is perfectly correct, as you can confirm by looking at the 'photograph' in Fig. 2). At which Fred would say "But the back of the carriage passed me at time '1' – it must have been well past me at time '3' ". From this they, and we, can quite correctly remark that the carriage they observed moving past them was measured to be shorter than their own. They have measured the length of a moving carriage, and found that it is shorter than a similar carriage (their own) which they can measure at leisure when stationary. This is *length contraction*.

Fred then says “I looked through the window at the clock at the back of the other carriage, and I noticed that it was reading ‘3’, when mine was reading ‘1’ – it was two seconds fast”. Barbara says “Well, I saw that *same* clock a bit later [in Fig. 2], and it was reading ‘11’, just like mine – it wasn’t fast at all.” They know that their own clocks were synchronised throughout the encounter (they can make sure that their clocks are synchronised at some point, and they know that they both go at the same rate), so they can only conclude (correctly) that the clock they both saw was going more slowly than theirs were. Time in the other carriage is passing more slowly than in their own.

The extraordinary thing is that Barbara and Fred’s counterparts in the other carriage would come to precisely the same conclusions. Because this setup is perfectly symmetrical, they would measure Barbara and Fred’s clocks to be moving slowly, and their carriage to be shorter. There is no sense in which one of the carriages is *absolutely* shorter than the other.

See Sect. 3.5.1 for the quantitative description of length contraction. Ex.3

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1.3 The light clock

Having persuaded ourselves (I hope) of the existence of time-dilation and length-contraction, it is easy to put numbers to the effects. idx:l

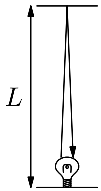

Figure 3: Light clock

Imagine you're on a train which is moving through a station at some relativistic speed. You throw a ball into the air and catch it again: how would you describe this? You'd say that the ball started off in your left hand, followed a parabolic path (like anything thrown), landed in your right hand, and that it took one second (say) to do it all. Now imagine you're on the platform watching this go on: how would you describe it now? You'd say that it started off at the start of the platform, landed a good way down the platform, and took one second to do it. These two perceptions agree that the ball follows a parabolic path (different parabolae, yes, but parabolae nontheless – this is the Relativity Principle at work), but they disagree on how far the ball travelled in flight. That disagreement is easily explained: from the point of view of the observer on the platform, the ball was travelling very quickly, since it had the train's speed as well as its own, so of *course* it covered more ground before it landed again.

None of this is mysterious. I've made it mysterious by the elaborate way in which I've described it. But I've described it that way to pull this perfectly normal situation into line with the next step, the *light clock*.

The light clock (see Fig. 3) is an idealised timekeeper, in which a flash of light leaves a bulb, bounces off a mirror, and returns – this is one 'tick' of the clock. If the mirror and the flashbulb are a distance L apart, and I, standing by the light clock, time the round trip as t' seconds, then, since the speed of light is the constant c,

$$2L = ct' \tag{3.1}$$

Note that t' here is the time on my watch, standing by, and moving with, the clock. Also note that the clock's mirrors are arranged perpendicular to the clock's motion, and both the stationary and the moving observer measure the same separation between them – there is no length contraction perpendicular to the motion. idx:l cont tion! abse

To see that this must be so, consider the following *reductio ad absurdum*. Imagine there were a perpendicular length contraction. Then observers measuring a train moving along a parallel railway track would see it getting narrower; specifically, the train's axles would get shorter, so that at some speed the train would be derailed with its wheels lying between the rails. However, from the point of view of the observers in the train, it is the rails which would be moving; therefore if there were a perpendicular length contraction, the measured distance between the rails would become shorter, and at some speed the train would be derailed with its wheels lying *outside* the rails. These statements must be either both true or both false; they contradict one another, so they cannot both be true, and they must therefore both be false; they are both consequences of the supposition that there exists a perpendicular length contraction; so that statement must in turn be false, demonstrating that there can be no such contraction.

Now imagine the light clock sitting on the train going through the station, as you watch it from the platform. The clock is in motion, at a speed v, and one tick is timed as t seconds (see Fig. 4), during which time the clock will have moved a distance vt down the platform.

How far has the light travelled? We know the light travelled at a speed c (the second axiom tells us that, no matter where or how the light was travelling – in a clock sitting beside us on the platform, or a clock zooming though the station in front of us – we would *always* measure its speed to be the same number c), and we timed its round trip at t seconds, so the light beam must have travelled a distance ct in the time that the clock itself travelled a

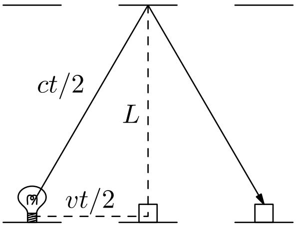

Figure 4: Light clock

distance vt. But from the figure,

$$\left(\frac{ct}{2}\right)^2 = L^2 + \left(\frac{vt}{2}\right)^2.$$

Substituting L from Eqn. (3.1), and rearranging, we find

$$t' = \frac{t}{\gamma}, \tag{3.2}$$

where the factor $\gamma = \gamma(v)$ is defined as idx:g

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{3.3}$$

Now, the important thing about this equation is that it involves t', the time for the clock to 'tick' as measured by the person standing next to it on the train, and it involves t, the time as measured by the person on the platform, and *they are not the same.* Ex.3

How can this possibly be? Why is this different from the perfectly reasonable behaviour of the ball thrown down the carriage? The difference is that when you watched the ball from the platform, you saw it move with the speed it was given plus the speed of the train – in other words, the person on the platform and the person on the train had a perfectly reasonable disagreement about the speed of the ball, which resulted in them agreeing on the time the ball was in flight. However, both of them *agree* on the speed of the light in the light clock, as the second axiom says they must. Something has to give, and the result is that the two observers disagree on how long the light takes for a circuit.

So at least one of the clocks is broken? They're both in perfect working order. They only work properly when they're stationary? No, the Relativity Principle tells us that there's no sense in which either of them is 'more stationary' than the other, so that the clocks work in exactly the same way whether they're moving or not. No…

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> *Both clocks are perfectly accurately measuring the passage of time; time is flowing differently for the two observers.*

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1.4 Is there anything I can hold on to?

At this point you may be feeling rather seasick. People tend to find relativity rather disorienting, as more and more pillars of their dynamical intuition are kicked away. You can end up in the situation where you trust none of your steps at all, and find yourself unable to move at all, for fear that the whole edifice will come tumbling down.

Distinguish between what you know, and what you intuit. In fact, rather little of what you know has changed: it comes down to not much more than the addition of relativistic velocities (loosely '$c + 0.5c = c$') and the relativity of simultaneity ('not everyone agrees that two events are simultaneous'), both of which are fairly direct consequences of the second axiom. Unfortunately, both of these eat away at our intuition of how moving objects behave.

Look at what has *not* changed, however. For example, in our discussion of the trains above, it was still true that the backward-moving light flash hit the rear of the carriage before the forward-moving one hit the front because, reasonably enough, the carriage rear was moving *into* the flash. It was still true that the order of events at a single point in space is absolutely fixed. It was still true that when Fred and Barbara looked at their own watches those watches told them the local time accurately, and when they looked at the nearby watches on the other train (which are, in principle, at the *same* point in space and time as Fred and Barbara), they could reasonably measure, or observe, what they saw there.

In this focus on (local) measurement, we can see the influence of the philosophical *positivism* which influenced Einstein in his development of SR. This focus can also help us understand what is going on. If we concentrate on working out what the participants would measure – that is, on what they would see happening at their own point in space and time – rather than on what we intuitively expect them to see, then when we find ourselves puzzled, saying 'that couldn't possibly happen, because…', we have someplace to start.

2 Spacetime

In Sect. 1 we saw that if a set of observers, in motion relative to each other, separately observe a common pair of events, they will make *differing* measurements of the events' coordinates in space and time. This is not simply due to insignificant differences such as having differing origins – having the observers' frames in standard configuration (see Sect. 1-1.4) deals with that. No, the spatial and temporal separations between the events (Sect. 1.2), and possibly even the order in which the separated events are observed to occur (Sect. 1.1), will differ for the different observers. idx:s

The relationship between the different measurements is not random – the coordinates obtained by one observer are systematically related to those obtained by another. Section 3 is concerned with the details of this relationship, and the *Lorentz transformation*, Eqn. (3.18), allows one to deduce the coordinates of an event in one frame, given the coordinates of the same event in another frame.

Before we can deduce this, however, we must look more carefully at the way in which time and space are interrelated.

