VESF School on Gravitational Waves

Cascina, Italy

May 25th – 29th 2009

An Introduction to General Relativity, Gravitational Waves and Detection Principles

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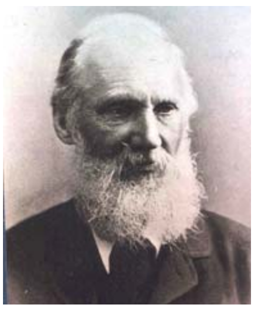
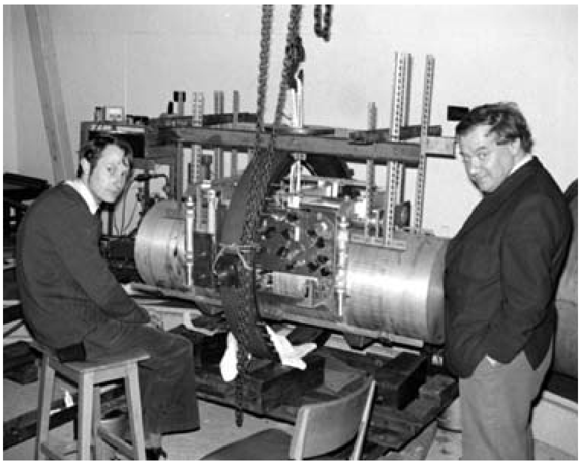
Who am I?…



William Thompson
(Lord Kelvin)
1824 - 1907

Who am I?...



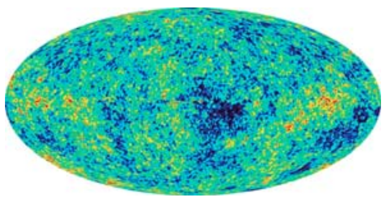
Jim Hough and Ron Drever, 1978

Institute for Gravitational Research

~40 research staff and students, with activity spanning advanced materials, optics and interferometry, data analysis, for ground- and space-based GW detectors.

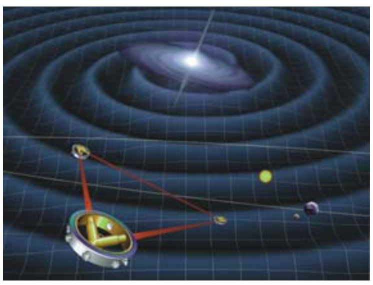
My Research Interests:

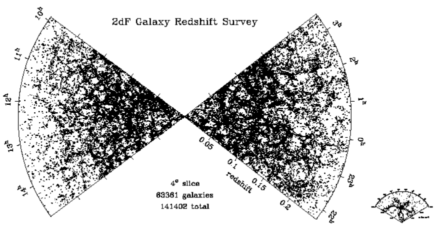
Cosmology:
- galaxy distance indicators
- galaxy redshift surveys
- cosmological parameters

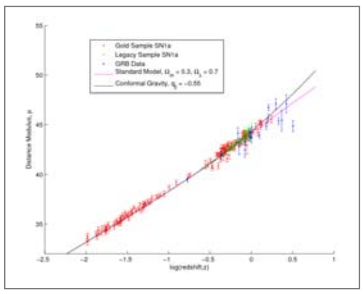


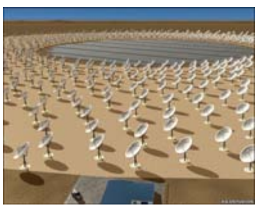
Gravitational wave data analysis:
- Bayesian inference methods
- LISA data analysis

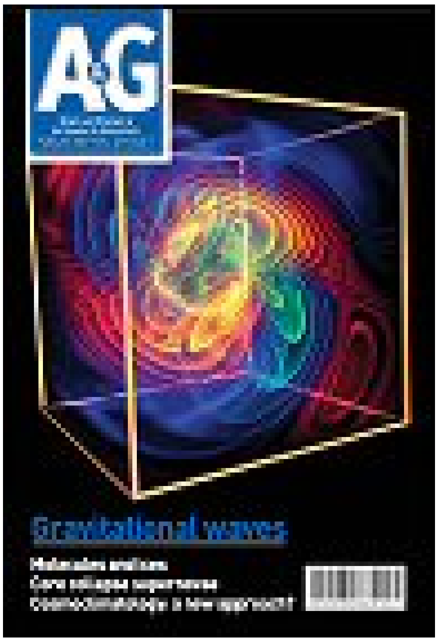
Multi-messenger astronomy











February 2007

Gravitational astrophysics

Martin Hendry and Graham Woan survey the astrophysical problems that may be illuminated by the detection of gravitational waves – and how planned instruments will do so.

ABSTRACT

Like the surface of a busy swimming pool, spacetime is awash with waves generated by the local and distant motions of mass and, in principle, much of this activity can be reconstructed by analysing the waveforms. However, instrumentation with a reasonable chance of directly detecting these gravitational waves has only become available within the past year, with the LIGO detectors now running at design sensitivity. Here we review the burgeoning field of observational gravitational astrophysics: using gravitational wave detectors as telescopes to help answer a wide range of astrophysical questions from neutron-star physics to cosmology. The next generation of ground-based telescopes should be able to make extensive gravitational observations of some of the more energetic events in our local universe. Looking only slightly further ahead, the space-based LISA observatory will reveal the gravitational universe in phenomenal detail, supplying high-quality data on perhaps thousands of sources, and tackling some of the most fascinating questions in contemporary astronomy.

