Classical mechanics from the mountaintop

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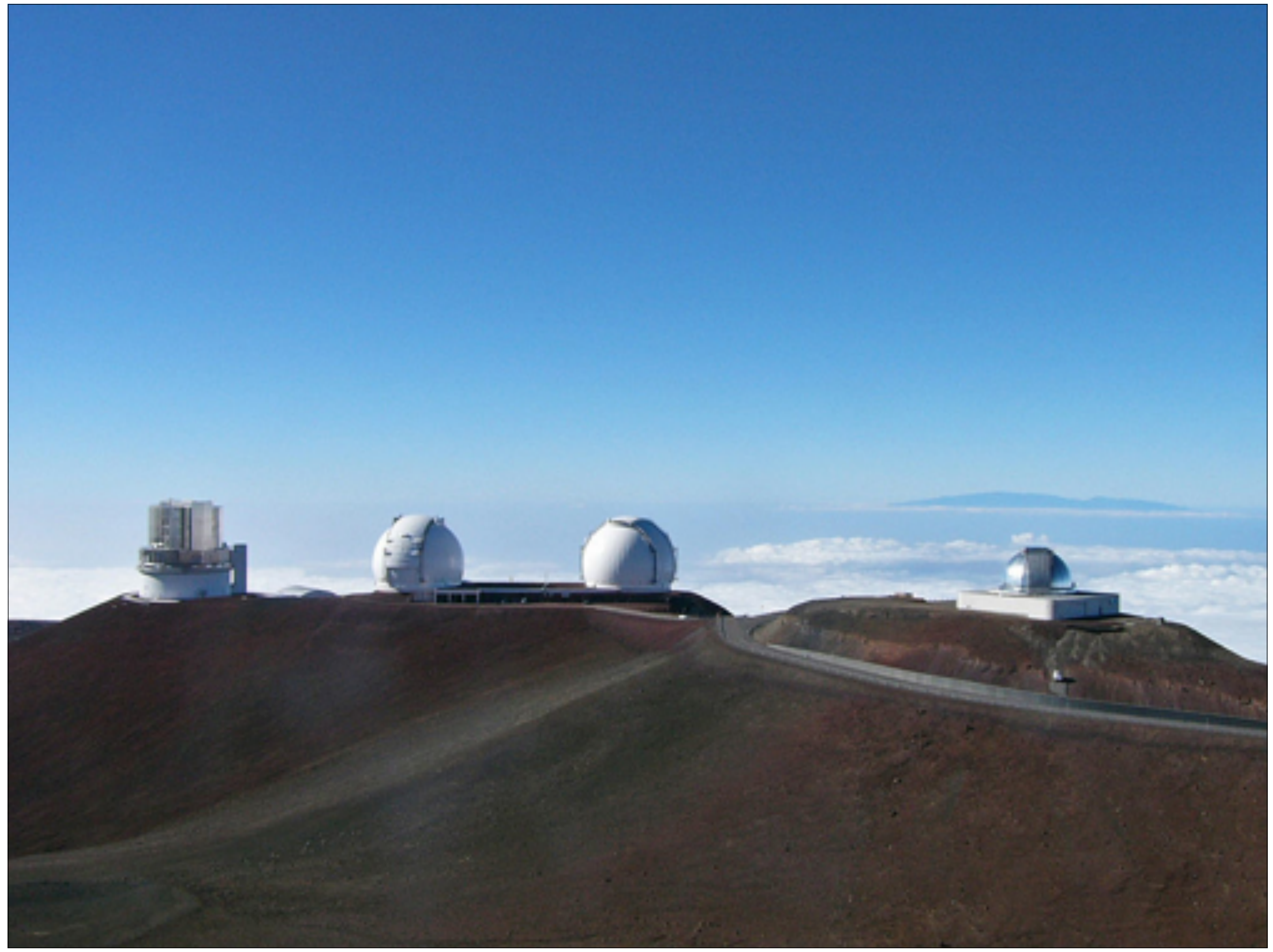


AXIOMATA,

SIVE

LEGES MOTUS.

LEX I.

(*) *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.*

Projectilia perseverant in motibus suis, nisi quatenus a resistentiâ aëris retardantur, et vi gravitatis impelluntur deorsum. Trochus, cujus partes cohærendo perpetuò retrahunt sese a motibus rectilineis, non cessat rotari, nisi quatenus ab aëre retardatur. Majora autem Planetarum et Cometarum corpora motus suos et progressivos et circulares in spatiis minus resistentibus factos conservant diutius.

LEX II.