2.1 Natural units

Because of the high speed of light, metres and seconds are absurdly mismatched units of distance and time. Much better is to measure the two in the *same* units, metres, with one metre of time being the time it takes light to travel a distance of one metre (we did this, in fact, in Sect. 1.1). These are referred to as *natural units*. We could alternatively measure distances in seconds, and we do this naturally when we talk of the Earth being 8 light-minutes from the sun, or the nearest star being a little more than 4 light-years away. In fact, since 1981 or so, the International Standard definition of the metre is that it is the distance light travels in 1/299792458 seconds; that is, the speed of light is $299\,792\,458\,\mathrm{m\,s^{-1}}$ *by definition*, and so c is therefore demoted to being merely a conversion factor between two different units of time. In the same sense, the inch is defined as $2.54\,\mathrm{cm}$, and this figure of 2.54 is merely a conversion factor between two different, and only historically distinct, units of length.

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Thus, we can say $1\,\mathrm{ns} = 30\,\mathrm{cm}$: we can read this as a rather odd way of writing the statement that light travels $30\,\mathrm{cm}$ in $1\,\mathrm{ns}$, or else, as here, read it as a way of converting between two units of distance or time. Similarly, we can write $1 = 2.54\,\mathrm{cm/in}$ as an odd way of writing the unit conversion, or $1 = 3 \times 10^8\,\mathrm{m\,s^{-1}}$ as another odd way of writing the unit conversion between metres and seconds.

There are several advantages to all this. (i) As we shall learn below, space and time are not really distinct, but having different units for the two 'directions' can obscure this. (ii) If we measure time in metres, then we no longer need the conversion factor c in our equations, which are consequently simpler. (iii) In these units, light travels a distance of one metre in a time of one metre, giving the speed of light as an easy-to-remember, and dimensionless, $c = 1$. We also quote other speeds in these units of metres per metre, so that all speeds are dimensionless and less than one.

It is easy, once you have a little practice, to convert values and equations between the different systems of units. Throughout the rest of this course, I will quote equations in units where $c = 1$, so that the factor c disappears from the equations. Most relativity textbooks do not do this, and quote equations in physical units instead (that is, SI units).

For example, to convert $10\,\mathrm{J} = 10\,\mathrm{kg\,m^2\,s^{-2}}$ to natural units, we could proceed in two ways. Since $c = 1$, we have $1\,\mathrm{s} = 3 \times 10^8\,\mathrm{m}$, and so $1\,\mathrm{s}^{-2} = (9 \times 10^{16})^{-1}\,\mathrm{m}^{-2}$. So $10\,\mathrm{kg\,m^2\,s^{-2}} = 10\,\mathrm{kg\,m^2} \times (9 \times 10^{16})^{-1}\,\mathrm{m}^{-2} = 1.1 \times 10^{-16}\,\mathrm{kg}$.

Alternatively, we can write $1 = 3 \times 10^8\,\mathrm{m\,s^{-1}}$, or $1 = (3 \times 10^8)^{-1}\,\mathrm{s\,m^{-1}}$. Thus

$$\begin{aligned} 10\,\mathrm{J} &= 10\,\mathrm{kg\,m^2\,s^{-2}} \times (1)^2 \\ &= 10\,\mathrm{kg\,m^2\,s^{-2}} \times (3 \times 10^8)^{-2}\,\mathrm{s^2\,m^{-2}} \\ &= 1.1 \times 10^{-16}\,\mathrm{kg}. \end{aligned}$$

Ex.3

In General Relativity, people tend to work in units where mass has the same units as distance and time, and the gravitational constant $G = c = 1$. In relativistic quantum mechanics, likewise, units are chosen so that $\hbar = c = 1$.

2.2 Plane rotations

But first a quick review of plane geometry. idx:r

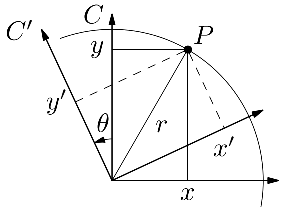

Figure 5: Rotation in the plane

In Fig. 5 we see frames C and C', where the latter has been rotated by an angle θ with respect to the former. The point P has coordinates (x, y) in the frame C. The *same* point P has coordinates (x', y') in the frame C'; what is the relationship between the two sets of coordinates (x, y) and (x', y').

A little geometry (write the angle between OP and the x-axis as α, express x, y, x' and y' in terms of α, θ and r, and eliminate r and α) gives

$$x' = x\cos\theta + y\sin\theta \tag{3.4a}$$

$$y' = -x\sin\theta + y\cos\theta. \tag{3.4b}$$

The point P is distance r from the origin. Pythagoras' theorem tells us that the coordinates in S are such that $r^2 = x^2 + y^2$; the transformation in Eqn. (3.4) is such that $r^2 = x'^2 + y'^2$ in addition (this can be seen directly from the figure, and also demonstrated by explicitly calculating $x'^2 + y'^2$ using Eqn. (3.4)). That is, although the coordinates of the point P in the two frames are different, the distance r is the same in both frames – it is an idx:P *invariant of the transformation*:

$$x^2 + y^2 = r^2 = x'^2 + y'^2. \tag{3.5}$$

Further, the curve in Fig. 5 is the locus of points which have the same distance r from the origin. If only the x- and y-axes were calibrated, we could transfer this calibration to the x'- and y'-axes using this curve.

2.3 The Minkowski diagram

How can we visualise motion? idx:

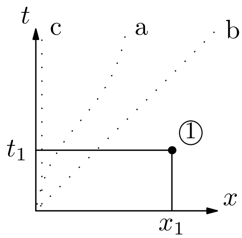

Figure 6

Any event (say event ①) has a set of four coordinates (t_1, x_1, y_1, z_1). Since we will almost always restrict our attention to frames in Standard Configuration, and since there is no transverse contraction, the y and z coordinates are uninteresting. We can plot the remaining two coordinates (t_1, x_1) on a diagram, to obtain something like Fig. 6. Now imagine a flickering bulb moving along the x-axis: each flash is a separate event, and so each will have a particular x and t coordinate. Now plot these coordinate pairs on the same diagram. The sequence of points will form a slanting line on the diagram; this is the *worldline* of the bulb, line (a) in Fig. 6. At any point, this worldline will have slope $\Delta t/\Delta x = 1/v$. If an object moves at speed c, then in 1 m of time it will travel 1 m along the x-axis, so its worldline will be a line at 45° to the axes (line (b)). If an object does not move, but instead stays stationary at the position $x = 0$, it still has a worldline, lying along the t-axis (line (c)). This diagram, characteristically with the x-axis horizontal and the t-axis vertical, is a *Minkowski diagram*, and the space it describes is referred to as *Minkowski space*. idx:

Consider now a train moving along at speed v, with a flashbulb in the centre, and a mirror at each end, as illustrated in Fig. 7. Call the train's frame S', and locate its spatial

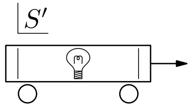

Figure 7

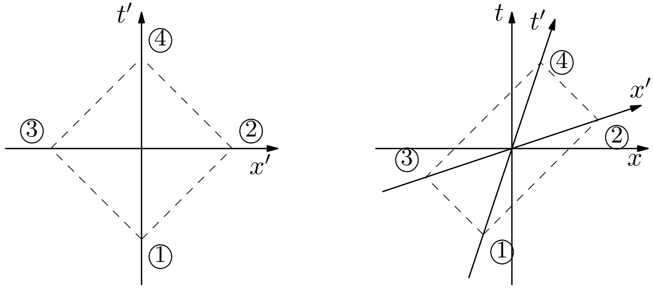

Figure 8: (a) Reflection in S' and (b) in S

origin (that is, coordinate $x'_1 = 0$) at the flashbulb. The bulb goes off at time $t'_1 = -3\,\mathrm{m}$; call this event ①. Events ② and ③ are that flash being reflected from mirrors on the x'-axis, at positions $x'_2 = -3\,\mathrm{m}$ and $x'_3 = +3\,\mathrm{m}$ (these reflections will happen simultaneously in this frame, at $t'_2 = t'_3 = 0$). Event ④ is these reflected flashes reaching the centre again, at $t'_4 = +3\,\mathrm{m}$ and $x'_4 = 0$ (all these times, remember, are obtained simply from 'distance is speed times time'). The worldline of the flashbulb is simple, and is a straight line lying along the t'-axis, like worldline (c) in Fig. 6. We can plot these on a Minkowski diagram, to obtain Fig. 8. What does this sequence of events look like on a Minkowski diagram for frame S?

