# Classical mechanics from the mountaintop: supplementary notes

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This document contains a few extra notes to accompany the slides for my 'mechanics' talk on 14 January. They're not expected to be standalone, but simply to reiterate or expand on the remarks I made at the time.

As regards reading, I'd enthusiastically endorse Alex's recommendation of Feynman's *The character of physical law*. Some of the lecture video is online at http://research.microsoft.com/apps/tools/tuva/. I'll also, slightly diffidently, add Peter Atkins, *Creation Revisited*, Freeman (1993) isbn:978-0716745006. This book is a mixture of the slightly batty and deeply insightful; frequently startlingly lyrical, and more frequently extremely memorable.

Some of the remarks below *are written differently to indicate that they contain slightly more mathematical detail, which might be of interest to those with the lingo, but can be happily skipped by others.*

## 1 Newton and differential equations

Part of the point of mentioning Newton is to draw attention to the difference between the approach that he represents – which is an approach based on causal links between bits of 'physics' in contact with each other – and the 'whole-system' approach based on the Lagrangian, the Hamiltonian, and minimisation principles. The former tends to be more immediately practical; the latter affords more insight. *The former is (broadly) characterised by differential equations (if you happen to know what those are), as the beating pulse of mathematical physics; the latter by integral equations.*

## 2 Fermat's and Hamilton's principles

Fermat's principle[1] says that the light 'chooses' the path that gets it to its destination in the minimal time (as opposed to minimal distance). Hamilton's principle[2] generalises this beyond light, to say that the particle (or other object) described by a Lagrangian[3] 'chooses' the path which minimises the action: the 'action' is the sum of the values of the Lagrangian at all the points along the path (*that is, the integral along the path,* '$S = \int L\,\mathrm{d}t$').

The Lagrangian, recall, is defined as $L = T - V$, the difference between the kinetic energy of (typically) a particle, $T$ and its potential energy, $V$. The kinetic energy is the energy the particle has by virtue of its motion; the potential energy is the energy the particle has 'in reserve', perhaps by virtue of being raised up, or from some internal source such as elastic energy. The expression for the Hamiltonian is formally derived from the Lagrangian in a somewhat abstract way, but in most cases it turns out to be simply $H = T + V$, which is the kinetic plus the potential energy.

[1] http://en.wikipedia.org/wiki/Pierre_de_Fermat
[2] http://en.wikipedia.org/wiki/William_Rowan_Hamilton
[3] http://en.wikipedia.org/wiki/Joseph_Louis_Lagrange

When I quoted Hamilton's equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$
$$\dot{q}_i = +\frac{\partial H}{\partial p_i},$$

the goal was simply to show how (relatively) simple they are. *The first of the two is simply saying that the change in the momentum of a particle (a heavier or a faster particle has more momentum) is dependent on how much the Hamiltonian changes with position (the position coordinate is written $q_i$ rather than, say, $x$, because it's not quite the same thing in all circumstances): or, a particle going 'downhill' gets faster. The fact that the same is true for the other equation is part of the symmetry of these equations, and partly why the coordinate $q$ is not quite the same as the obvious $x$, $y$ and $z$.*

# 3 Quantum mechanics

One illustration of the power of the Hamiltonian approach (as I mentioned, it's often more fiddly than a more straightforward approach, for ordinary calculations) is that it provides a fundamental starting point for excursions into other areas. One approach to the development of quantum mechanics is to start with a Hamiltonian description of classical mechanics, and identify the ways in which the equations have to be adjusted to be 'quantised'. The link is mathematically somewhat oblique, but if if you squint, you can possibly see a correspondence between the first of Hamilton's equations, above, and the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi,$$

which describes how the wave function $\psi$ evolves under the control of the Hamiltonian operator.

# 4 Quantum field theory and Noether's theorem

A lot of particle theory – which is really applied quantum field theory, which is in turn a quantised version of classical field theory, which is a version of Hamiltonian mechanics adapted to continuous media rather than particles – is concerned with the identification of symmetries in the Lagrangian, and Noether's theorem[4] is a key first step here. Noether's theorem has been claimed as "certainly one of the most important mathematical theorems ever proved in guiding the development of modern physics."

The ideas of the 'eightfold way'[5] and the Goldstone Boson[6] are built on this preoccupation with symmetry.

[4] http://en.wikipedia.org/wiki/Emmy_Noether
[5] http://en.wikipedia.org/wiki/Eightfold_Way_(physics)
[6] http://en.wikipedia.org/wiki/Goldstone_boson