(*) *Mutationem motus proportionalem esse vi motrici impressæ, et fieri secundùm lineam rectam quâ vis illa imprimitur.*

Si vis aliqua motum quemvis generet; dupla duplum, tripla triplum generabit, sive simul et semel, sive gradatim et successivè impressa fuerit. Et hic motus (quoniam in eandem semper plagam cum vi gener-

(*) 24. Ex hac primâ lege quam (9) demonstravimus, sequitur omnem motum esse naturâ suâ æquabilem et rectilineum, adeóque nec illius velocitatem retardari, nec directionem mutari, nisi aliquod obstaculum mobili afferatur; Unde etiam projectilia motum suum amittunt, quoniam est aliqua hujusce retardationis causa. Cum autem corpora projecta vel per medium resistens deferantur, vel etiam super aliorum corporum superficies scabras incedant, et vi gravitatis deorsum semper urgeantur, necesse est ut eam amittant motus sui partem quam in hisce obstaculis superandis continuò absumunt, ac proindè quo major vel minor erit medii resistentia, eò majus vel minus decrementum accipiet corporis projecti velocitas. Ex his igitur patet majora planetarum et cometarum corpora nullam sensibilem in spatiis cœlestibus experiri resistentiam, cùm motus suos diutissime conservent.

(*) 25. Si corpus vi activâ, qualis est vis gravitatis, secundùm eandem aut parallelam directionem continuò urgeatur, motus illius continuò acceleratur; nam per leg. 1., manet celeritas acquisita, et per leg. 2. nova complementi continuò additur. Si verò aliqua vis in corpus jam motum contrariâ directione perpetuò agat, motus illius continuò retardatur, per leg. 2. Si vis [illegible] continuò ac uniformiter agat, id est, si constans sit, corpus eâ vi impulsum, æqualibus temporibus æqualia accipit celeritatis incrementa, seu motu uniformiter accelerato fertur, et celeritates vi illâ acquisitæ, sunt ut tempora quibus

strice determinatur) si corpus antea movebatur, motui ejus vel conspiranti additur, vel contrario subducitur, vel obliquo obliquè adjicitur, et cum eo secundùm utriusque determinationem componitur.

(*) LEX III.

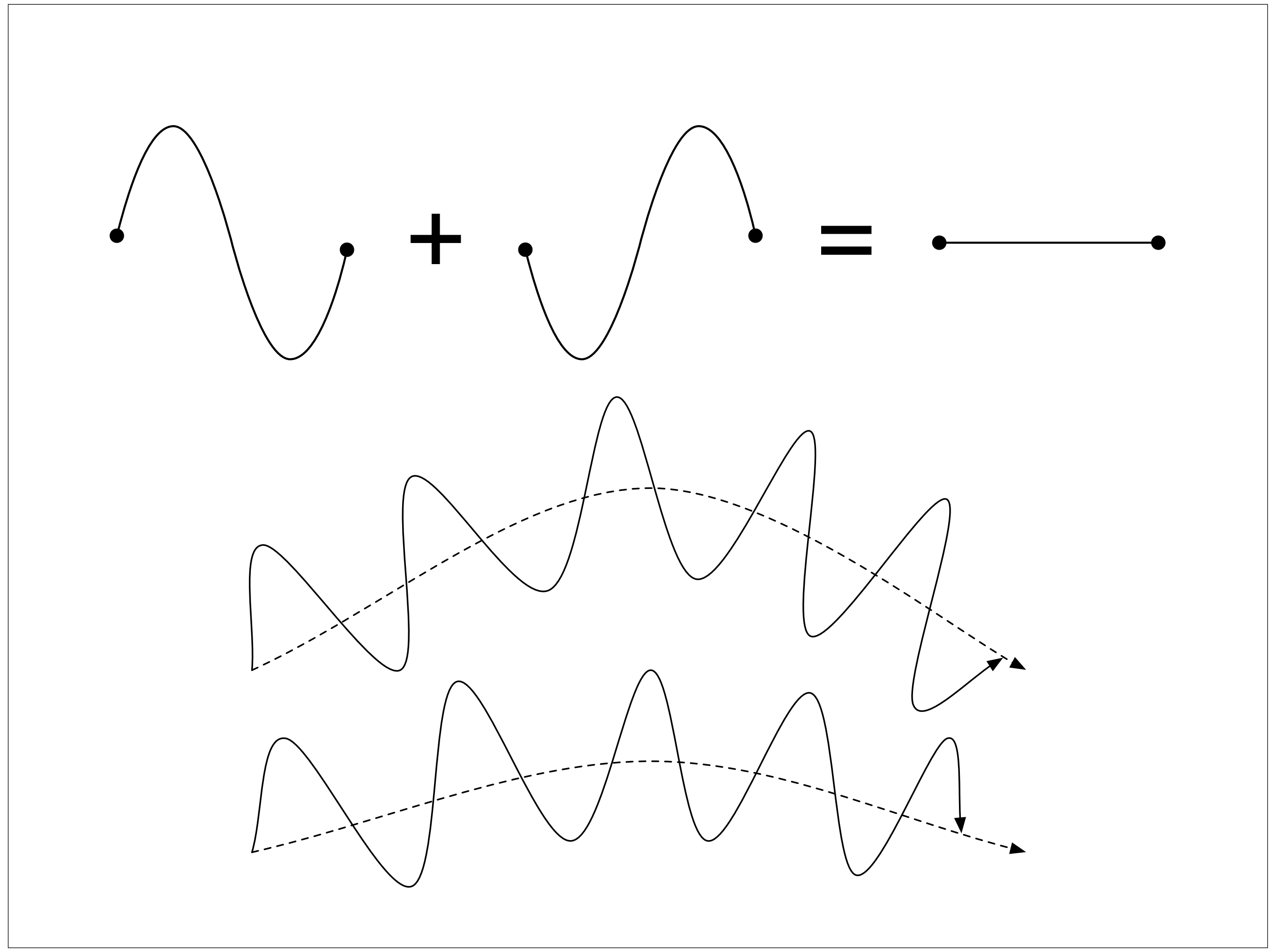
Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.