Firstly, the worldline of the flashbulb is a slanted line in this frame; we know from the previous paragraph that this worldline lies along the t'-axis, so we can draw that axis in immediately. The events ① and ④ happen at the location of the flashbulb, therefore they are located on the worldline of the flashbulb an equal distance from the origin. So where are the events ② and ③? The light which travels from event ① to events ② and ③ must have a worldline which is angled at 45° (the fact that this is true *in all frames* is another statement of the second axiom); and the reflected light which travels from events ② and ③ back to event ④ must also be angled at 45°. The point where these light worldlines 'turn around' is the point where they are reflected, that is, at events ② and ③. We therefore see that the events ② and ③, which take place on the x'-axis, *do not take place on the x-axis* (that is, the events are simultaneous in the frame S' – have the same t'-coordinate – but are not simultaneous in the frame S; this is yet another illustration of the relativity of simultaneity discussed in Sect. 1.1).

The t'-axis joins the events ① and ④ in Fig. 8, and the x'-axis joins events ② and ③. This must also be true for the newly-discovered locations of these events in the S frame. This means that we can show, in Fig. 8, the positions of the four events ① to ④ as they would be measured in frame S, as well as the positions of the axes of the S' frame (the t'-axis is the worldline of the moving flashbulb, and the x'-axis is the line of simultaneity in the moving frame).

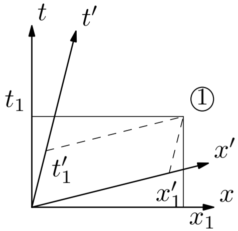

Figure 9

Even when you cannot entirely solve a problem using a Minkowski diagram, they are extremely useful when you are trying to visualise a problem, prior, perhaps, to turning it into a set of events to manipulate using the LT. Although it may seem a rather abstract way of noting down the information in a problem, with practice, you can often qualitatively solve the problem before plugging in any numbers at all.

In Fig. 8, events which happen at the same time in frame S' – that is, which have the same t'-coordinate – are connected by a line parallel to the x'-axis. This remains true in Fig. 8. Therefore in any Minkowski diagram, we can indicate the coordinates of a marked event, in each of the frames displayed in the diagram. We can see this in Fig. 9, where the event ① has coordinates (t_1, x_1) in frame S, and the *same* event has coordinates (t'_1, x'_1) in frame S'.

2.4 The invariant interval

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Figure 9 is starting to look very much like Fig. 5. In both cases, we have a single position, or event, described by two sets of coordinates in two frames. In Fig. 5, the two sets of coordinates are linked by the fact that the distance $x^2 + y^2$ is an *invariant* under the transformation which relates the two frames. That is, given any pair of points on the plane (x_1, y_1) and (x_2, y_2), we can define the separations $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$, and the distance $\Delta x^2 + \Delta y^2$ is an invariant under the transformation. Can we find a similar invariant for the transformation between frames which is represented by the Minkowski diagram, Fig. 9? It turns out that we can, and that the quantity $s^2 = \Delta t^2 - \Delta x^2$ is an invariant of the transformation which relates frames S and S'.

In principle, this is only a guess at this stage, though we are about to demonstrate (in Sect. 2.5) that this must be true, and we will show (in Sect. 3.4) that it is reasonable.

This quantity is referred to as the *interval*, or sometimes, interchangeably, as the *squared interval* or the *invariant interval*. It is also sometimes written as Δs^2, but since the interval is always a difference, the Δ is somewhat redundant. Note that if two events are further separated in space than they are in time, so that Δx^2 is greater than Δt^2, then the quantity s^2 can be negative. Do not let this disturb you: there is no significance to the quantity s which is the square root of this expression, and it is best simply to think of the typographical object s^2 as a single unparseable symbol.

Consider, for example, the events shown in Fig. 3. Let event ① be the light being emitted from the flashbulb at the bottom, and event ② the light being received there again, after its round trip. As before, we take the primed frame, S', to be fixed to the clock, and we can decide to put its space origin at the location of the bulb and its time origin at the instant the bulb flashes. This means that the coordinates of the two events are, respectively, $(t'_1, x'_1) = (0, 0)$ (by the definition of our frame's origin) and $(t'_2, x'_2) = (2L, 0)$ (this is a restatement of Eqn. (3.1) with $c = 1$, combined with the statement that ② happens at the spatial origin). The interval between these two events is therefore

$$\begin{aligned} s'^2_{12} &= \Delta t'^2 - \Delta x'^2 \\ &= \left(t'_2 - t'_1\right)^2 - \left(x'_2 - x'_1\right)^2 \\ &= (2L)^2. \end{aligned} \tag{3.6}$$

What are the coordinates of these *same* events in the frame S, through which frame S' is moving? Because we assume that the two frames are in standard configuration (Sect. 1-1.4), we know immediately that an event which happened at the (spatial and temporal) origin of frame S' happened at the origin of frame S also, so the coordinates in frame S of event ① are $(t_1, x_1) = (0, 0)$. For definiteness, let us suppose that the light clock has size $L = 1\,\mathrm{m}$ and is moving at speed $v = 2/5$. Thus the interval between the two events is, according to Eqn. (3.6), $s'^2_{12} = 4\,\mathrm{m}^2$ (this isn't in any sense an 'area', but we shall be well-behaved and include the units in numerical expressions below). Equation (3.2) tells us that $t_2 = 2.5\,\mathrm{m}$ (since $t'_2 = 1\,\mathrm{m}$ and $v = 2/5$), and Fig. 4 reminds us that $x_2 = vt_2$, or $x_2 = 1.5\,\mathrm{m}$. Putting these together, we find

$$\begin{aligned} s^2_{12} &= \Delta t^2 - \Delta x^2 \\ &= (t_2 - t_1)^2 - (x_2 - x_1)^2 \\ &= (2.5\,\mathrm{m})^2 - (1.5\,\mathrm{m})^2 \\ &= 4\,\mathrm{m}^2. \end{aligned}$$

Thus we see explicitly that the interval s^2_{12} has the *same* value when worked out using the S frame's coordinates, as it has when worked out using the S' frame's coordinates. In other words, it is an *invariant* of the transformation of the coordinates from their values in one frame, to their values in the other. Ex.3

Ex.3

2.5 Invariance of the interval

You do not have to recall the following argument in detail, but you should attempt to appreciate the overall logic. This is adapted from the argument in Schutz [1, §1.6].

To prove that the interval we have just defined is an invariant, we need two things.

First, we need to see that $\Delta s^2 = 0$ implies $\Delta s'^2 = 0$. Consider two events connected by a light signal – for example the event ① in the previous section, and the event consisting of the arrival of the light flash at the mirror. Whatever their coordinates in frame S', we know something about the *difference* in their coordinates, namely that $\Delta t'^2 = \Delta x'^2$, since this (or rather $|\Delta t'| = |\Delta x'|$) is immediately deducible from the statement that the light flash connecting them moves at speed $c = 1$; in other words, $s'^2 = 0$. But, by the second postulate, this is also true in *any* other frame, S, from which we observe this pair of events: $|\Delta t| = |\Delta x|$. That is, $s'^2 = 0$ if and only if $s^2 = 0$.

Seconly, we need to note that the coordinate differences $(\Delta t', \Delta x')$ in the primed frame are *linear* functions of the differences $(\Delta t, \Delta x)$ in the unprimed frame, say $\Delta t' = a_{tt}\Delta t + a_{tx}\Delta x$ and $\Delta x' = a_{xt}\Delta t + a_{xx}\Delta x$, where the coefficients a are unknown functions of the velocity v of S' relative to S. If this were not so – if the primed-frame differences were some more general function of the unprimed-frame differences – then we would not be free to move the origin of our frame from point to point in spacetime; since we manifestly do have this freedom, the relationship between the coordinates can be no more than linear.

It follows from the second statement that

$$\Delta s'^2 = \Delta x'^2 - \Delta t'^2 = (a_{xt}\Delta t + a_{xx}\Delta x)^2 - (a_{tt}\Delta t + a_{tx}\Delta x)^2 \tag{3.7}$$

$$\begin{aligned} &= (a_{xx}^2 - a_{tx}^2)\Delta x^2 + (a_{xt}^2 - a_{tt}^2)\Delta t^2 \\ &\quad + 2(a_{xt}a_{xx} - a_{tt}a_{tx})\Delta x\Delta t. \end{aligned} \tag{3.8}$$

This is true in general, so it will also be true in particular for the case $\Delta s^2 = 0$. This, as we know, implies both $\Delta t^2 = \Delta x^2$ and $\Delta s'^2 = 0$. Thus Eqn. (3.8) turns into

$$0 = \big((a_{xx}^2 - a_{tx}^2) + (a_{xt}^2 - a_{tt}^2)\big)\Delta x^2 + 2(a_{xt}a_{xx} - a_{tt}a_{tx})\Delta x\Delta t. \tag{3.9}$$

