Astronomy 2 – Special Relativity – Part 2
The axioms

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In this part, I am going to introduce the two axioms of Special Relativity. These two axioms are, to an extent, the only new *physics* introduced in this course: once we have introduced them and made them plausible, the rest of the course is devoted to examining their consequences, and the way in which they change the physics we are already familiar with.

The structure of this part is simple:

Aims and objectives for part 2

Aims: You should

1. understand the two axioms of SR.
2. understand the ideas of a coordinate transformation, and of the covariance of an equation under a coordinate transformation.

Objectives: You should be able to

1. quote the two axioms of SR, explain their meaning, and mention corroborating experiments.
2. draw simple conclusions from the axioms.
3. apply a coordinate transformation to an expression (as illustrated near Eqn. (2.4)), and demonstrate that an expression is or is not covariant.

1 The first postulate: the Relativity Principle

A form of the Principle of Relativity was described very clearly by Galileo, and his account is both clear and charming enough to quote at length:

> SALVATIUS: Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all direction; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though there is no doubt that when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction oposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue

> their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. *Galileo Galilei, Dialogue Concerning the Two Chief World Systems, quoted in [1, §3.1]*

This is a very vivid account of the Relativity Principle (RP), which we shall state more precisely at the end of this section. Another way of putting it is that 'you can't tell if you're moving' – there's no experiment you can do which would allow you to distinguish between a moving and a stationary frame.

The Relativity Principle as quoted here, and discussed below, is also sometimes referred to as the Equivalence Principle (for SR). However, when you go on to study GR, you will discover that it has an Equivalence Principle of its own (and if one is reading a careful account, one discovers even weak, strong, and semistrong variants of it). To avoid confusion, it seems best to leave the term '*The* Equivalence Principle' to GR: the RP is deeply linked to the Equivalence Principle, so the issue here is more one of terminology than physics. The various equivalence principles are discussed at illuminating length in Rindler [2, ch.1].

The RP as quoted here implicitly refers only to mechanics. However, given that you need mechanical components to do any electomagnetic experiment, and given that all mechanical objects are held together by (atomic) electromagnetic forces, it would seem unavoidable that it must apply to electromagnetism as well. See also Einstein's remarks quoted below, and Rindler [2, §1.12].

From the RP, one can show that, with certain obvious (but, as we shall discover, wrong) assumptions about the nature of space and time, one could derive the (apparently also rather obvious) *Galilean transformation* (GT)

$$x' = x - Vt \tag{2.1a}$$
$$y' = y \tag{2.1b}$$
$$z' = z \tag{2.1c}$$
$$t' = t \tag{2.1d}$$

between two frames in the standard configuration of Sect. 1-1.4. This transformation relates the coordinates of an event (t, x, y, z), measured in frame S, to the coordinates of the *same* event (t', x', y', z') in frame S'. Differentiating these, we find that

$$v'_x = v_x - V \tag{2.2a}$$
$$v'_y = v_y \tag{2.2b}$$
$$v'_z = v_z \tag{2.2c}$$
$$a' = a, \tag{2.2d}$$

where v_x is the x-component of velocity, and so on.

If you take the RP as true, then it follows that any putative law of mechanics which *does* appear to allow you to distinguish between reference frames cannot in fact be a law of physics. That is, the RP, *in classical mechanics*, demands that all laws of mechanics be *covariant under the Galilean Transformation.* What that means is that physical laws take the same form whether they are expressed in the coordinates S or S', related by a GT. Certain physical quantities are also invariant under a change of frame, meaning that they take the same value in both coordinate systems. Ex.2