Quicquid premit vel trahit alterum, tantundem ab eo premitur vel trahitur. Si quis lapidem digito premit, premitur et hujus digitus a lapide. Si equus lapidem funi alligatum trahit, retrahetur etiam et equus (ut ita dicam) æqualiter in lapidem: nam funis utrinque distentus eodem relaxandi se conatu urgebit equum versus lapidem, ac lapidem versus equum; tantumque impediet progressum unius quantum promovet progressum alterius. Si corpus aliquod in corpus aliud impingens, motum ejus vi suâ quomodocunque mutaverit, idem quoque vicissim in motu proprio eandem

generantur. At si vis constans contrariâ directione in corpus motum continuò agat; æqualibus temporibus æqualia fiunt celeritatis decrementa, et corpus motu uniformiter retardato movebitur. Generaliter tandem, si corpus quiescens quálibet vi sive constanti sive variabili continuò urgeatur, et deinde eâ celeritate quam vis illius actione continuâ acquisivit, contrâ directionem vis illius magnitudinis projiciatur, ut vestigia sua relegat, corpus illud in itu et reditu suo eandem habebit celeritatem, ubi ad eadem viæ suæ puncta, eundo et redeundo pervenerit; adeoque motum redeundo non amittet, nisi cùm pervenerit ad punctum ex quo cœpit eundo moveri; nam eadem vis in itu et reditu corporis, æqualibus temporibus æquales celeritatis gradus generat et extinguit (8).

26. Corpora gravia in terræ viciniâ, sublatâ medii resistentiâ, motu uniformiter accelerato descendunt, et motu uniformiter retardato ascendunt...... Demonstratio..... Sublatâ medii resistentiâ idem est ejusdem corporis pondus, sive eadem illius in subjectum planum pressio, tum in vertice, tum in radice montis; est autem pondus, seu vis motrix (12) ut massa in vim gravitatis acceleratricem ducta: ergò cum ejusdem corporis massa eadem in vertice et in radice montis permaneat, manebit etiam eadem vis acceleratrix gravitatis. Itaque corpora gravia in radice et vertice montis æqualia spatia æqualibus temporibus percurrunt, sublatâ aëris resistentiâ, ut accuratissimis notum est experimentis (12): constans est igitur vis acceleratrix, et per lineas ad horizontem perpendiculares (3) uniformiter agit; gravia ergò motu uniformiter accelerato descendunt, et uniformiter retardato ascendunt (25). Q. e. d.

