NOTES ON QUATERNIONS, AND POINTING

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The 'qp' program manages rotations by expressing positions in 3-d, and rotations, as quaternions. This document briefly introduces quaternions, and discusses some specific formulae which are used in the 'qp' program.

There is an excellent and compact summary of the core results in the book [1], which goes far beyond the applications to 3-d rotations. The book [2] is considerably chattier, and is much more practically focused (to the extent that it has almost an engineering feel). The two books are rather indigestible in different ways. Girard's has a very mathematical tone: it is clear, but it generally leaves applications and insight to the exercises; Kuipers is all about applications, but spends so much time on these, and on contrasting the quaternion approach with trigonometric and matrix approaches, that it can be hard to find the useful quaternion results. The articles [3] and [4] have less detail than the books, but being by the same authors, they give a good illustration of their respective styles.

1. Quaternions

Quaternions are a generalisation (rather, one of the various generalisations) of the complex numbers. Rather than there being a single complex element i, there are three new elements i, j and k, which do not commute, and which have the properties:

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1 \tag{1}$$

$$\mathrm{ij} = -\mathrm{ji} = \mathrm{k}, \quad \mathrm{jk} = -\mathrm{kj} = \mathrm{i}, \quad \mathrm{ki} = -\mathrm{ik} = \mathrm{j}. \tag{2}$$

The set of such quaternions is labelled $\mathbb{H}$. With the quaternion multiplication illustrated here, they form a group. Quaternion multiplication is distributive (thus $a(b + c) = ab + ac$), but not associative ($a(bc) \neq (ab)c$).

As with the complex numbers, a general quaternion may be written

$$\begin{aligned} q &= q_0 1 + q_1 \mathrm{i} + q_2 \mathrm{j} + q_3 \mathrm{k} \\ &= (q_0, \mathbf{q}), \end{aligned}$$

where the second notation breaks apart the 'scalar' part of the quaternion and the 'vector' part. There is a precise correspondence between these vectors and the $(q_1, q_2, q_3) \in \mathbb{R}^3$, enough that we can use the term 'vector' to apply to both; also we will use the notation $\mathbf{q}$ to refer to either the vector part of a quaternion q, or to the quaternion itself if its scalar part is zero. The set of quaternions with zero scalar part is labelled $\operatorname{Vec}\mathbb{H}$.

We can define the quaternion conjugate $\overline{q}$ as

$$\overline{q} = q_0 1 - q_1 \mathrm{i} - q_2 \mathrm{j} - q_3 \mathrm{k}.$$

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Through application of the multiplications of Eq. (1), we can see that

$$|q|^2 \equiv q\overline{q} = \overline{q}q = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

If $|q|^2 = 1$, the quaternion is a 'unit quaternion'.

Again applying the rules in Eq. (1) to the product ab, we find that

$$\begin{aligned} ab &= 1(a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) \\ &\quad + \mathrm{i}(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2) \\ &\quad + \mathrm{j}(a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3) \\ &\quad + \mathrm{k}(a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1) \qquad (3) \\ &= -(a_0b_0 + \mathbf{a}\cdot\mathbf{b}) + a_0b + b_0a + \mathbf{a}\times\mathbf{b}, \qquad \forall a,b \in \mathbb{H} \qquad (4) \end{aligned}$$

after a little rearrangement. Here, $\mathbf{a}$ and $\mathbf{b}$ are the vector parts of the corresponding quaternions, and the dot and cross products on these are the usual vector operations. We can define inner and cross products on quaternions as follows:

$$a \times b \equiv \frac{1}{2}(ab - ba) = \mathbf{a}\times\mathbf{b} \qquad \forall a,b \in \mathbb{H} \qquad (5)$$

$$\begin{aligned} (a,b) &\equiv -\frac{1}{2}(ab + ba) = (a_0b_0 + \mathbf{a}\cdot\mathbf{b}) - a_0b - b_0a \qquad \forall a,b \in \mathbb{H} \qquad (6) \\ &= \mathbf{a}\cdot\mathbf{b} \qquad \forall a,b \in \operatorname{Vec}\mathbb{H}, \qquad (7) \end{aligned}$$

and we can observe that

$$\begin{aligned} ab &= -(a,b) + a \times b \qquad \forall a,b \in \mathbb{H} \qquad (8) \\ &= -\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\times\mathbf{b}, \qquad \forall a,b \in \operatorname{Vec}\mathbb{H}. \qquad (9) \end{aligned}$$