This will be true irrespective of the value of Δx^2, so the first coefficient must be zero; and it will be true for $\Delta t = \Delta x$ *and* for $\Delta t = -\Delta x$, so the second coefficient must be zero also. That is

$$(a_{xx}^2 - a_{tx}^2) + (a_{xt}^2 - a_{tt}^2) = 0 \tag{3.10}$$

$$a_{xt}a_{xx} - a_{tt}a_{tx} = 0. \tag{3.11}$$

Using Eqn. (3.11) to eliminate a_{xt} in Eqn. (3.10), we find

$$\begin{aligned} 0 &= a_{xx}^2 - a_{tt}^2 - a_{tx}^2\left(1 - \frac{a_{tt}^2}{a_{xx}^2}\right) \\ &= (a_{xx}^2 - a_{tt}^2)\left(1 - \frac{a_{tx}^2}{a_{xx}^2}\right). \end{aligned}$$

Thus either $a_{tx}^2 = a_{xx}^2$ or $a_{xx}^2 = a_{tt}^2$. If the former is true, then, from Eqn. (3.11) $a_{xt}^2 = a_{tt}^2$ and so, from Eqn. (3.7), $\Delta s'^2 \equiv 0$ (that is, identically zero for *all* values of Δx and Δt), which we can be sure is not the case. Therefore we must instead have $a_{xx}^2 = a_{tt}^2$ whence, from Eqn. (3.10), $a_{xt}^2 = a_{tx}^2$ and, from Eqn. (3.8), we have

$$\begin{aligned} \Delta s'^2 &= (a_{xx}^2 - a_{tx}^2)(\Delta x^2 - \Delta t^2) \\ &= \phi(\mathbf{v})\Delta s^2, \end{aligned} \tag{3.12}$$

where $\phi(\mathbf{v})$ is some as-yet unknown function of v.

Now consider the interval between two events as viewed in three frames, the original S, a frame S' moving with speed $\mathbf{v}$ relative to S, and a frame S'' moving with speed $-\mathbf{v}$ relative to S' (which is, of course, equivalent to frame S). From the above argument we must have

$$\Delta s''^2 = \phi(-\mathbf{v})\Delta s'^2 = \phi(-\mathbf{v})\phi(\mathbf{v})\Delta s^2,$$

and, since frame S'' is equivalent to S, this means

$$\Delta s^2 = \phi(\mathbf{v})\phi(-\mathbf{v})\Delta s^2,$$

or $\phi(\mathbf{v})\phi(-\mathbf{v}) = 1$. What else can we find out about the function $\phi(\mathbf{v})$?

Consider finally frames S' and S'', moving with velocities $+\mathbf{v}$ and $-\mathbf{v}$ respectively with respect to frame S, and again consider the interval between two events as observed in these three frames. We have $\Delta s'^2 = \phi(\mathbf{v})\Delta s^2$ and $\Delta s''^2 = \phi(-\mathbf{v})\Delta s^2$. Since the situation is symmetric between frames S' and S'' we must have $\phi(-\mathbf{v}) = \phi(\mathbf{v})$, so that $\phi(\mathbf{v})^2 = 1$ and $\phi(\mathbf{v}) = \pm 1$.

It cannot be the case that the sign of Δs^2 is swapped even for infinitesimally small v, so we must choose the positive solution. Therefore so $\phi(\mathbf{v}) = 1$, and so

$$\Delta s'^2 = \Delta s^2$$

for all frames S and S'. That is, the interval Δs^2 has the same numerical value, irrespective of the frame used to calculate it. idx:i inter

2.6 Length contraction and time dilation

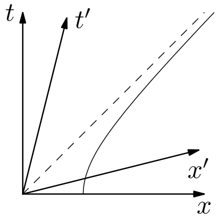

Figure 10

In Fig. 5, we drew an arc indicating the locus of points (x, y) with the same value of the invariant r. Can we do a similar thing with the invariant s^2 on the Minkowski diagram? Yes: in Fig. 10 we have drawn the locus of points for which $t^2 - x^2 = -1$. This makes very clear how different is the geometry of a Minkowski diagram from that on the plane in Fig. 5; all the points on the hyperbola in the figure are the *same* distance from the origin.

We are now ready to derive the Lorentz Transformations, but before we do that, there are a couple of other remarks we can make about Minkowski space.

idx:l
cont
idx:t
tion
idx:M

2.7 Worldlines and causality

In Fig. 9, the t'-axis indicates the locus of positions of a particle at the origin of the S' frame, as that frame moves through S. This line, which is straight in this case but would be curved for a more general motion of a point, is known as the *worldline* of the particle. As suggested above, the worldline of a photon, or anything else moving at the speed of light, always appears as a 45° diagonal line on a Minkowski diagram.

idx:v
idx:c
idx:M

Since the interval is just $\Delta s^2 = \Delta t^2 - \Delta x^2$, we can see that all the points on a photon's worldline have $\Delta s^2 = 0$, all the points below that have $\Delta s^2 < 0$, and all those above the diagonal have $\Delta s^2 > 0$. Such separations are termed, respectively, *null* (or *lightlike*), *spacelike* and *timelike* separations.

Ex.3

idx:t
sepa
idx:s
sepa
idx:l
sepa

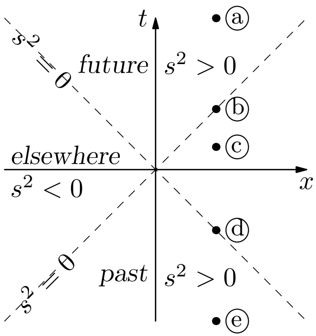

Figure 11

We can illustrate these terms in Fig. 11. Events ⓐ and ⓔ have timelike separations from the origin, events ⓑ and ⓓ have null separations from the origin, and event ⓒ has a spacelike separation from the origin. We can immediately categorise the five events in terms of the type of separation between the event and an event at the origin:

- ⓐ timelike separation from the origin, a slower-than-light signal could travel from the origin to ⓐ;
- ⓑ null separation, only a speed-of-light signal could make it from the origin to ⓑ;
- ⓒ spacelike separation, no signal could travel between the origin and ⓒ;
- ⓓ null separation, a speed-of-light signal could travel from ⓓ to the origin;
- ⓔ timelike separation, a slower-than-light signal could get from ⓔ to the origin.

Although we are familiar with time being simply divided into a future and a past, we can see from this Minkowski diagram that spacetime is divided into *three* regions, the familiar 'future' and 'past', plus a region termed 'elsewhere', consisting of events which cannot interact causally with events at the origin. Ex.3

3 The Lorentz transformation

In Sect. 1.2 we saw how observers could make measurements of lengths and times in mutually moving frames, and reasonably disagree about the results – the phenomena of length idx:I contraction and time dilation. In Sect. 1.3, we were able to put numbers to this and derive a quantitative relation, Eqn. (3.2), between the duration of a 'tick' of the light clock as measured in two frames. We want to do better than this, and find a way of relating the coordinates of any event, as measured in any pair of frames in relative motion. That relation – a transformation from one coordinate system to another – is the *Lorentz transformation* (LT) (the derivation in Sect. 3.1 has a lot in common with the accounts given in [2] and [3]).

3.1 The derivation of the Lorentz transformation

Consider two frames in standard configuration, and imagine an event such as a flashbulb going off; observers in each of the two frames will be able to measure the coordinates of this event. Those observers will of course produce different numbers for those coordinates – they will disagree about the precise time and location of the event – with those in frame S producing coordinates (t, x, y, z), and those in S' producing (t', x', y', z'). It is our task now to calculate the relationship between those two sets of numbers.

idx:s
conf
Ex.3

First of all, we can note that $y' = y$ and $z' = z$, since this is just an restatement of the lack of a perpendicular length contraction, as discussed in Sect. 1.3. Therefore we can unproblematically make things easier for ourselves by supposing that the event takes place on the x-axis, so that $y = z = 0$ (problem: construct for yourself the argument that there is no loss of generality here). Therefore our problem has diminished into that of obtaining (t', x') for this event, given (t, x).

Now imagine a second event, located at the origin with coordinates $(0, 0)$ in frame S; since the frames are in standard configuration, we can immediately deduce that $(0, 0)$ in frame S' also. Since we have two events, we have an interval between them, with the value $s^2 = (t - 0)^2 - (x - 0)^2 = t^2 - x^2$ in frame S. Since the interval is frame-independent (which is what Sect. 2.5 showed), the calculation of this interval done by the observer in the primed frame will produce the same value:

$$t^2 - x^2 = s^2 = t'^2 - x'^2. \tag{3.13}$$

Thus the relationship between (t, x) and (t', x'), which we already know must be linear (Sect. 2.5), must additionally be one for which Eqn. (3.13) is true.