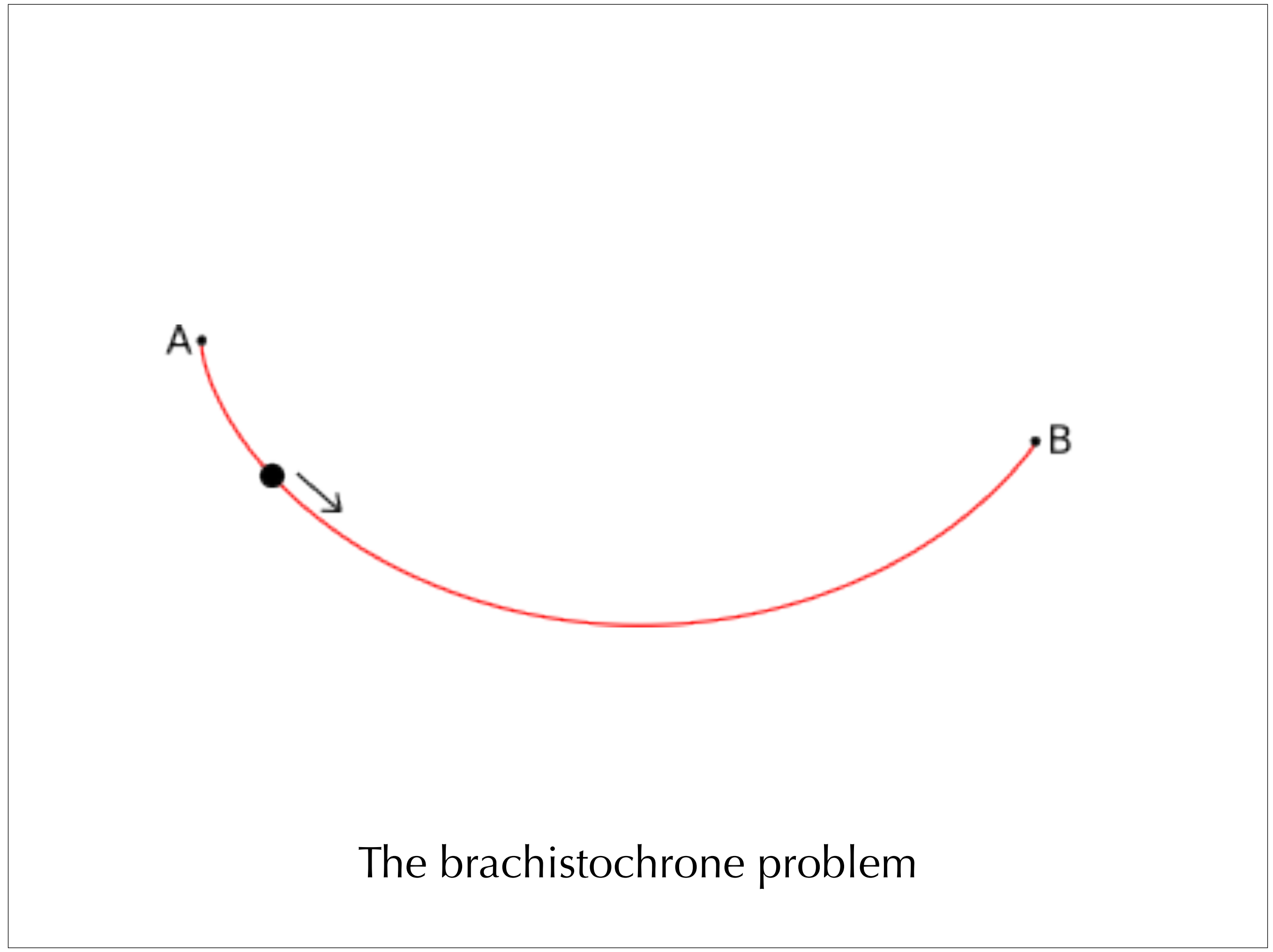
27. Sublatâ medii resistentiâ in terræ viciniâ, spatia quæ corpus è quiete cadendo percurrit, sunt ut quadrata temporum quibus percurruntur .... Dem. .... Recta S K, repræsentet spatium quod grave cadendo percurrit; T C, T c, T B, exponant tempora quibus describuntur spatia S P, S p, S K; et C L, c l, B D, ad T B, normales, exhibeant celeritates temporibus T C, T c, T B, per spatia S P, S p, S K, acquisitas; quia in motu uniformiter accelerato, celeritates sunt ut tempora, (25), erit T C : T c = C L : c l; et T C : T B = C L : B D, adeóque recta, T D, transit per puncta L, et l, et triangula T C L, T c l, T B D, similia sunt. Jam fingamus lineam, c l, motu sibi semper parallelo ità accedere ad lineam C L, ut tandem cum ipsâ coincidat; evanescente temporculo C c, celeritas, c l, non differt a celeritate C L, adeóque per temporculum infinitè parvum seu evanescens C c, celeritas, C L, uniformis censeri potest. Porrò spatia motu æquabili descripta sunt ut celeritas in tempus ducta (1), ergò spatium P p, quod temporculo, C c, percurri supponitur, erit ut rectangulum, C L × C c = C d; quare si totum tempus, T C, in temporcula innumera æqualia C c, divisum concipiatur, et similiter spatium S P, tempore T C, percursum in totidem spatiola evanescentia, singulis temporculis correspondentibus percursa dividatur, erit summa rectangulorum C d, hoc est area trianguli T C L, ut summa spatiolorum P p, id est ut S P; et eodem modo demonstratur aream trianguli T B D, esse ut spatium S K, tempore T B, percursum. Est igitur triangulum T C L : T B D = S P : S K. Sed triangulorum similium areæ T C L, T B D, sunt ut quadrata laterum homologorum,



Fermat’s principle

Travel time gets shorter as you go from the first to the second,
then it gets longer.
Ie minimum at the shortest path

Catenary
Can describe it in terms of balance of forces
but also minimising total potential energy



Brachistochrone problem
Very similar to the tautochrone problem, solved by Huygens in 1659
Johann Bernoulli, 1696 (first published ‘integral’)
Led to the calculus of variations