Note that $a \times b \in \operatorname{Vec}\mathbb{H}, \forall a, b$, and that if $(a,b) = 0$, then a and b anticommute: $ab = -ba$. Also if $a, b \in \operatorname{Vec}\mathbb{H}$, then typically $ab \notin \operatorname{Vec}\mathbb{H}$ (unless $(a,b) = 0$).

In the case where a is a unit vector, then $a\overline{a} = 1$ and $aa = -1$, from which $\overline{a} = -a$, and $(a,a) = \mathbf{a}\cdot\mathbf{a} = 1$.

It can be shown that the vector triple product in quaternion form is

$$\begin{aligned} (a \times b) \times c &= -c \times (a \times b) \\ &= -a(c,b) + b(c,a), \qquad \forall a,b,c \in \mathbb{H}. \qquad (10) \end{aligned}$$

The scalar triple product $(a, b \times c)$ does not have a similarly compact form, but we can note that, not unexpectedly,

$$(a \times b, a) = (a \times b, b) = 0. \qquad (11)$$

For the proofs of these relations, and a few more properties, see [1, Sect. 1.5.2].

If r is a unit quaternion, then we can decompose it into

$$r = \cos\frac{\theta}{2} + \mathbf{u}\sin\frac{\theta}{2},$$

where $\mathbf{u}$ is a unit vector (that is, $\mathbf{u}\cdot\mathbf{u} = 1$ and $\mathbf{u} \in \operatorname{Vec}\mathbb{H}$) and $\theta \in [0, 2\pi)$. It can then be shown that for all $q \in \operatorname{Vec}\mathbb{H}$, the quaternion

$$q' = rq\overline{r} \qquad (12)$$

is also in $\operatorname{Vec}\mathbb{H}$, and corresponds to a rotation of the vector $\mathbf{q}$ by an angle θ clockwise about the axis $\mathbf{u}$.

This is the key result for our present purposes, and is why the algebra of quaternions finds current application in computer graphics and video games, and in spacecraft and aviation control systems. The important advantages of the quaternion representation are that the quaternions remain well-behaved, and in particular are linear, in all orientations including the poles. It is also very straightforward to compose rotations, with good control over numerical accuracy. Some of these advantages attach to a representation in terms of rotation matrices, but these are slightly more complicated to set up, and a straightforward representation requires more storage space and calculation.

1.1. **Quaternions and rotation groups.** Neither the terminology nor the concepts of group theory are required for the following notes, but it may be interesting to point out the group structure of what we have so far.

If a quaternion q is multiplied by another, a, the result is another quaternion (as we have seen above); if the quaternion is multiplied by two others ab, then there is a further quaternion $c = ab$ which has the same effect. This indicates that the quaternions have a group structure. The same can be said of the quaternions in $\mathrm{Vec}\,\mathbb{H}$, and the operation of Eq. (12), and in this case the quaternions are a 'representation' of the group SU(2) (that is to say, they have the same structure as the abstract group with that name, another representation of which is the set of 2×2 complex-valued unit-norm matrices). A further group is the group of rotations of the 3-d sphere, SO(3) (any such rotation takes a point on the sphere into another point on the sphere, and the result of two successive rotations corresponds to some other single rotation; thus this is a group), and it turns out that the group SU(2) contains two copies of the group SO(3), which is why, ultimately, the operation of Eq. (12) is able to represent a rotation of the 3-d sphere.