It's not just fundamental physical laws that take the same form in different frames, but their consequences as well. Consider the constant-acceleration equation

$$x = v_0 t + \tfrac{1}{2} a t^2. \tag{2.3}$$

Transform this to the moving frame by replacing unprimed quantities by primed ones using Eqn. (2.1), and we find

$$x' = v_0' t' + \tfrac{1}{2} a' t'^2. \tag{2.4}$$

That is, we find exactly the same relation, as if we had simply put primes on each of the quantities in Eqn. (2.3). This is known as 'form invariance', or sometimes 'covariance', and indicates that the expression Eqn. (2.4) has exactly the same *form* as Eqn. (2.3), with the only difference being that we have different numerical values for the coefficients and coordinates (in general, though, $a' = a$ and $t' = t$ according to the GT). Barton [3, §2.3.3] discusses this usefully; see also Example 2.3. Another way of putting this is that if you throw a ball whilst on a moving train, you see it follow a parabola; someone watching this ball from a station platform *also* sees a parabola: the ball is moving at a different speed and a different angle, and it moves a longer distance – it is a different parabola – but it remains a parabola nonetheless. Ex.2

Everything, therefore, seems to be rosy.

Everything, in fact, *was* rosy, until the end of the nineteenth century. Around then, physicists were investigating *Maxwell's Equations*, one of the highpoints of nineteenth-century physics, which unified all of the phenomena of electricity and magnetism into a single formalism of tremendously insightful power and overwhelmingly successful application – Maxwell's Equations worked. Unfortunately, they were not invariant under a Galilean Transformation. The wave equation, and Maxwell's equations, do not transform into themselves under a GT. Ex.2

This is fairly easy to show for the wave equation, slightly more involved for Maxwell's Equations. Bell [4, ch.9] discusses this, or rather the Lorentz transformation of Maxwell's Equations, in some depth. More advanced textbooks on electromagnetic theory also tend to have sections on SR, which make this point more or less emphatically.

It appeared that Maxwell's Equations had their simplest form only in a frame which was *not moving* – the fact that the equations of electromagnetism were not invariant under the GT appeared to indicate that, whenever you watched an electromagnetic experiment (such as an ammeter, or a microwave oven) in a moving frame, it should work differently from that same experiment in a stationary frame. Specifically, it suggests that there actually exists such an absolutely stationary frame, which is otherwise rendered unnecessary by the RP: saying 'you can't tell if you're moving' is another way of saying 'there is no (experimentally accessible) standard of absolute rest'.

Einstein noted that electrodynamics appeared to be concerned only with relative motion, and did not take a different form when viewed in a moving frame. His 1905 paper is very clear on this point, and begins:

> It is known that Maxwell's electrodynamics – as usually understood at the present time – when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certian definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is staionary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise – assuming equality of relative motion in the two cases discussed – to electric currents of the same path and intensity as those produced by the electric forces in the former case. [5]

Another, linked, problem was that of the aether. Since light is an electromagnetic wave, it seems obvious that, like water waves or sound waves, there must be something that light is a wave *in*. This 'light medium' was named the aether, and had the apparently contradictory properties of being both very rigid (so that it could sustain the very high frequencies of light) and very tenuous (so that objects such as planets could move through it freely). The aether would be a clear candidate for the frame of absolute rest.

The Earth moves around the sun in its orbit, with a constantly changing velocity. It followed, therefore, that there was some point in its orbit at which it had a maximum, and another point at which it had its minimum, velocity with respect to the putative aether. Although this velocity is rather slow compared to the speed of light, it should have been possible to *measure* the change in the velocity of the Earth with respect to the aether or the absolute rest frame. There was therefore a series of experiments in the late nineteenth and early twentieth centuries which attempted to measure this relative velocity: the Michelson-Morley aether-wind experiment attempted to measure the different light-travel times for beams directed along and across the flow of the aether; the Fizeau experiment and Lodge's experiments attempted to detect the extent to which the aether could be dragged along by fast-moving objects on Earth. All of them failed: no-one was able to detect the Earth's movement through the aether, or the movement relative to the absolute rest frame, which the Galilean Transformation and the apparently necessary properties of an apparently necessary aether demanded.