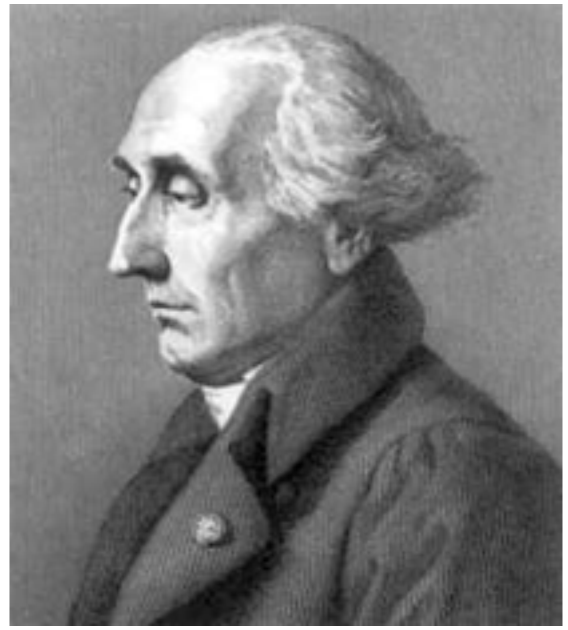
The Lagrangian

$$L = T - V$$

$$S = \int L \, \mathrm{d}t$$

The action

Hamilton's principle



Joseph Louis Lagrange (1736–1813)
William Rowan Hamilton (1805–1865)

$$H = T + V$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

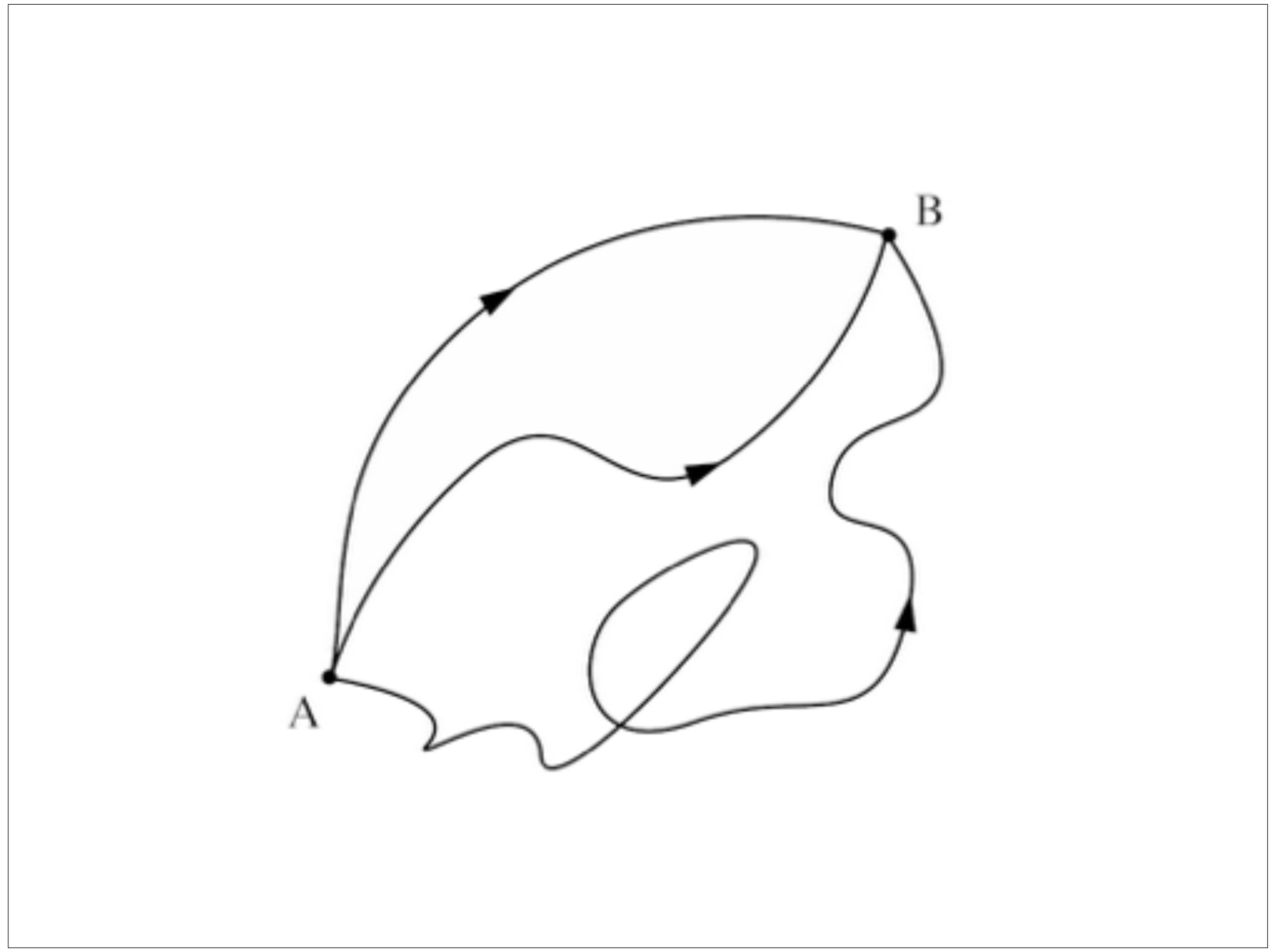
$$\dot{q}_i = +\frac{\partial H}{\partial p_i}$$