Equation (3.13) is strongly reminiscent of Eqn. (3.5), and we can make it more so by writing $l = \mathrm{i}t$ and $l' = \mathrm{i}t'$, so that Eqn. (3.13) becomes

$$l^2 + x^2 = -s^2 = l'^2 + x'^2. \tag{3.14}$$

This strongly suggests that the pairs (l, x) and (l', x') can be related *via* the analogue of Eqn. (3.4), so that

$$x' = x\cos\theta + l\sin\theta \tag{3.15a}$$
$$l' = -x\sin\theta + l\cos\theta, \tag{3.15b}$$

for some angle θ, which depends on v, the relative speed of frame S' in frame S. That is, this specifies a linear relation for which Eqn. (3.13) is true. If we finally write $\theta = \mathrm{i}\phi$ (since l is pure imaginary, so is θ, so that ϕ is real), and recall the trigonometric identities $\sin\mathrm{i}\phi = \mathrm{i}\sinh\phi$ and $\cos\mathrm{i}\phi = \cosh\phi$, then

$$x' = x\cosh\phi - t\sinh\phi \tag{3.16a}$$
$$t' = -x\sinh\phi + t\cosh\phi. \tag{3.16b}$$

A further thing we know is that the coordinates of (an event at) the origin of the moving frame are $(t', x' = 0)$ and $(t, x = vt)$, for all t, in which case Eqn. (3.16a) can be rewritten as

$$\tanh\phi(v) = v. \tag{3.17}$$

Since we now have ϕ as a function of v, we now have, in Eqn. (3.16), the full transformation between the two frames; but combining Eqn. (3.16) and Eqn. (3.17) with a little hyperbolic trigonometry (remember $\cosh^2\phi - \sinh^2\phi = 1$), we can rewrite Eqn. (3.16) in the more usual form

$$t' = \gamma(t - vx) \tag{3.18a}$$
$$x' = \gamma(x - vt). \tag{3.18b}$$

Above, we deduced the trivial transformations for y and z, which we repeat here to complete the Lorentz transformation

$$y' = y \tag{3.18c}$$
$$z' = z. \tag{3.18d}$$

Here γ is (as in Eqn. (3.3) but now with $c = 1$) idx:g

$$\gamma = \left(1 - v^2\right)^{-1/2}. \tag{3.19}$$

A more direct, but less physically illuminating, route to the LT is to note that the transformation from the unprimed to the primed coordinates must be linear, if the equations of physics are to be invariant under a shift of origin. That is, we must have a transformation like $t' = Ax + By + Cz + Dt$, and similarly (with different coefficients) for x', y' and z'. By using the Relativity Principle and the constancy of the speed of light, one can deduce the transformation given in Eqn. (3.18). See Rindler [2, §2.6], Taylor & Wheeler [4, §§L.4–5], or Barton [5, §4.3] for details. Rindler [2, §2.17] shows an even more powerful consequence of the same ideas.

If frame S' is moving with speed v relative to S, then S must have a speed $-v$ relative to S'. Swapping the roles of the primed and unprimed frames, the transformation from frame S' to frame S is exactly the same as Eqn. (3.18), but with the opposite sign for v:

$$t = \gamma(t' + vx') \tag{3.20a}$$
$$x = \gamma(x' + vt'), \tag{3.20b}$$

which can be verified by direct solution of Eqn. (3.18) for the unprimed coordinates. Ex.3.

3.2 Addition of velocities

Adding and subtracting the expressions in Eqn. (3.16), and recalling that $e^{\pm\phi} = \cosh\phi \pm \sinh\phi$, we find idx:v of

$$t' - x' = e^{\phi}(t - x) \tag{3.21a}$$
$$t' + x' = e^{-\phi}(t + x), \tag{3.21b}$$

as yet another form (once we add $y' = y$ and $z' = z$) of the LT.

Consider now three frames, S, S' and S''. If S and S' are in standard configuration with relative velocity v_1, and S' and S'' are in standard configuration with relative velocity v_2 (implying that $\mathbf{v}_1$ is parallel to $\mathbf{v}_2$), then frames S and S'' will also be in standard configuration with some third velocity v; we obviously cannot presume that $v = v_1 + v_2$, as it would be under a Galilean transformation. However, applying Eqn. (3.21a) twice shows us that

$$t'' - x'' = e^{\phi}(t - x) = e^{\phi_1+\phi_2}(t - x), \tag{3.22}$$

where ϕ, ϕ_1 and ϕ_2 are the hyperbolic velocity parameters corresponding to v, v_1 and v_2. This shows us how to add velocities: Eqn. (3.17) plus a little more hyperbolic trigonometry ($\tanh(\phi_1 + \phi_2) = (\tanh\phi_1 + \tanh\phi_2)/(1 + \tanh\phi_1 \tanh\phi_2)$) produces

$$v = \frac{v_1 + v_2}{1 + v_1 v_2}. \tag{3.23}$$

Ex.3

The form of the LT shown in Eqn. (3.21), and the addition law in Eqn. (3.22), conveniently indicate three interesting things about the LT: (i) for any two transformations performed one after the other, there exists a third with the same net effect (ie, the LT is 'transitive'); (ii) there exists a transformation (with $\phi = 0$) which maps (t, x) to themselves (ie, there exists an identity transformation); (iii) for every transformation (with $\phi = \phi_1$, say) there exists another transformation (with $\phi = -\phi_1$) which results in the identity transformation (ie, there exists an inverse). These three properties are enough to indicate that the LT is an example of a mathematical 'group', known as the 'Lorentz group'.

Any two IFs, not just those in standard configuration, may be related via a sequence of transformations, namely a translation to move the origin, a rotation to align the axes along the direction of motion, a LT, another rotation, and another translation. The transformation which augments the LT with translations and rotations is known as a *Poincaré transformation*, and it is a member of the *Poincaré group*.

3.3 The invariant interval and the geometry of spacetime

Is the interval merely a mathematical curiosity? No; it is the key to a profound reappraisal of our picture of space and time. idx:i

The LT as represented by Eqn. (3.16) *looks* like a rotation – indeed like the rotation which inter Eqn. (3.15a) would represent were l and θ real. The hyperbolic functions and the pattern of signs tell us that it is not exactly the same, but the similarities are instructive.

A crucial – indeed the defining – feature of a rotation in a plane is that it preserves Euclidean distance. If you have a point (x, y) in the plane, and you rotate the axes so that the same point is assigned with coordinates $(x', y') = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$, then you know that the distance $x^2 + y^2$ will be equal to $x'^2 + y'^2$. There's nothing which marks out one pair (x, y) as more fundamental in any sense than another pair (x', y'), nor even any important difference between the x- and y-axes – they're just a pair of directions on the plane. The only fixed thing here is the Euclidean distance.

In the same way, the crucial feature of the LT is that it preserves the interval Δs^2. Just as in the previous paragraph, there is nothing fundamental about the particular pair of coordinates (t, x) which our clocks and measuring rods pick out for us; the coordinates (t', x') which are natural for another, moving, observer, are just as fundamental. Similarly, and astonishingly, there is no real difference between the t- and x-axes – they're just different directions in the relevant geometrical space. This is really an astonishingly radical view of space and time, and drove Minkowski to remark [6] idx:M

> The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality. idx:s

In the Newtonian ontology, Space was a three-dimensional geometrical space, which idx:N obeyed Euclid's geometrical axioms, and specifically preserved the Euclidean length. Time of sp entered in a simple way: history consisted of this three-dimensional space moving gracefully through Time, with each instant of history having its 'own' space associated with it. SR teaches us that the world is not like this after all: instead we live in a *four*-dimensional space, with three Space dimensions and one Time dimension – *Spacetime.* Spacetime does not obey Euclidean geometry, since a 'rotation' in Spacetime (ie, a Lorentz Transformation) preserves, not Euclidean distance, but the interval Δs^2. The geometry of Spacetime contains a great deal more structure than this, which you will learn much more about if you study General Relativity.

3.4 Proper time and the invariant interval

In some frame S, consider two events, one at the origin, and one on the x-axis at coordinates (t, x). That is, these two events are separated by a distance x and by a time t in this frame, but observers in a different frame would measure *different* separations between the events, so there is nothing fundamental to the numbers x and t obtained in this frame.

Consider now a clock moving at a constant velocity v, chosen so that it is present at both events (we assume for the moment that the two events are close enough that this is possible for a clock moving at a sub-light speed). Of all the time and space coordinates which could be used to describe the motion of this clock, there is one we can pick out as special: the frame, S'', in which the clock is at rest at the origin. In this frame, the clock's position is $x'' = 0$, both events happen at the origin, and the time coordinate, t'', is the time which the clock actually shows on its face. We write this particular time coordinate as τ, and call it the *proper time*. Given any other frame (in standard configuration) in which the clock has coordinates (t, x), we can use Eqn. (3.18a) to transform to the clock's rest frame and write idx:p
time

$$\tau = \gamma(t - vx). \tag{3.24}$$

This calculated proper time would be agreed on by all the observers who could not agree on the spatial or temporal separations of the two events. In other words, this number τ is invariant under a LT – it is a *Lorentz scalar*. idx:L

Noting that the velocity required for this scenario is simply $v = x/t$, we can put this velocity into Eqn. (3.24) to find $\tau^2 = t^2 - x^2$. But since this is true of any frame S, it is also true of a third frame S', so that

$$\tau^2 = t^2 - x^2 = t'^2 - x'^2, \tag{3.25}$$

agreed upon by observers in every frame, even though they have quite different values of the separations as measured in their own frames.