That there are *two* copies of SO(3) in SU(2) can be seen by considering 'rotation' in Eq. (12) by more than 2π:

$$\begin{aligned} r_{2\pi+\theta} &= \cos\frac{2\pi+\theta}{2} + \mathbf{u}\sin\frac{2\pi+\theta}{2} \\ &= -\cos\frac{\theta}{2} - \mathbf{u}\sin\frac{\theta}{2} \\ &= -r_\theta, \end{aligned}$$

with the result that

$$q' = r_{2\pi+\theta}q\bar{r}_{2\pi+\theta} = r_\theta q\bar{r}_\theta.$$

Thus, there are two distinct operations corresponding to Eq. (12) which correspond to the same rotation by θ of the sphere.

2. The quaternions and spherical polar coordinates

To illustrate the use of quaternions for rotation, we can identify the three basis quaternions i, j and k with the three unit vectors of 3-d space, and calculate the effect of a rotation r_φ by an angle φ about k, followed by a rotation r_θ by an angle θ clockwise about the *new* axis $\mathrm{j}' = r_\varphi \mathrm{k}\bar{r}_\varphi$.

Although it is not a conventional notation, it is very convenient in these calculations to write $\lfloor\theta = \cos(\theta/2)$ and $\lceil\theta = \sin(\theta/2)$, after which $\lfloor\theta^2 + \lceil\theta^2 = 1$, $\lfloor\theta^2 - \lceil\theta^2 = \cos\theta$ and $2\lfloor\theta\lceil\theta = \sin\theta$. We interpret $\lfloor\theta^2$ as $\cos^2(\theta/2)$.

Thus

$$\begin{aligned} S = r_\theta r_\varphi &= (\lfloor\theta + \mathrm{j}'\lceil\theta) r_\varphi \\ &= \lfloor\theta(\lfloor\varphi + \mathrm{k}\lceil\varphi) + \lceil\theta r_\varphi \mathrm{j} && \text{since } \overline{r}_\varphi r_\varphi = 1 \\ &= \lfloor\theta\lfloor\varphi - \mathrm{i}\lceil\theta\lceil\varphi + \mathrm{j}\lceil\theta\lfloor\varphi + \mathrm{k}\lfloor\theta\lceil\varphi. \end{aligned}$$

Applying this rotation to the z-axis, k, for example, we obtain

$$\begin{aligned} Sk\overline{S} &= (\lfloor\theta\lfloor\varphi - \mathrm{i}\lceil\theta\lceil\varphi + \mathrm{j}\lceil\theta\lfloor\varphi + \mathrm{k}\lfloor\theta\lceil\varphi)\mathrm{k}(\lfloor\theta\lfloor\varphi + \mathrm{i}\lceil\theta\lceil\varphi - \mathrm{j}\lceil\theta\lfloor\varphi - \mathrm{k}\lfloor\theta\lceil\varphi) \\ &= (\lfloor\theta\lfloor\varphi - \mathrm{i}\lceil\theta\lceil\varphi + \mathrm{j}\lceil\theta\lfloor\varphi + \mathrm{k}\lfloor\theta\lceil\varphi)(\lfloor\theta\lceil\varphi + \mathrm{i}\lceil\theta\lfloor\varphi + \mathrm{j}\lceil\theta\lceil\varphi + \mathrm{k}\lfloor\theta\lfloor\varphi) \\ &= \mathrm{i}\sin\theta\cos\varphi + \mathrm{j}\sin\theta\sin\varphi + \mathrm{k}\cos\theta, \end{aligned}$$

on expanding the brackets and simplifying.

2.1. **Conversion from equatorial to horizontal coordinates.** If an object has RA and Dec (α, δ), what are its Alt-Az coordinates (θ, φ), at an observatory with co-latitude $\overline{l}$ and at local sidereal time θ_s?

In equatorial coordinates, the object is in direction

$$T = \mathrm{i}\cos\alpha\sin\overline{\delta} + \mathrm{j}\sin\alpha\sin\overline{\delta} + \mathrm{k}\cos\overline{\delta} \tag{13}$$

(writing $\overline{\delta} = \pi/2 - \delta$). We rotate this vector by angle $-\theta_s$ clockwise about k to bring it into the right-handed coordinate system where the longitudinal coordinate is zero at the meridian, and then by angle $-\overline{l}$ about j to bring it from this system into the right-handed horizontal system. Note that here we are replacing the change of coordinate system by a rotation of the target vector T in a fixed coordinate system (which is in effect, in this case, the horizontal system).