Describes motion in a 2n-dimensional space
But this tells us that only 2x3=6 numbers completely characterise motion in 3-d space.
Also, this is true for more than just particle mechanics.
Leads to Newton's equations, Maxwell's, Schrödinger's; electrical and acoustic systems.
Not always useful for actual calculations, though

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi$$

$$\hat{H}\psi = E\psi$$

Schrödinger equation (one form of it)
Similarly, it’s the classical theory of fields (based on continuous Lagrangians) which is used as the basis for quantum field theory



Feynman path-integral formulation
(building on work by Dirac)
Each path is weighted by exp(iS/hbar)

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = +\frac{\partial H}{\partial p_i}$$

If H indep of q, then p is conserved

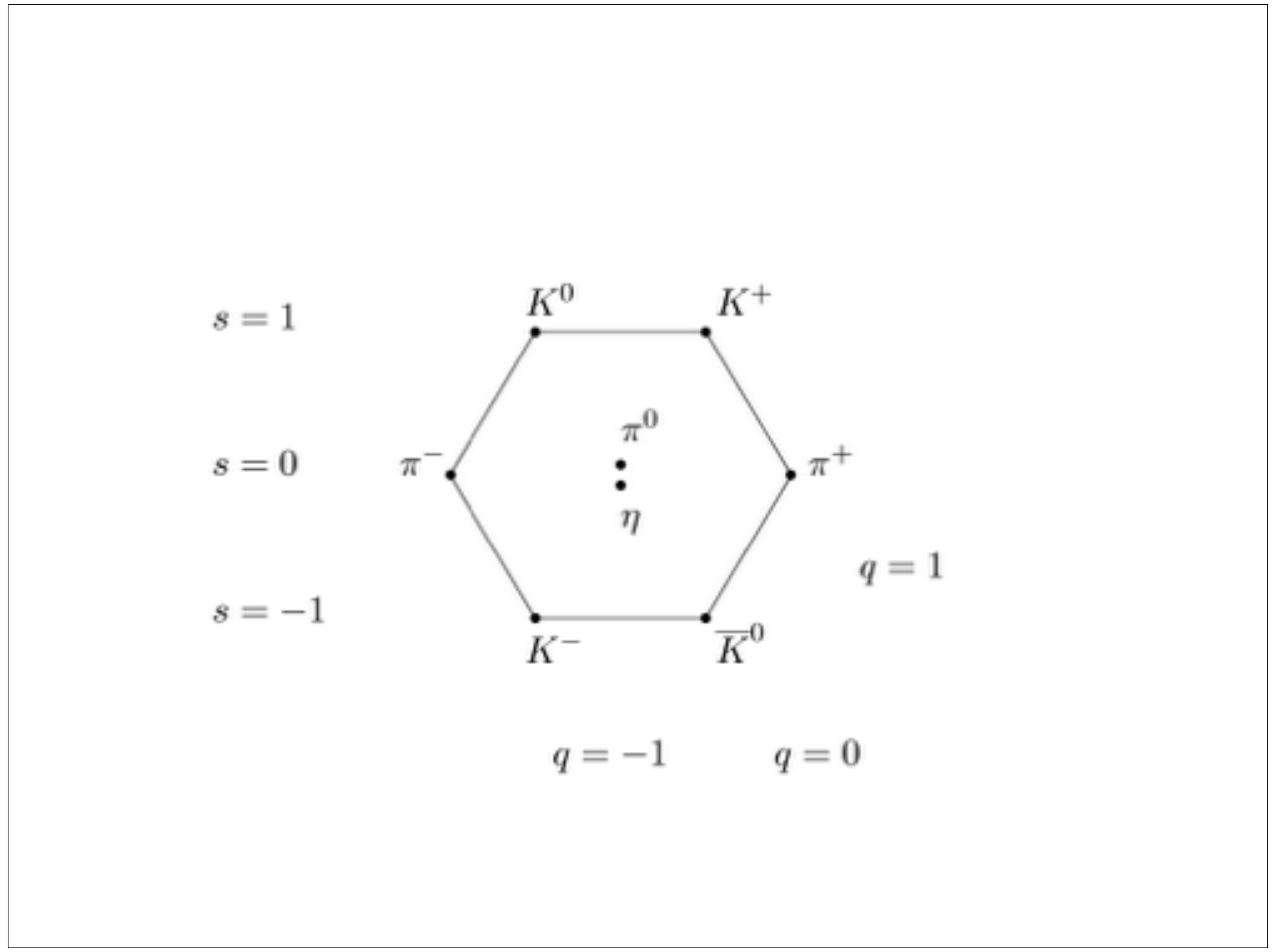
Emmy Noether (1882–1935)