We can generalise this slightly in several ways. Firstly, since frames are isotropic[1], we can generalise this to events not on the x-axis, and replace the term x^2 by $x^2 + y^2 + z^2$; secondly, since the frames are also homogeneous and the LT is linear, we can replace the coordinates (t, x, y, z), which are distances from the origin, by separations $(\Delta t, \Delta x, \Delta y, \Delta z)$ between arbitrary pairs of events; thirdly, we can formally generalise this to the case where τ^2 is negative – that is, to the case where no clock could move fast enough to be at both the events in question – and define $s^2 = \tau^2$. We end up with

$$s^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2. \tag{3.26}$$

This is the interval which we defined in Sect. 2.4: though the definition in that section was logically a guess, the proof that it was an invariant of the LT, in Sect. 2.5, showed that it was consistent to use this, and the rederivation in this section has shown, further, that it is natural.

The pattern of coefficients $(+1, -1, -1, -1)$ defines a *distance* between two events separated by coordinate separtions $(\Delta t, \Delta x, \Delta y, \Delta z)$. It is the first appearance of the *metric* of General Relativity. idx:r

Some authors (Schutz for example) define the interval with the opposite sign, as $s^2 = x^2 + y^2 + z^2 - t^2$. This is logically self-consistent, as long as suitable changes are made to the definitions of 'spacelike' and 'timelike' above. The sum of the signs of the terms is known as the *signature*, so that Eqn. (3.26), with signs $(+, -, -, -)$, has signature -2, and the above alternative definition, with signs $(-, +, +, +)$, has signature $+2$. idx:g relat

[1]The term *isotropic* means 'the same in all directions', indicating that there is no preferred direction; and the term *homogeneous* means 'the same at all points', meaning that there are no special points. For example, a uniform electric field would be homogeneous but not isotropic, since there is a preferred direction, towards higher or lower voltages, but this preferred direction can be detected at all points.

3.5 Applications of the LT

The applications of the Lorentz Transformations are basically all the same: given a set of coordinates in one frame, what are the coordinates of the same event or events in another, moving, frame.

3.5.1 Length contraction

A rod of length L_0 lies along the x'-axis of frame S'; what is its length as measured in the S frame? idx:l

Lay the rod along the x'-axis so that one end is at the origin. Let the two frames be in standard configuration. At time $t = 0$, the S observers who are opposite the ends of the rod let off two bangers – we can find the length of the rod by considering the coordinates of these two events in the two frames. The two events happen at t-coordinates $t_1 = t_2 = 0$, and x-coordinates $x_1 = 0$ and $x_2 = L$ (this is the definition of the length of the rod, L, as measured in the stationary frame – see Sect. 1-1.3). Since the origins were coincident at $t = t' = 0$, we know that $x_1 = x_1' = 0$. From Eqn. (3.20), we can write down that cont

$$x_2 = \gamma\left(x_2' + vt_2'\right) \tag{3.27a}$$

$$0 = t_2 = \gamma\left(t_2' + vx_2'\right). \tag{3.27b}$$

Eqn. (3.27b) tells us that $t_2' = -vx_2'$; substituting this into Eqn. (3.27a) and writing $x_2 = L$ and $x_2' = L_0$, we have

$$L = \gamma\left(1 - v^2\right)L_0 = \frac{L_0}{\gamma}, \tag{3.28}$$

showing that the moving rod is measured, by observers in the stationary frame, to be shorter than its 'rest length' L_0. Compare Sect. 1.2. Ex.3

3.5.2 A worked example: The trains again

As in Sect. 1.1, consider a train moving past a station platform, this time at a specific speed $v = 0.5$ (this is not the speed of the carriage in Fig. 1 – see also Example 3.13); the carriage is 6 m long, and a flashbulb is set off in the centre of the carriage at time $t' = 0$. (i) What time does the flash reach the back of the carriage as measured in the carriage? (ii) And as measured on the platform? (iii) At the instant when the flash reaches the back of the carriage, what is the location of the front of the wavefront, as measured on the platform? (iv) And as measured in the carriage? (v) Where is the front of the carriage at this instant (measured in the platform's frame)? (vi) So what is the reading, at this instant, on the front clock?

First, we have to settle on the frames we want to use. Let the platform's frame be S and the carriage's be S', with the two frames in standard configuration and the origin of the S' frame at the centre of the carriage.

(i) This one's easy. The flash goes off at time $t' = 0$ and moves backwards and forwards at speed $c = 1$. The back of the carriage is at coordinate $x'_1 = -3\,\text{m}$, so the flash reaches the back at time $t'_1 = 3\,\text{m}$. Call this event 1.

(ii) The inverse LT is, from Eqn. (3.20), $t = \gamma(t' + vx')$, $x = \gamma(x' + vt')$, where $\gamma = 1/\sqrt{1 - v^2} = 1/\sqrt{1 - \frac{1}{4}} = 2/\sqrt{3}$. Therefore we have $x_1 = -\sqrt{3}\,\text{m}$, $t_1 = +\sqrt{3}\,\text{m}$.

(iii) Also easy. The flash started at the origin $x = 0$ of the S frame at time $t = 0$ and moves at speed $c = 1$, so at time $t_2 = t_1 = \sqrt{3}\,\text{m}$ the wavefront must be at position $x_2 = +\sqrt{3}\,\text{m}$. Call this event 2.

(iv) We need the coordinates of event 2 in the moving frame. From Eqn. (3.18), $x'_2 = \gamma(x - vt) = 1\,\text{m}$ and $t'_2 = \gamma(t - vx) = 1\,\text{m}$ (since the flash is moving at speed $c = 1$, we must have $x'_2 = t'_2$, so this result is reassuring).

(v) The front of the carriage is at (constant) coordinate $x' = L' = 3\,\text{m}$ in the S' frame and so at coordinate $x = L = L'/\gamma$ at time $t = 0$ (by length contraction or from Eqn. (3.18b)). Therefore the front of the carriage, moving at speed v, is afterwards at coordinate $x = L + vt$. At time $t_3 = t_2 = \sqrt{3}$, the coordinate of the front will therefore be $x_3 = (3\,\text{m})/\gamma + \sqrt{3}/2 = 2\sqrt{3}$ (event 3).

(vi) Using the result of part (v), we immediately obtain $t'_3 = \gamma(t_3 - vx_3) = 0$, from Eqn. (3.18a). Alternatively, we have $t_3 = \sqrt{3}\,\mathrm{m}$ and $x'_3 = 3\,\mathrm{m}$ and want t'_3; we can therefore rearrange Eqn. (3.20a) to find $t' = t/\gamma - vx'$ and get the same result without going via step (v).

Note that we used the LT only in parts (ii), (iv) and (vi). In parts (i), (iii) and (v) we used nothing more sophisticated than ‘distance is speed times time’, plus a bit of careful thought about what distance and what time we meant. Ex.3

Ex.3

idx:I

4 Paradoxes

SR has numerous paradoxes – situations where a relativistic analysis leads to conclusions which seem absurd, but which are nonetheless true. Resolving these paradoxes delivers substantial insights into relativistic arguments. If you *really* understand why the two following 'paradoxes' are in fact perfectly reasonable, you will have made considerable progress in understanding the rest of the course.

4.1 The twins

This is a famous one. Imagine two intrepid scientists, Odysseus and Penelope. Odysseus idx:p leaves Earth in a spaceship to travel to a star 45 light-years away, and does so at a speed $v = 0.9$; then turns round and comes back. How long is he away for as timed by Penelope on earth, and how long in the frame of the spaceship? If he travels a total distance of 90 light-years at a speed $v = 0.9$, then the journey will take 100 years as measured by Penelope. However, we know that the time measured on board the spaceship will be less, according to Eqn. (3.2) above: for $v = 0.9$, we have $\gamma = 2.29$, and the journey will take only 43.6 years for Odysseus, who will consequently be that much younger than Penelope when he returns to Earth.