For example, consider Rigel (RA=$5^{\mathrm{h}}14^{\mathrm{m}}32^{\mathrm{s}}$, Dec=$-8°12'6''$), as observed from Glasgow (latitude $55°54'8''$) at LMST=$3^{\mathrm{h}}27^{\mathrm{m}}29^{\mathrm{s}}$. We have

$$\begin{aligned} T &= \mathrm{i}0.195 + \mathrm{j}0.970 - \mathrm{k}0.143 \\ r_\varphi &= \cos(-51^{\circ}\!.9/2) - \mathrm{k}\sin(51^{\circ}\!.9/2) \\ r_l &= \cos(-34^{\circ}\!.1/2) - \mathrm{j}\sin(34^{\circ}\!.1/2) \\ r_l r_\varphi &= 0.860 + \mathrm{i}0.128 - \mathrm{j}0.264 - \mathrm{k}0.418 \\ r_l r_\varphi T \overline{r}_\varphi \overline{r}_l &= \mathrm{i}0.812 + \mathrm{j}0.446 + \mathrm{k}0.377. \end{aligned}$$

We can turn this last result back into polar coordinates by comparison with Eq. (13), which tells us that $\tan\varphi = q_2/q_1$ and $q_3 = \cos\overline{\theta}$. Remembering to turn φ into azimuthal coordinate $\pi - \varphi$, we end up with Rigel having azimuth $151^{\circ}\!.2$ and elevation $22^{\circ}\!.1$.

This is a mapping from one angular system to another, through a transformation expressed in angular terms, and doing the calculation in terms of quaternions is indirect to the point of perversity. The example illustrates the relationship between angular measure and quaternions, but the examples below might illustrate the power of the quaternion approach more clearly.

3. Various relations involving quaternions

The point of using a quaternion representation is to make it easy, and computationally efficient, to perform a variety of calculations involving motions on the sphere.

3.1. **Tracking.** Our goal here is to describe the motion of a point across the celestial sphere due to sidereal motion, in horizontal coordinates. In equatorial coordinates, this consists of rotation by an angle $h = \omega t$. For sidereal rotation, $\omega = 15'/\mathrm{min} \times 1.00274$ and t is in UT. Thus the rotation in horizontal coordinates, of a target point described by a quaternion $T \in \mathrm{Vec}\,\mathbb{H}$, is generated by

$$S_h = r_l(\lfloor h + \mathrm{k}\lceil h)\bar{r}_l, \tag{14}$$

where r_l is the rotation by observatory co-latitude, described above. Specifically, at a time t, we must rotate T to

$$T_t = ST\overline{S}. \tag{15}$$

Consider now an infinitesimal rotation, by an angle $\mathrm{d}h$. We can write

$$T \to T' = \mathrm{d}ST\bar{\mathrm{d}}S, \qquad \mathrm{d}S = 1 + \mathbf{u}\frac{\mathrm{d}h}{2},$$

where $\mathbf{u}$ is the axis of rotation (in horizontal coordinates). Expanding this, and retaining only first order in $\mathrm{d}h$, we obtain

$$\begin{aligned} T' &= T + \frac{1}{2}(\mathbf{u}T - T\mathbf{u})\mathrm{d}h \\ &= T + \mathbf{u} \times T\mathrm{d}h \\ &\equiv T + \mathrm{d}T. \end{aligned}$$