Noether's theorem

For each symmetry of the Lagrangian,
there is a conserved quantity

$$
\begin{gathered}
L=-\frac{1}{2}\partial_\nu g^a_\mu\partial_\nu g^a_\mu-g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu-\frac{1}{4}g^2_s f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu+\frac{1}{2}ig^2_s(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu+\bar{G}^a\partial^2 G^a+ \\
g_s f^{abc}\partial_\mu\bar{G}^a G^b g^c_\mu-\partial_\nu W^+_\mu\partial_\nu W^-_\mu-M^2 W^+_\mu W^-_\mu-\frac{1}{2}\partial_\nu Z^0_\mu\partial_\nu Z^0_\mu-\frac{1}{2c^2_w}M^2 Z^0_\mu Z^0_\mu-\frac{1}{2}\partial_\mu A_\nu\partial_\mu A_\nu- \\
\frac{1}{2}\partial_\mu H\partial_\mu H-\frac{1}{2}m^2_h H^2-\partial_\mu\phi^+\partial_\mu\phi^--M^2\phi^+\phi^--\frac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0-\frac{1}{2c^2_w}M\phi^0\phi^0-\beta_h[\frac{2M^2}{g^2}+\frac{2M}{g}H+ \\
\frac{1}{2}(H^2+\phi^0\phi^0+2\phi^+\phi^-)]+\frac{2M^4}{g^2}\alpha_h-igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu-W^+_\nu W^-_\mu)-Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu- \\
W^-_\mu\partial_\nu W^+_\mu)+Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu-W^-_\nu\partial_\nu W^+_\mu)]-igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu-W^+_\nu W^-_\mu)-A_\nu(W^+_\mu\partial_\nu W^-_\mu- \\
W^-_\mu\partial_\nu W^+_\mu)+A_\mu(W^+_\nu\partial_\nu W^-_\mu-W^-_\nu\partial_\nu W^+_\mu)]-\frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu+\frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu+ \\
g^2 c^2_w(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu-Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu)+g^2 s^2_w(A_\mu W^+_\mu A_\nu W^-_\nu-A_\mu A_\mu W^+_\nu W^-_\nu)+ \\
g^2 s_w c_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu-W^+_\nu W^-_\mu)-2A_\mu Z^0_\mu W^+_\nu W^-_\nu]-g\alpha[H^3+H\phi^0\phi^0+2H\phi^+\phi^-]- \\
\frac{1}{8}g^2\alpha_h[H^4+(\phi^0)^4+4(\phi^+\phi^-)^2+4(\phi^0)^2\phi^+\phi^-+4H^2\phi^+\phi^-+2(\phi^0)^2H^2]-gMW^+_\mu W^-_\mu H- \\
\frac{1}{2}g\frac{M}{c^2_w}Z^0_\mu Z^0_\mu H-\frac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^--\phi^-\partial_\mu\phi^0)-W^-_\mu(\phi^0\partial_\mu\phi^+-\phi^+\partial_\mu\phi^0)]+\frac{1}{2}g[W^+_\mu(H\partial_\mu\phi^-- \\
\phi^-\partial_\mu H)-W^-_\mu(H\partial_\mu\phi^+-\phi^+\partial_\mu H)]+\frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0-\phi^0\partial_\mu H)-ig\frac{s^2_w}{c_w}MZ^0_\mu(W^+_\mu\phi^-- \\
W^-_\mu\phi^+)+igs_wMA_\mu(W^+_\mu\phi^--W^-_\mu\phi^+)-ig\frac{1-2c^2_w}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^--\phi^-\partial_\mu\phi^+)+igs_wA_\mu(\phi^+\partial_\mu\phi^-- \\
\phi^-\partial_\mu\phi^+)-\frac{1}{4}g^2W^+_\mu W^-_\mu[H^2+(\phi^0)^2+2\phi^+\phi^-]-\frac{1}{4}g^2\frac{1}{c^2_w}Z^0_\mu Z^0_\mu[H^2+(\phi^0)^2+2(2s^2_w-1)^2\phi^+\phi^-]- \\
\frac{1}{2}g^2\frac{s^2_w}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^-+W^-_\mu\phi^+)-\frac{1}{2}ig^2\frac{s^2_w}{c_w}Z^0_\mu H(W^+_\mu\phi^--W^-_\mu\phi^+)+\frac{1}{2}g^2s_wA_\mu\phi^0(W^+_\mu\phi^-+ \\