The aether drift experiments are discussed in most relativity textbooks. French [6], for example, gives clear accounts. For an interesting sociological and historical take on the Michelson-Morley experiments, and the context in which they were interpreted, see also Collins & Pinch [7, ch. 2].

So there was a conflict between the predictions of Maxwell's equations plus the GT plus the Relativity Principle, and what was observed in experiment. At this stage there were a number of options.

(i) Perhaps Maxwell's equations were wrong – perhaps light didn't behave as a powerfully simplifying theory, and a huge number of successful experiments, suggested it should.

(ii) Perhaps the Relativity Principle was wrong, although one possible result of the repeated failures to measure the Earth's motion through the aether could, I imagine, have been the weakening of the idea of an absolute rest-frame.

(iii) Perhaps the GT was wrong, though this transformation seems so obvious, and so bound up with our other preconceptions that it would be difficult to see *how* it could be wrong.

(iv) Perhaps there was some further physics at work. There were suggestions that an experimental apparatus, or even the Earth itself, might be able to drag the aether along with it enough to wipe out any detection from the Michelson-Morley experiment. Lorentz managed to find a transformation – now known as the Lorentz Transformation – under which Maxwell's equations *are* invariant, but was then left with the problem of explaining why those equations were apparently uniquely subject to a different transformation law from everything else. Attempts to explain the latter transformation included suggestions that objects might inexplicably change their lengths, and time be distorted, when moving head-on into the aether; but this in turn would require further explanation (see Lorentz's papers in [8] for an account of the latter attempts). It would have been clear that something was very wrong.

The resolution was that (i) Maxwell's equations are right, (ii) the Relativity Principle is right, (iii) the Galilean Transformation is inadequate and (iv) Special Relativity is the new physics to come out of this.

Einstein explains this as clearly as anyone. In his 1905 paper which introduced SR ('*On the Electrodynamics of Moving Bodies*'), he opens with the paragraph above commenting that only *relative* motion is important in Maxwell's equations, and then goes on to say, with magisterial finality:

> Examples of this sort, together with the unsuccessful attempts to discover any motion of the Earth relatively to the 'light medium', suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the 'Principle of Relativity') to the status of a postulate… [5]

In other words, he is saying that Galileo's Relativity Principle, which had really only been a statement about mechanical experiments, now applied to *all* of physics.

We can recast the Relativity Principle, the first postulate of Special Relativity, as follows:

> *The Principle of Relativity:* All inertial frames are equivalent for the performance of *all* physical experiments.

There is no physical (or chemical or biological or sociological or musical) experiment I can do which will have a different result when I'm moving from when I'm stationary. There is therefore no need for even the idea of a standard of absolute rest. Ex.2

2 The second postulate: the constancy of the speed of light

The passage I quoted above, from Einstein's 1905 paper, goes on to say:

> We will raise this conjecture (the purport of which will hereafter be called the 'Principle of Relativity') to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. [5]

This is the second postulate of Special Relativity.

This doesn't seem particularly remarkable at first reading; after all, we *know* that 'the speed of light' is one of Nature's fundamental constants, at $c = 299792458\,\mathrm{ms}^{-1}$. The sting is in the final remark, 'independent of the state of motion of the emitting body'. At first thought, there would seem to be three things that 'the speed of light' could mean:

1. The speed relative to the emitter (like a projectile);
2. The speed relative to the transmitting medium (like water or sound); or
3. The speed relative to the detector.

Option 2 is ruled out by the first postulate: if this were true then the frame in which light had this special value would be picked out as special; the RP also incidentally excludes the notion of the aether. Option 1 also turns out not to be the case. The GT (Eqn. (2.2a)) says that velocities add, so that a (classical) projectile emitted from a moving object has not only the speed it was fired with, but also the speed of the emitter. This is not so for light, or indeed any object moving at a significant fraction of the speed of light, and this can be amply verified by observing the light coming from Jupiter's moons, or binary stars, or other astrophysical objects moving at great speeds. Ex.2

No, option 3 is the case, so that, no matter what sort of experiment you are doing, whether you are directly observing the travel-time of a flash of light, or doing some interferometric experiment, the speed of light relative to your apparatus will always have the same numerical value. This is perfectly independent of how fast you are moving: it is independent of whichever inertial frame you are in, so that another observer, measuring the *same* flash of light from their moving laboratory, will measure the speed of light relative to *their* detectors to have exactly the same value.