That seems a little odd, but we're used to peculiarity in SR, now. However, some bright spark then points out that, relative to Odysseus, it is *Penelope* who has been moving, so shouldn't the whole situation be symmetrical, just as with the trains in Sect. 1.1 above, forcing us to conclude that Penelope will be younger than Odysseus? This is nonsensical – although two trains can be mutually measured each to be shorter than the other, two clocks (Odysseus and Penelope) can't logically each be showing less time than the other.

The paradox is resolved as soon as we point out that Odysseus *cannot* conclude that it is Penelope that is moving, because only Odysseus experiences the change of direction at the distant star – SR discusses the relationships between inertial frames, and Odysseus does not (indeed cannot) remain in a single inertial frame for the entire round trip.

This becomes clearer when we consider the Minkowski diagram. Before that, imagine sailing between two points, say from Troy to Ithaca. Your direct route would follow the y-axis in Fig. 12. Contrast this with a route which takes a diversion, such as the dog-leg route in the figure. Ignoring the detail of the Earth's curvature, we can take this to be a flat Euclidean surface, on which surface it is possible to prove that a straight line is the shortest distance between two points, so that the direct route to Ithaca is, of course, shorter than the indirect one. If, in order to measure the distance sailed, you towed a propellor behind you through the wine-dark sea, it would turn a smaller number of times if you took the direct route, than if you

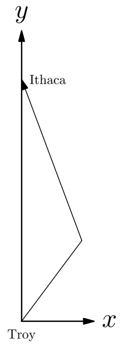

Figure 12

took the indirect one. So far, so obvious.

Contrast this with Fig. 13, which is the path which Odysseus takes through spacetime. It appears that Odysseus's route is the longer, but remember that the plane of the Minkowski diagram is *not* a Euclidean surface, so that our intuitions about lengths and angles are not reliable. It is possible to show, in fact, that a straight line is the *longest* distance between two points in Minkowski space, so that Penelope, taking the straight route from O to X, has travelled a greater distance through spacetime than has Odysseus. Since distance through spacetime, s^2, is the same as the proper time τ^2 between events (Sect. 3.4), we can say that the proper time along Penelope's path is greater than the proper time along Odysseus's, which is to say that a clock carried with her will show a greater time elapsed than one carried by Odysseus, which is to say that Penelope is older than Odysseus when they re-meet at X.

There is an alluring blind alley here, prompted by the presence of the acceleration at the turning-point, even prompting some folk to insist that the 'Twins Paradox' needs GR to resolve it. In fact, there is no need to talk about any actual acceleration: rather than actually turning

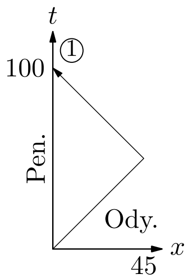

Figure 13

round at the star, Odysseus could simply synchronise the clock of another traveller he meets there, who is already travelling at the right speed in the return direction. Our conclusion would then be to do with the total elapsed time on the two legs of the journey rather than one miraculously unaged traveller, but this is exactly the same conclusion as above, merely in a less vivid form.

4.2 The pole in the barn

A farmer with a 20 m ladder holds it horizontally and runs toward a barn which is 10 m deep. idx:p
The farmer's wife, standing by the barn door, sees him running at a speed at which $\gamma = 2$. The in th
ladder is therefore length-contracted to have a measured length of 10 m in the barn's frame, so that the ladder will fit entirely into the barn, and the farmer's wife can slam the door behind him, with the '20 m' ladder entirely (and briefly!) within the 10 m barn (in other words, the farmer's wife is 'measuring the length of the ladder' using essentially the procedure described in Sect. 1-1.3).

However, in the farmer's frame, the barn is rushing towards him with the same γ-factor, so that the barn is measured to have a length of only 5 m, making it even more outlandish that the ladder should fit in the barn.

The reason this sounds so bizarre is that we are forgetting the central insight of this part of the course: spatially separate events which are simultaneous in one frame will not be simultaneous in another. When we say 'the ladder is entirely within the barn' we *mean* something like 'there is at least one pair of events, located at each end of the ladder while those ends are within the barn, which are simultaneous in the frame of the barn'. However, this pair of events are *not* simultaneous in the frame of the ladder. In other words, the paradox idx:s
comes about because of the seemingly innocuous remark 'the ladder will fit entirely into the barn' which we smuggled into the the description above; this remark is meaningful *only* when it is associated with a particular reference frame: it is true within the farmer's wife's frame, but is simply not true within the farmer's.

We can look at the problem another way, by asking the question 'how does the end of the ladder know when to stop?' We cannot assume that the ladder is completely rigid; thus when the front of the ladder hits the back wall of the barn, the shock wave – the *information* that this has happened – takes a finite time to make it to the back of the ladder, so that the back of the ladder keeps moving inwards even after the front of the ladder has halted at the back wall. Very shortly after the barn door has been slammed shut, the ladder's recoil will smash through the door.

What this example shows is that any conclusion which you correctly reach in one frame must be reachable in any other frame, even though the detailed mechanisms might be different.

Example 3.15, though it is framed in different language, works through this problem in illuminating detail. Ex.3

Questions

What would happen if the trains in Fig. 1 were moving in the same direction, but at different speeds? Exactly the same – all that matters is *relative* motion. If you wanted, you could consider the motion of the trains as seen from an inertial frame moving at the average speed of the trains: from *that* frame, the motion would look exactly like Fig. 1 again.

I thought events 'light reaches back at time 3' and 'light reaches front at time 3' were simultaneous, not times '3' and '1'. They *are* indeed simultaneous, *in the carriage frame* – the first paragraph in Sect. 1.1 shows that. However, the next paragraph shows that *in the platform frame* they cannot possibly be simultaneous. This is startling, but this is what the section is all about: two spatially separate events which are simultaneous in one frame, cannot possibly be simultaneous in another frame moving relative to the first. Whenever you see the word 'simultaneous', you should automatically ask 'simultaneous for whom?'.

Won't the folk in the carriage measure the carriage getting shorter, so that the light will no longer take three units to get to the end? Remember that only *relative* motion is detectable: since there is no sense in which the carriage is 'really stationary' or 'really moving', any experiment (which includes measurements) purely within the carriage must get the same results in both cases. The other way to think about this is that if the folk in the carriage could measure any change in length, they would be able to tell they were moving, violating the first postulate.

What are the units of the speed v? The units of a speed are the units of distance divided by the units of time. Normally, this is metres divided by seconds, but in relativity we decide to measure time in units of metres (we could call these units 'relseconds', say, and define one 'relsecond' to be $1/(3 \times 10^8)$ s, but we're as well just renaming that 'relsecond' as 'metre'), so that metres divided by metres is a pure number. Another way of thinking about this is that v in relativity is measured as a fraction of the speed of light.

Is ϕ in Eqn. (3.16) a rotation? No: although I've deliberately made it *look* like a rotation, in order to highlight the similarity between Eqn. (3.16) and Eqn. (3.15a), there's no (intuitive, spatial) rotation involved. The quantity ϕ is merely a parameter of the transformation between two frames in standard configuration.

The twins paradox is dispelled by saying 'Odysseus knows he's moving'. Doesn't this violate the first postulate? You have to be very careful about the precise statement of the postulate – 'you can't tell you're moving' is rather woolly, and needs to be completed by '…in an inertial frame'. During both the outbound and return journeys, Odysseus is in an inertial frame, and both he and Penelope could equally justifiably regard themself as being stationary. However, while Penelope stays in a single inertial frame for her entire journey through spacetime between the departure and the return, Odysseus *cannot* stay in a single inertial frame for both legs of the journey; he *must* switch frames at the turnround point, either by slowing down and accelerating in the opposite direction, or perhaps by jumping ship, onto another spaceship handily heading earth-wards. This change of frames is the asymmetry, since there can be no doubt which of Odysseus and Penelope has changed frames.

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Examples

Example 3.1 (section 1.2)

Which of the following statements are true?

1. Fred and Barbara's watches stay synchronized with each other.
2. Fred and Barbara's watches stay synchronized with the clocks in the other carriage.
3. Fred and Barbara measure their carriage to get shorter when they're moving.
4. Fred and Barbara measure the other carriage to get shorter when it's moving relative to them.
5. Fred and Barbara measure the speed of light in the other carriage to be less than c.
6. By measuring the Doppler shift of a light signal sent from the back of the carriage to the front, the two observers can determine the carriage's velocity to any desired accuracy.

Example 3.2 (section 1.3)

Repeat this analysis for a ‘tennis-ball clock’. Balls are fired horizontally across a carriage (moving at a non-relativistic speed): the time it takes them to bounce back to the starting point is one ‘tick’ of the clock. What is the crucial point at which the two analyses diverge?

Example 3.3 (section 1.3)

Consider the following argument.

Consider two frames S and S' in standard configuration; there is a rod of length L_0 laid along the x' axis with one end at the origin. What is the length of this rod, L, as measured in the S frame?