Consider now this rotation expressed as infinitesimal rotations of angle φ about k and θ about $\mathrm{j}' = \widehat{\mathrm{k} \times T} = \mathrm{k}\times T/|\mathrm{k}\times T|$, the unit vector in the direction perpendicular to both k and T. These correspond to rotations in longitude (about the z-axis) and in co-latitude respectively. As above, we have

$$\begin{aligned} T' &= \left(1 + \mathrm{k}\frac{\mathrm{d}\varphi}{2}\right)\left(1 + \mathrm{j}'\frac{\mathrm{d}\theta}{2}\right)T\left(1 - \mathrm{j}'\frac{\mathrm{d}\theta}{2}\right)\left(1 - \mathrm{k}\frac{\mathrm{d}\varphi}{2}\right) \\ &= T + \mathrm{d}\varphi\mathrm{k} \times T + \mathrm{d}\varphi\mathrm{j}' \times T. \end{aligned}$$

Taking the inner product of this with, in turn, k and j', we find

$$\begin{aligned} (\mathrm{d}T, \mathrm{j}') &= (\mathrm{d}\varphi\mathrm{k} \times T + \mathrm{d}\theta\mathrm{j}' \times T, \mathrm{j}') \\ &= \mathrm{d}\varphi(\mathrm{k} \times T, \mathrm{k} \times T)\frac{1}{|\mathrm{k} \times T|} \\ &= \mathrm{d}\varphi|\mathrm{k} \times T|, \end{aligned}$$

and using

$$\mathrm{j}' \times T = \frac{1}{|\mathrm{k} \times T|}(\mathrm{k} \times T) \times T = \frac{1}{|\mathrm{k} \times T|}\left[-\mathrm{k}|T|^2 + T(\mathrm{k}, T)\right]$$

we find

$$\begin{aligned} (\mathrm{d}T, \mathrm{k}) &= (\mathrm{d}\varphi\mathrm{k} \times T + \mathrm{d}\theta\mathrm{j}' \times T, \mathrm{k}) \\ &= \mathrm{d}\theta(\mathrm{j}' \times T, \mathrm{k}) \\ &= \frac{\mathrm{d}\theta}{|\mathrm{k} \times T|}\left[(\mathrm{k}, T)^2 - 1\right], \end{aligned}$$

since $(\mathrm{k}, \mathrm{k}) = |T|^2 = 1$. Thus, writing $\mathrm{d}T = (\mathbf{u} \times T)\mathrm{d}h$, Note: angvel

$$\dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{(\mathbf{u}\times T, \mathrm{k}\times T)}{|\mathrm{k}\times T|^2}\dot{h} \tag{16a}$$

$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{(\mathbf{u}\times T, \mathrm{k})|\mathrm{k}\times T|}{(\mathrm{k}, T)^2 - 1}\dot{h}. \tag{16b}$$

Thus for a given rotation angle h about the equatorial axis, we can calculate S_h by Eq. (14), and hence $\mathbf{u}$; we can use this to calculate the expected pointing, T_t, by Eq. (15), and the expected velocities in azimuth and co-elevation at that point using Eq. (16a) and Eq. (16b), where the T in these expressions is the T_t calculated for the expected or actual position of the telescope.

The inner product of two vectors, (a, b), requires three multiplications and two additions, and the cross products $a \times b$ require six multiplications and three additions. On the same theme, the ordinary quaternion product, for two quaternions not necessarily in $\mathrm{Vec}\,\mathbb{H}$, requires 16 multiplications and 11 additions; a little algebra allows rotation Eq. (12) to be done more directly, as

Note: rotation

$$\begin{aligned}
rq\overline{r} = {} & q_0(r_0^2 + r_1^2 + r_2^2 + r_3^2) \\
& + \mathrm{i}\left[q_1(r_0^2 + r_1^2 - r_2^2 - r_3^2) + 2q_3(r_0r_2 + r_1r_3) + 2q_2(r_1r_2 - r_0r_3)\right] \\
& + \mathrm{j}\left[q_2(r_0^2 - r_1^2 + r_2^2 - r_3^2) + 2q_1(r_0r_3 + r_1r_2) + 2q_3(r_2r_3 - r_0r_1)\right] \\
& + \mathrm{k}\left[q_3(r_0^2 - r_1^2 - r_2^2 + r_3^2) + 2q_2(r_0r_1 + r_2r_3) + 2q_1(r_1r_3 - r_0r_2)\right].
\end{aligned} \tag{17}$$

With suitable reusing of intermediate results, this requires a little less than twice the above number of operations, at 23 multiplications and 24 additions.