> *Constancy of c:* There exists a finite constant speed $c = 299792458\,\mathrm{m\,s^{-1}}$, such that anything which moves at this speed in one inertial frame is measured to move at that speed in all other inertial frames.

Barton [3, §3.1] gives a wonderfully pedantic expression of this. There is no real way of *justifying* this postulate: it is simply a truth of our universe, and we can do nothing more than simply *demonstrate* its truth through experiment.

This experimental corroboration most typically takes the form of a measurement of the speed of light emitted from an orbiting body, at the phases in its orbit when it is moving directly towards or away from us. The orbiting body can be a particle in an accelerator, or a binary star orbiting its companion, but in either case the measured light speed is determined to be independent of the speed of the emitter, to impressively high accuracy.

For further discussion of this experimental support, and references to further reading, see French [6], chapters 2 and 3, and Barton [3] section 3.4.

The constancy of the speed of light is not the only second postulate you could have. You could take alternatives such as 'Maxwell's Equations are true', or 'Moving clocks run slow according to…', or any other statement which picked out the phenomena of Special Relativity, and you could still derive the results of SR, including, for an encore, the constancy of the speed of light. However, this particular second postulate is a particularly simple and fundamental one, which is why it is much the best choice.

Alternatively, you could choose as a second postulate something like 'The Galilean Transformation is true', and derive from the pair of results most (all?) of the laws of mechanics. The point here is that such a pair of postulates would give you a perfectly consistent theory – a perfectly possible world – but one which does not happen to match *our* world.

Taking a more mathematical tack, Rindler [2, §2.17] (and Barton [3, §4.3], less abstractly) shows that the *only* linear transformations consistent with the Euclidicity and isotropy of inertial frames are the Galilean and Lorentz transformations. A second postulate consisting of 'there is no upper limit to the speed of objects' picks out the Galilean transformation; the statement 'there is an upper speed limit' picks out the Lorentz transformation with an arbitrary constant, and saying '…and that speed limit is c' sets the value of the constant. See also Rindler's other remarks [2, §2.7] on the properties of the Lorentz Transformation.

Questions

Does time move slower when you're moving, as Example 2.4 suggests? Yes! Part 3 starts to discuss this.

How would you measure the speed of light emitted from, say, binary stars? Perhaps using some interference experiment, similar to the way that Michelson and Morley hoped to measure the speed of light relative to the aether. See [3, §3.4] and references there, for details of how you perform such measurements.

If you were moving at c, how would you measure c? This is a good question, but probably an unanswerable one. Firstly, it is impossible for any object with a non-zero mass to be accelerated all the way to speed c. Further, even a mysteriously zero-mass entity couldn't measure this, since *no time passes* for such an entity, and it passes through all the points on its worldline at the same time.

What is the difference between the Lorentz Transformation you mentioned, and the Galilean Transformation? Both transformations relate an event's coordinates in one frame to the coordinates of the same event as measured in another frame. The difference is only in the way that the sets of coordinates are related. We cover this in the next part.

If the speed of light is constant with respect to the detector (as discussed in Sect. 2), does that mean it's not constant with respect to the source or the medium? It's not *defined* to be a constant with respect to either, but it will be, as you can see by imagining a detector fixed to the source. The same would be true if there were a medium through which the light were travelling. That is, light has a constant speed with respect to the detector *and* the source, even if they are in relative motion. This, as you can imagine, leads to a number of startling consequences, which we explore in the next part.