W^-_\mu\phi^+)+\frac{1}{2}ig^2s_wA_\mu H(W^+_\mu\phi^--W^-_\mu\phi^+)-g^2\frac{s_w}{c_w}(2c^2_w-1)Z^0_\mu A_\mu\phi^+\phi^--g^1s^2_wA_\mu A_\mu\phi^+\phi^-- \\
\bar{e}^\lambda(\gamma\partial+m^\lambda_e)e^\lambda-\bar{\nu}^\lambda\gamma\partial\nu^\lambda-\bar{u}^\lambda_j(\gamma\partial+m^\lambda_u)u^\lambda_j-\bar{d}^\lambda_j(\gamma\partial+m^\lambda_d)d^\lambda_j+igs_wA_\mu[-(\bar{e}^\lambda\gamma^\mu e^\lambda)+ \\
\frac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j)-\frac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)]+\frac{ig}{4c_w}Z^0_\mu[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda)+(\bar{e}^\lambda\gamma^\mu(4s^2_w-1-\gamma^5)e^\lambda)+(\bar{u}^\lambda_j\gamma^\mu(\frac{4}{3}s^2_w- \\
1-\gamma^5)u^\lambda_j)+(\bar{d}^\lambda_j\gamma^\mu(1-\frac{8}{3}s^2_w-\gamma^5)d^\lambda_j)]+\frac{ig}{2\sqrt{2}}W^+_\mu[(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)e^\lambda)+(\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)]+ \\
\frac{ig}{2\sqrt{2}}W^-_\mu[(\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda)+(\bar{d}^\kappa_jC^\dagger_{\lambda\kappa}\gamma^\mu(1+\gamma^5)u^\lambda_j)]+\frac{ig}{2\sqrt{2}}\frac{m^\lambda_e}{M}[-\phi^+(\bar{\nu}^\lambda(1-\gamma^5)e^\lambda)+\phi^-(\bar{e}^\lambda(1+ \\
\gamma^5)\nu^\lambda)]-\frac{g}{2}\frac{m^\lambda_e}{M}[H(\bar{e}^\lambda e^\lambda)+i\phi^0(\bar{e}^\lambda\gamma^5e^\lambda)]+\frac{ig}{2M\sqrt{2}}\phi^+[-m^\kappa_d(\bar{u}^\lambda_jC_{\lambda\kappa}(1-\gamma^5)d^\kappa_j)+m^\lambda_u(\bar{u}^\lambda_jC_{\lambda\kappa}(1+ \\
\gamma^5)d^\kappa_j]+\frac{ig}{2M\sqrt{2}}\phi^-[m^\lambda_d(\bar{d}^\lambda_jC^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j)-m^\kappa_u(\bar{d}^\lambda_jC^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j]-\frac{g}{2}\frac{m^\lambda_u}{M}H(\bar{u}^\lambda_ju^\lambda_j)- \\
\frac{g}{2}\frac{m^\lambda_d}{M}H(\bar{d}^\lambda_jd^\lambda_j)+\frac{ig}{2}\frac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5u^\lambda_j)-\frac{ig}{2}\frac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5d^\lambda_j)+\bar{X}^+(\partial^2-M^2)X^++\bar{X}^-(\partial^2- \\
M^2)X^-+\bar{X}^0(\partial^2-\frac{M^2}{c^2_w})X^0+\bar{Y}\partial^2Y+igc_wW^+_\mu(\partial_\mu\bar{X}^0X^--\partial_\mu\bar{X}^+X^0)+igs_wW^+_\mu(\partial_\mu\bar{Y}X^-- \\
\partial_\mu\bar{X}^+Y)+igc_wW^-_\mu(\partial_\mu\bar{X}^-X^0-\partial_\mu\bar{X}^0X^+)+igs_wW^-_\mu(\partial_\mu\bar{X}^-Y-\partial_\mu\bar{Y}X^+)+ \\
igc_wZ^0_\mu(\partial_\mu\bar{X}^+X^+-\partial_\mu\bar{X}^-X^-)+igs_wA_\mu(\partial_\mu\bar{X}^+X^+-\partial_\mu\bar{X}^-X^-)-\frac{1}{2}gM[\bar{X}^+X^+H+ \\
\bar{X}^-X^-H+\frac{1}{c^2_w}\bar{X}^0X^0H]+\frac{1-2c^2_w}{2c_w}igM[\bar{X}^+X^0\phi^+-\bar{X}^-X^0\phi^-]+\frac{1}{2c_w}igM[\bar{X}^0X^-\phi^+- \\
\bar{X}^0X^+\phi^-]+igMs_w[\bar{X}^0X^-\phi^+-\bar{X}^0X^+\phi^-]+\frac{1}{2}igM[\bar{X}^+X^+\phi^0-\bar{X}^-X^-\phi^0]
\end{gathered}
$$

The standard model of particle physics



Eightfold way - meson octet.
Murray Gell-Mann