Imagine our line of observers stretching along the x axis, all with synchronized clocks (showing t). At time $t = 0$, when the origins of the two frames coincide (so that one end of the rod is at coordinate $x = 0$), precisely one of the observers in the S frame will be standing opposite the far end of the rod, located at coordinate $x' = L_0$ in the S' frame; this observer has coordinate $x = L$.

Let us arrange for a banger to go off at the origin at time $t = t' = 0$: travelling at speed c, the light from this will reach our observers at coordinates $x = L$ and $x' = L_0$ at times $t = L/c$ and $t' = L_0/c$. But, from Eqn. (3.2), we know that any two events separated by time t' in the moving frame would be measured to be separated by time $t = \gamma t'$ in the stationary one. Thus, the time intervals L/c and L_0/c are related, and so

$$L = \gamma L_0 \qquad \text{[WRONG!]}.$$

This conclusion is *wrong*, as comparison with Eqn. (3.28) shows, so the argument is fallacious. What's wrong with it?

Example 3.4 (section 2.1)

Convert the following to units in which $c = 1$: (a) 10 J; (b) lightbulb power,100 W; (c) Planck's constant, $\hbar = 1.05 \times 10^{-34}\,\mathrm{J\,s}$; (d) velocity of a car, $v = 30\,\mathrm{m\,s^{-1}}$; (e) momentum of a car, $3 \times 10^4\,\mathrm{kg\,m\,s^{-1}}$; (f) pressure of 1 atmosphere, $10^5\,\mathrm{N\,m^{-2}}$; (g) density of water, $10^3\,\mathrm{kg\,m^{-3}}$; (h) luminosity flux, $10^6\,\mathrm{J\,s^{-1}\,cm^{-2}}$.

Convert the following to physical units (SI): (i) velocity, $v = 10^{-2}$; (j) pressure $10^{19}\,\mathrm{kg\,m^{-3}}$; (k) time $10^{18}\,\mathrm{m}$; (l) energy density $u = 1\,\mathrm{kg\,m^{-3}}$; (m) acceleration $10\,\mathrm{m^{-1}}$; (n) Eqn. (3.18), $t' = \gamma(t - vx)$; (o) the 'mass-shell' equation $E^2 = p^2 + m^2$. (Example slightly adapted from Schutz [1, ch.1])

Example 3.5 (section 2.4)

Use the same method to show that the quantity $\Delta t^2 + \Delta x^2$ is *not* conserved; that is that $\Delta t^2 + \Delta x^2 \neq \Delta t'^2 + \Delta x'^2$.

Example 3.6 (section 2.4)

The demonstration of the invariance of the interval between successive ticks of the light clock, in the argument surrounding Eqn. (3.6), was worked out for specific values of the distance L and velocity v. Show that it is true in general.

Example 3.7 (section 2.7)

The usual train travels through the usual station at a speed $v = 1/5$, in units where $c = 1$. The tea urn in the buffet car explodes $500\,\mathrm{m}$ in time after the train has passed the station, when the train is $100\,\mathrm{m}$ further down the track (both distances measured in the station's frame). Draw a Minkowski diagram, indicating on it the worldlines of the station and the train, and the explosion event. At what time is the explosion visible at the station? [degree exam, 2001] (Objective 5)

Example 3.8 (section 2.7)

Define the squared interval, s^2, between two events. Consider three events, with coordinates $(t_1, x_1) = (1, 1)$, $(t_2, x_2) = (6, 4)$ and $(t_3, x_3) = (4, 6)$ (take $c = 1$). Calculate the intervals s_{12}^2, s_{13}^2 and s_{23}^2 between the three pairs of events, state whether each is timelike, spacelike or null, and in each case whether the earlier event could influence the later event through a signal travelling no faster than light [degree exam, 2001]. (Objective 4)

Example 3.9 (section 3.1)

With respect to Fig. 1, consider the following two frames. Frame S is attached to the top carriage and has its spatial origin $x = 0$ at the centre of the top carriage; frame S' is attached to the bottom carriage and has its spatial origin $x' = 0$ at the centre of the bottom carriage. Sketch the position of these carriages/frames at time $t = 0$. Are these frames in standard configuration? Can we use the Lorentz Transformation to relate the coordinates of these frames?

Example 3.10 (section 3.1)

A disco-spaceship cruises past the earth travelling at speed $v = \sqrt{3}/2$. The carefree young things on board are relaxing to music at 120 beats-per-minute (that is, 0.5s between beats). Explain how you would use a network of observers to measure the length of the spaceship and the interval between beats in the Earth frame. [5]

Use the time dilation formula to obtain the interval between beats, as measured on Earth. [5]

The Interplanetary Noise Abatement Society's rapid response unit sends a signal to the disco-ship 10^{10} m (a little under a minute) after the ship passes, as measured on Earth; the music stops at a time 3×10^{10} m after the flyby, as measured on the ship. Sketch these two events on a Minkowski diagram, obtain their coordinates, and calculate the invariant interval between them. State whether the end of the music could be attributed to the signal from the Noise Abatement Society. [10]

Example 3.11 (section 3.2)

What happens as we take the velocity v towards zero – the non-relativistic velocities of our everyday experience? From Eqn. (3.19), what is the limit of γ as v tends to zero? What, therefore, is the form of the LT, Eqn. (3.18) in this slow-speed limit? Do you recognise this?

From Eqn. (3.23), how do velocities add in this limit, where either v_1 or v_2 is small compared with $c = 1$? And what happens if one of the velocities is already the speed of light?

Example 3.12 (section 3.5.1)

The LT can be used to obtain the length-contraction formula (Eqn. (3.28)) directly. Use a similar argument to obtain the time-dilation result of Eqn. (3.2), using the LT of Eqn. (3.18). (Objective 3)

Example 3.13 (section 3.5.2)

What is the speed of the trains in Fig. 1? Remember that the train carriage is 6 m long, and that the clocks are showing times in units of metres. Hint: what two events are simultaneous in the platform frame?

Further, show that the time shown in Fig. 2 is correct. (Objective 3)

Example 3.14 (section 3.5.2)

The length contraction formula is

$$L = L'/\gamma,$$

and the Lorentz transformation is

$$\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt). \end{aligned}$$

Explain *precisely* the meaning of the symbols in these expressions, stating also the relationship between the two frames related by the Lorentz transformation.

A train enters a tunnel travelling at a speed $3/5$ (in units where $c = 1$). The tunnel is 500 m long, and the train 100 m long, both as measured in their rest frames. The train has a clock at the extreme front and back, which are synchronized in the train's frame. The rear clock is observed to show time 0 m at the instant the rear of the train disappears into the tunnel. The front clock is also observed when it emerges from the other end of the tunnel.

Draw a Minkowski diagram showing the train's motion, including the worldlines of the clocks and the tunnel ends, and the three events consisting of ① the rear clock disappearing into the tunnel, an event ② at the front of the train, simultaneous with ① in S', and an event ③ at the point where the front clock emerges from the tunnel.

By using the Lorentz Transformation and/or the length-contraction formula, or otherwise, deduce the time on the front clock when it emerges from the tunnel. [Degree exam, September 2001] (Objective 1, Objective 3, Objective 5)

[Note: This question has been mildly rewritten from the version which appeared in the first distributed version of the notes]

Example 3.15 (section 4.2)

Take two cars, moving along the x-axis at a speed v, separated by 1 km in their frame. The lead car passes first a checkpoint and then, 0.5 km further on in the checkpoint frame, a traffic policeman.

Sketch a Minkowski diagram of these events, indicating at least the worldlines of the cars, the checkpoint and the policeman, and marking the events ①, the lead car passing the checkpoint, ②, the position of the second car at the same time as ① in the cars' frame, ③, the position of the second car at the same time as ① in the checkpoint frame, ④, the second car reaching the checkpoint, and ⑤, the first car reaching the policeman. [It will probably help if you choose the frames such that event ① has coordinates $x_1 = t_1 = x'_1 = t'_1 = 0$, but if you want to do it another way, that's fine.]

Give expressions for the coordinates of these events (in either S or S' as appropriate), in terms of the speed v.

Give an expression for the interval s^2_{45} between events ④ and ⑤, and calculate numerical values for this in the cases (i) $v = 1/2$, (ii) $v = 3/5$ and (iii) $v = 4/5$. In each of the three cases, state, with an explanation, whether it is possible for the traffic policeman to signal to the checkpoint to lower a barrier before the second car arrives.

[Optional extra: draw Minkowski diagrams for the three cases (i), (ii) and (iii), which illustrate the answers you obtain above. This might illuminate both the answer to this question, and the pole-in-the-barn problem.] (Objective 2, Objective 3, Objective 4, Objective 5)