I've heard of experiments which found light travelling faster than c through some materials. How can this be? There are experiments using non-linear optical materials, which show part of a light-pulse exiting a slab of material before it is due to. There is no contradiction here, partly because of the distinction (which I elided above) between the 'group velocity' and 'phase velocity' of a light pulse. The constant c refers to the group velocity: it is possible for the phase velocity to be greater than c. But this is OK because the phase velocity carries no information. See the *Physics World* news items `http://physicsweb.org/article/news/4/7/8` and `http://physicsweb.org/article/news/6/1/13`, for articles which mention superluminal group velocities, and include references; in contrast, `http://physicsweb.org/article/news/6/1/7` is an article about *zero*-speed light. See a textbook on electromagnetism for further details about the distinction between group and phase velocities, and see Barton [3, §3.1] for a careful discussion of the distinction in the context of relativity, which makes this reasonably clear.

Examples

Example 2.1 (section 1)

Consider a rocket at rest ($v_0 = 0$) at the origin of a frame S. At time $t = 0$, it starts to fire its rockets so that it moves along the x axis, and at time $t = t_1$ we find the rocket at position $x = x_1$, moving at speed $v = v_1$. Consider a second frame S', moving at speed V along the x axis, such that frames S and S' are in standard configuration (so that $x = x' = 0$ when $t = t' = 0$). Work out the momentum of the rocket at $t = 0$ and $t = t_1$ in the two frames (that is, work out $p_0 = mv_0$, $p_1 = mv_1$, $p'_0 = mv'_0$ and $p'_1 = mv'_1$): is momentum frame-invariant? Work out the *change* in momentum in the two frames: is this frame-invariant? Work out the kinetic energy and the change in kinetic energy in the two frames: are these frame-invariant? (Objective 3)

Example 2.2 (section 1)

From the constant-acceleration equations, you learned, in first year, how to analyse projectile motion: you discovered that, for a projectile launched at speed u at an angle θ to the horizontal, the time aloft was $t = 2u \sin\theta/g$ and the range was $x = u^2 \sin 2\theta/g$.

(i) Consider a train moving through a station at a speed U. Draw a diagram of this situation, indicating the two reference frames S and S' in Standard Configuration.

(ii) Someone on the train (frame S') throws a ball vertically into the air at a speed u' (ie, $u'_x = 0$ and $u'_y = u'$). Calculate its time aloft, t', and range, x', measured in the moving frame, and reassure yourself that the latter is zero (ie, the ball comes vertically downwards and the person catches it again).

(iii) Do the same calculation from the point of view of a person standing on the platform (ie, work out the initial u_x and u_y for the ball, and work out the consequent projectile motion).

(iv) Finally, take your answer from part (ii), and replace primed quantities by unprimed quantities using the GT, Eqn. (2.1). Confirm that you get the same answers as in part (iii). (Objective 3)

Example 2.3 (section 1)

Consider the wave equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0. \tag{i}$$

Take

$$\begin{aligned}\frac{\partial}{\partial x}&=\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}+\frac{\partial t'}{\partial x}\frac{\partial}{\partial t'}\\ \frac{\partial}{\partial t}&=\frac{\partial x'}{\partial t}\frac{\partial}{\partial x'}+\frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}\end{aligned}$$

and so on, and using the GT, Eqn. (2.1), show that Eqn. (i) does *not* transform into the same form under a GT. (Objective 3, Objective 3)

Example 2.4 (section 1)

I have a friend moving past me in a rocket at a relativistic speed, and I observe her watch to be moving slowly with respect to mine (as we will discover later). She examines my watch as I do this: is it moving faster or slower than hers? (Objective 2)

Example 2.5 (section 2)

You are on a train moving through a station at $50\,\mathrm{ms}^{-1}$, and you throw a ball forwards at $10\,\mathrm{ms}^{-1}$: what is the speed of the ball as measured by someone on the platform?

Now you are on a train moving at half the speed of light, and you shine a torch forwards: what is the speed of the light from the torch as measured by someone on the platform? (Objective 2)