3.2. **Composition of velocities.** Suppose we have two rotations $r_\sigma = \lfloor\sigma + \mathbf{u}_\sigma\lceil\sigma$ and $r_\rho = \lfloor\rho + \mathbf{u}_\rho\lceil\rho$, and we wish to identify a single rotation $r_h = \lfloor h + \mathbf{u}_h\lceil h$ which has the same effect as r_ρ followed by r_σ. For finite angles, this is straightforward:

$$r_h = r_\sigma r_\rho = \lfloor\sigma\lfloor\rho + \lceil\sigma\lfloor\rho\mathbf{u}_\sigma + \lfloor\sigma\lceil\rho\mathbf{u}_\rho + \lceil\sigma\lceil\rho\mathbf{u}_\sigma\mathbf{u}_\rho.$$

If we instead rotate by infinitesimal angles $\mathrm{d}\rho$ and $\mathrm{d}\sigma$, then the composite rotation is

$$r_h = 1 + \mathbf{u}_h\frac{\mathrm{d}h}{2} = \left(1 + \mathbf{u}_\sigma\frac{\mathrm{d}\sigma}{2}\right)\left(1 + \mathbf{u}_\rho\frac{\mathrm{d}\rho}{2}\right).$$

Much as above, this rotation will rotate T into T', where

$$T' = r_hT\overline{r}_h = T + (\mathbf{u}_h\mathrm{d}h) \times T \tag{18}$$

$$= \left(1 + \mathbf{u}_\sigma\frac{\mathrm{d}\sigma}{2}\right)\left(1 + \mathbf{u}_\rho\frac{\mathrm{d}\rho}{2}\right)T\left(1 - \mathbf{u}_\rho\frac{\mathrm{d}\rho}{2}\right)\left(1 - \mathbf{u}_\sigma\frac{\mathrm{d}\sigma}{2}\right)$$

$$= T + (\mathbf{u}_\sigma\mathrm{d}\sigma + \mathbf{u}_\rho\mathrm{d}\rho) \times T, \tag{19}$$

Note: vcompose

and comparing Eq. (18) and Eq. (19) we can see that

$$\mathbf{u}_h\dot{h} = \mathbf{u}_\sigma\dot{\sigma} + \mathbf{u}_\rho\dot{\rho},$$

where $\dot{h}$ is obtained by the demand that $\mathbf{u}_h$ be unit. This can be decomposed into longitudinal and latitudinal velocities as in Eq. (16).

3.3. **Rotating one quaternion into another.** Consider $A, B \in \mathrm{Vec}\,\mathbb{H}$, with $A\overline{A} = B\overline{B} = 1$. We wish to calculate the rotation which will bring A into B, paying some attention to numerical efficiency.

3.3.1. *Direct 'trigonometric' method.* Write $\mathbf{a} = A \times B \in \mathrm{Vec}\,\mathbb{H}$. If we define $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$, and note $|a \times b| = |a||b| \sin\alpha$, then the rotation from A to B is about the axis $\hat{\mathbf{a}}$, by an angle $\alpha = \arcsin|a \times b|$, and we can perform this rotation by constructing the generator $r_\alpha = \lfloor\alpha + \hat{\mathbf{a}}\lceil\alpha$. Note: rotations

We can also find $\lfloor\alpha$ and $\lceil\alpha$ without using the trigonometric functions by noting that $\cos\alpha = (1 - |a \times b|^2)^{1/2}$, and that

$$\left.\begin{aligned}\lfloor\alpha &= \cos\alpha/2\\ \lceil\alpha &= \sin\alpha/2\end{aligned}\right\} = \left(\frac{1 \pm \cos\alpha}{2}\right)^{1/2}. \tag{20}$$

