# NOTES ON THE KALMAN FILTER

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These notes discuss the specific Kalman filter used in the 'qp' program. It is not intended to be a general introduction to Kalman filters, since there exist multiple such introductions, including the valuable [**?**] and [**?**]. Instead, this document describes the way in which the general technique described there is specialised to the particular case being modelled. We here use the notation of [**?**].

In the case of the SRT, the longitudinal and latitudinal degrees of freedom are independent, and so we can manage them completely separately. Therefore we handle both directions using a 1-d filter. Because of this, in this application we don't have to care about the sense of these coordinate – east of north/south – and beyond noting that $\omega$, below, is dimensionless, we don't have to care about their units.

The angles that are provided to the filter are the actual angles in radians. The 'speed' that is provided to the filter is the speed as a fraction of a reference speed which is chosen to be the maximum speed of the corresponding motor (that is, the speed at which it moves in fact, when commanded to move at speed 255). The occasional 'measurements' arise from the telescope driving moving past one or other of a sequence of reed switches, at each of which we produce a 'click'.

## 1. The filter

The state vector is

$$\mathbf{x} = \begin{pmatrix} \theta \\ V \end{pmatrix},$$

where $\theta$ is one or other of the angular coordinates, and $V$ is the maximum motor speed in the corresponding axis. Specifically, $V$ is the speed in rad/s corresponding to commanded speed 255.

At each step, we require the commanded speed in the preceding timestep to have been constant. Refer to the commanded speed as $\omega \in [-1, 1]$, so that the actual speed is $V\omega$ (rad/s).

### 1.1. The predictor step.

The state transition matrix $\mathbf{F}_t$ is such that

$$\hat{\mathbf{x}}_{t|t-1} = \begin{pmatrix} \theta_{t|t-1} \\ V_{t|t-1} \end{pmatrix} = \mathbf{F}_t \hat{\mathbf{x}}_{t-1|t-1} = \begin{pmatrix} 1 & \omega\Delta t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \theta_{t-1|t-1} \\ V_{t-1|t-1} \end{pmatrix}$$

(this assumes that the drive motors are strong enough that we can ignore the time taken to accelerate to the commanded speed).

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The covariance matrix is updated similarly

$$\begin{aligned}
P_{t|t-1} &= \mathbf{F}_t P_{t-1|t-1} \mathbf{F}_t^\top + \mathbf{Q}_t \\
&= \begin{pmatrix} 1 & \omega\Delta t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}_{t-1|t-1} \begin{pmatrix} 1 & 0 \\ \omega\Delta t & 1 \end{pmatrix} + \begin{pmatrix} \delta\theta^2 & 0 \\ 0 & \delta V^2 \end{pmatrix} \\
&= \mathbf{P}_{t-1|t-1} + \begin{pmatrix} \omega\Delta t(p_{01} + p_{10} + \omega\Delta t p_{11}) + \delta\theta^2 & \omega\Delta t p_{11} \\ \omega\Delta t p_{11} & \delta V^2 \end{pmatrix}.
\end{aligned}$$

There are no control inputs, so no control vector $\mathbf{u}_t$, and consequently no control matrix $\mathbf{B}$. The state parameters $\theta$ and $V$ are independent, so the noise matrix $\mathbf{Q}$ is diagonal.

1.2. **The corrector step.** There are two cases here, where we do and where we don't make a measurement of angle.

We only 'measure' the position of the telescope drives when the drive passes one of the reed switches, and the system receives a 'click'. Nonetheless, the system is updated periodically, either with or without a change in speed. These can be handled easily as a notional measurement with infinite $\mathbf{R}_t$, for which the Kalman gain $\mathbf{K}_t = 0$, so that

$$\begin{aligned}
\hat{\mathbf{x}}_{t|t} &= \hat{\mathbf{x}}_{t|t-1} && \text{('missing' measurements)} \\
\mathbf{P}_{t|t} &= \mathbf{P}_{t|t-1}.
\end{aligned}$$

For the steps which include a click, we can update the state vector and covariance using

$$\begin{aligned}
\hat{\mathbf{x}}_{t|t} &= \hat{\mathbf{x}}_{t|t-1} + \mathbf{K}_t(z_t - \mathbf{H}_t \hat{\mathbf{x}}_{t|t-1}) \\
\mathbf{P}_{t|t} &= \mathbf{P}_{t|t-1} - \mathbf{K}_t \mathbf{H}_t \mathbf{P}_{t|t-1}.
\end{aligned}$$

Here, we compare the measured 'click' coordinate $z_t$ (which will be an integer multiple of the interval in radians between reed switches) with the predicted state position $z_{t|t-1}$, via

$$\hat{\mathbf{z}}_{t|t-1} = \mathbf{H}_t \hat{\mathbf{x}}_{t|t-1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ V \end{pmatrix}_{t|t-1} = \theta_{t|t-1}.$$

The 'matrix' $\mathbf{z}_t$ is therefore simply a $1 \times 1$ matrix. The uncertainty in position – arising from perhaps drive slop or other systematic positioning errors – will be a constant $\delta z$, resulting in the $1 \times 1$ error matrix $\mathbf{R}_t = \delta z^2$. With this, the Kalman gain is

$$\begin{aligned}
\mathbf{K}_t &= \frac{1}{A} \mathbf{P}_{t|t-1} \mathbf{H}^\top, \\
&= \frac{1}{A} \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}_{t|t-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{A} \begin{pmatrix} p_{00} \\ p_{10} \end{pmatrix},
\end{aligned}$$

where the denominator $A$ is

$$\begin{aligned} A &= \mathbf{H}_t P_{t|t-1} \mathbf{H}_t^\top + \mathbf{R}_t \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}_{t|t-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \delta z^2 \\ &= p_{00|t-1} + \delta z^2. \end{aligned}$$

Thus, the correction step for the state vector is given by

$$\begin{pmatrix} \theta \\ V \end{pmatrix}_{t|t} = \begin{pmatrix} \theta \\ V \end{pmatrix}_{t|t-1} + \frac{1}{A} \begin{pmatrix} p_{00} \\ p_{10} \end{pmatrix} (z_t - \theta_{t|t-1}),$$

and that of the covariance matrix is

$$\begin{aligned} \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}_{t|t} &= \mathbf{P}_{t|t-1} - \mathbf{K}_t \mathbf{H}_t \mathbf{P}_{t|t-1} \\ &= \mathbf{P}_{t|t-1} - \frac{1}{A} \mathbf{P}_{t|t-1} \mathbf{H}_t^\top \mathbf{H}_t \mathbf{P}_{t|t-1} \\ &= \mathbf{P}_{t|t-1} - \frac{1}{A} \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}_{t|t-1} \\ &= \mathbf{P}_{t|t-1} - \frac{1}{A} \begin{pmatrix} p_{00}^2 & p_{00} p_{01} \\ p_{10} p_{00} & p_{10} p_{01} \end{pmatrix}_{t|t-1} \end{aligned}$$