3.3.2. *'No-trig' method.* Alternatively – requiring more algebra but less arithmetic – we can proceed as follows. Write

$$r = r_0 + \rho\mathbf{a}, \tag{21}$$

where $r\overline{r} = 1$, so $r_0^2 - \rho^2\mathbf{a}\mathbf{a} = 1$. What are r_0 and ρ so that

$$A' \equiv rA\overline{r} = B \quad ?$$

We have

$$\begin{aligned}A' &= rA\overline{r}\\ &= (r_0 + \rho\mathbf{a})A(r_0 - \rho\mathbf{a})\\ &= r_0^2 A + 2\rho r_0 \mathbf{a} \times A - \rho^2 \mathbf{a}A\mathbf{a}.\end{aligned} \tag{22}$$

Using the vector triple product formula, Eq. (10),

$$\begin{aligned}\mathbf{a} \times A &= (A \times B) \times A\\ &= -A(A, B) + B(A, A) = B + A(AB + BA)/2\\ &= (B + ABA)/2 \qquad\qquad \text{since } AA = -1.\end{aligned}$$

Now, $ABA = -AB\overline{A}$ is a rotation of B about A by π radians, negated, and therefore corresponds to a reflection of B in the plane perpendicular to A, thus $ABA = B - 2A(A, B)$; alternatively

$$\begin{aligned}ABA &= (ABA + AAB) - AAB\\ &= -2A(A, B) + B \qquad\qquad \text{since } AA = -1.\end{aligned}$$

Since $(\mathbf{a}, A) = 0$, we have $\mathbf{a}A = -A\mathbf{a}$, so that Eq. (22) becomes

$$\begin{aligned}A' &= (r_0^2 + \rho^2\mathbf{a}\mathbf{a})A + 2\rho r_0\big(B - A(A, B)\big)\\ &= \big(2r_0^2 - 1 - 2\rho r_0(A, B)\big)A + 2\rho r_0 B.\end{aligned} \tag{23}$$

This is equal to B if $2\rho r_0 = 1$ and $r_0^2 = \big(1 + (A, B)\big)/2$. This fixes both of the coefficients in Eq. (21).

The 'obvious' route from $\mathbf{a}$ to the generator r, using the angle α explicitly, uses the square-root, arcsine, sine and cosine functions once each. The route using Eq. (20) uses the square-root function four times (to calculate $|\mathbf{a}|$ and $\cos\alpha$, and twice in Eq. (20)); the route via Eq. (23) uses that only once (to calculate $\sqrt{r_0^2}$). The speed differences are unlikely to be terribly significant in other than very constrained environments. There *may* be some accuracy advantages in the route with less arithmetic, but these also are unlikely to be important in most circumstances.

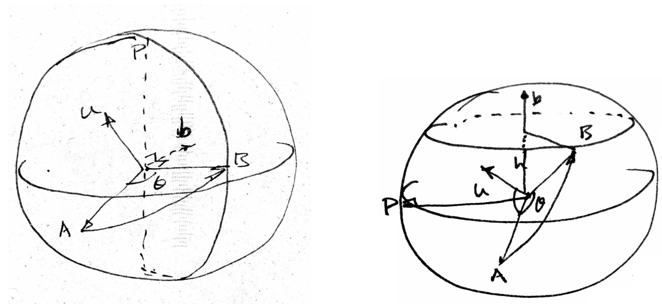

FIGURE 1. (a, left) A quaternion A being rotated about an axis u until it reaches a point B on a plane through P with normal b. (b, right) The same, but where the terminal plane cuts the vector b at a distance h from the origin.

3.4. **Detecting when a rotation crosses a meridian.** Consider Fig. (1)a, which shows a quaternion A being rotated along a great circle about u. Given A, u and b, we want to find B or, equivalently, the angle θ through which the point A is rotated until it hits the plane PB. More specifically, we are interested in the meridian drawn out by rotating P by π about b. All of A, u, b and B are unit quaternions, and all but u are vectors.

Note: meridians

The resulting direction B is

$$\begin{aligned} B &= (\lfloor\theta + u\lceil\theta)A(\lfloor\theta - u\lceil\theta) \\ &= \lfloor\theta^2 A + 2\lfloor\theta\lceil\theta(u\times A) + \lceil\theta^2 uA\overline{u} \\ &= \lfloor\theta^2 A + \lceil\theta^2 A^u + 2\lfloor\theta\lceil\theta(u\times A), \end{aligned} \tag{24}$$

where we have defined $v^q \equiv qv\overline{q} = -qvq$, for $v, q \in \mathrm{Vec}\,\mathbb{H}$. Comparing with Eq. (12), we can see that v^q corresponds to a rotation of v by an angle π about q, which is equivalent to a reflection of v in the line generated by the vector q. As a special case, when u is perpendicular to A, then $A^u = -A$ (and $B = \cos\theta A + \sin\theta(u\times A)$).

Define a set of basis quaternions to be $k = P$, $j = b = \widehat{P\times B}$ and $i = -P\times b$; it is natural to think of the point P as being the pole, but this is not necessary to the construction below. We can expand each of the quaternions in the problem in terms of this basis, thus $B = B_1\mathrm{i} + B_2\mathrm{j} + B_3\mathrm{k}$. From this,

$$B^{\mathrm{i}} = -\mathrm{i}B\mathrm{i} = B_i\mathrm{i} - B_2\mathrm{j} - B_3\mathrm{k} \Rightarrow \frac{1}{2\mathrm{i}}(B + B^{\mathrm{i}}) = B_1,$$

and similarly for the other basis quaternions.

When B is in the plane in Fig. (1)b, the projection of B onto b will be such that $(B, b) = B_2 = h$.

When B is in the PB plane, we will have $B_2 = 0$, or

$$\begin{aligned} h = B_2 &= \frac{1}{2\mathrm{j}}(B + B^{\mathrm{j}}) \\ &= \frac{1}{2\mathrm{j}}\Big(\lfloor\theta^2(A + A^{\mathrm{j}}) + \lceil\theta^2(A^u + (A^u)^{\mathrm{j}}) + 2\lfloor\theta\lceil\theta\big(u\times A + (u\times A)^{\mathrm{j}}\big)\Big). \end{aligned}$$

Restoring $\mathrm{j} = b$ in this expression, we note that each of these terms is a multiple of the vector b, so that we can divide throughout by $\lfloor\theta^2$, take the inner product of each coefficient with b, use $1/\lfloor\theta^2 = (1 + \tan^2\theta/2)$, and write

$$t_2 \tan^2\frac{\theta}{2} + 2t_1\tan\frac{\theta}{2} + t_0 = 0,$$

where

$$\begin{aligned} t_2 &= (A^u + (A^u)^b, b) - 2h \\ t_1 &= (u \times A + (u \times A)^b, b) \\ t_0 &= (A + A^b, b) - 2h, \end{aligned}$$

so that

$$\tan\frac{\theta}{2} = \frac{1}{t_2}\left(-t_1 \pm \sqrt{t_1^2 - t_2 t_0}\right). \tag{25}$$

From this point we can obtain the components in Eq. (24) without trigonometry via

$$\begin{aligned} \sec^2\frac{\theta}{2} &= \frac{1}{\lfloor\theta^2} = 1 + \tan^2\frac{\theta}{2} \\ \operatorname{cosec}^2\frac{\theta}{2} &= \frac{1}{\lceil\theta^2} = 1 + \cot^2\frac{\theta}{2} \\ 1/(2\lfloor\theta\lceil\theta)^2 &= \frac{1}{4}\left(2 + \tan^2\frac{\theta}{2} + \cot^2\frac{\theta}{2}\right) \end{aligned}$$

where we take the sign of $2\lfloor\theta\lceil\theta$ to match the sign of the corresponding $\tan\theta/2$. Notice that these require multiplications, divisions and additions, but no trigonometric functions.

Given the zero, one, or two solutions for $\tan\theta/2$, we have corresponding numbers of values of Eq. (24). Of these, we want the first B which corresponds to a rotation of P about b by an angle less than π, which is to say that B which is such that $P \times B$ is in the same direction as b, ie, $(P \times B, b) > 0$.

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