Observational astrophysics is often presented as the ultimate remote-sensing problem. There is no possibility of travelling to the majority of targets to interact with them and carry out experiments. Furthermore, many of our most closely studied targets have long since entirely ceased to exist, and can be studied only by reconstruction via their radiation fields. The tools at our disposal, though powerful, are limited in scope. Indeed, the vast majority of what we know about the universe comes from studying a single measure: the second moment (variance) of the electric component of the electromagnetic radiation field. This single statistic has delivered nearly all the imaging, spectroscopy and multiwavelength astrophysics that shapes our current view of the universe. We give it many names, such as apparent magnitude, fringe visibility and photon counts, but essentially we are studying a statistic of the ensemble electric field from many independent charged particles in a remote environment. Astrophysics based on the electromagnetic field will always be so. Although large-scale coherent electromagnetic processes do exist (in, for example, astrophysical masers and pulsar radio emission) the majority of what we see is the result of the random motions of charged particles in similar environments and the nature of the electromagnetic interaction means that small-scale emission mechanisms dominate. The field itself is also random but can be characterized by its statistics, themselves dependent on the distribution of environmental conditions in the source. It is from these statistics, rather than the field itself, that we learn about the environment.

This may seem a perverse way of describing observational astrophysics, but it highlights limitations of our techniques and gives a flavour of what we may be missing. Forms of non-electromagnetic astronomy exist, notably based on cosmic-ray and neutrino-particle counts, but the last (as far as we know) truly great challenge

coherent gravitational waves that directly reflect the motion. Electromagnetic radiation is usually generated by small-scale motions of charged particles, but powerful gravitational radiation can arise on stellar scales and with correspondingly long wavelengths, where the interstellar medium is electromagnetically opaque.

Unique view

about 380 000 years after the Big Bang, the cosmic gravitational background radiation (CGBR) probably did so after only about 10^{-35} s. CGBR observations would therefore see further back in time and reveal the mass distribution in the universe at this earliest instant, though their detection will be a significant challenge.

Given its clear attractions, why is gravitational astronomy not a standard technique in the astro-

Einstein's "Annus Mirabilis": 1905

ON THE ELECTRODYNAMICS OF MOVING BODIES

BY A. EINSTEIN

June 30, 1905

It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.

Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good.[1] We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell's theory for stationary bodies. The introduction of a "luminiferous ether" will prove to be superfluous inasmuch as the view here to be developed will not require an "absolutely stationary space" provided with special properties, nor

[1]The preceding memoir by Lorentz was not at this time known to the author.

1

einstein year

World Year of PHYSICS 2005

University of Glasgow SUPA

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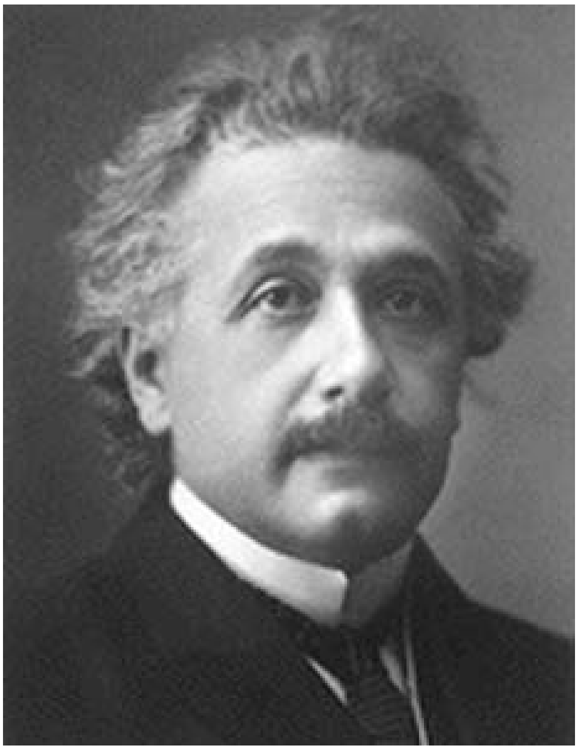
IGR



518 A. Einstein, Annalen der Physik, Band 49, 1916

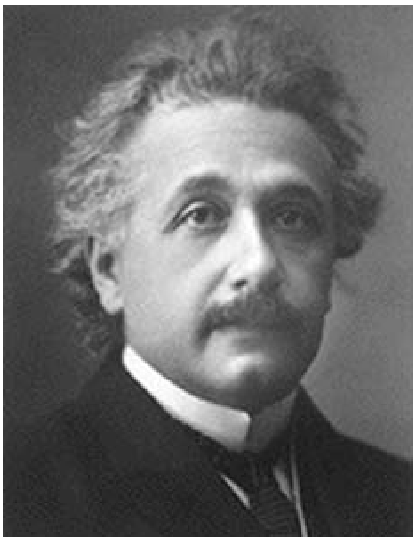
1916. № 7.

ANNALEN DER PHYSIK.
VIERTE FOLGE. BAND 49.

1. *Die Grundlage der allgemeinen Relativitätstheorie;*
von A. Einstein.

Die im nachfolgenden dargelegte Theorie bildet die denkbar weitgehendste Verallgemeinerung der heute allgemein als „Relativitätstheorie" bezeichneten Theorie; die letztere nenne ich im folgenden zur Unterscheidung von der ersteren „spezielle Relativitätstheorie" und setze sie als bekannt voraus. Die Verallgemeinerung der Relativitätstheorie wurde sehr erleichtert durch die Gestalt, welche der speziellen Relativitätstheorie durch Minkowski gegeben wurde, welcher Mathematiker zuerst die formale Gleichwertigkeit der räumlichen Koordinaten und der Zeitkoordinate klar erkannte und für den Aufbau der Theorie nutzbar machte. Die für die allgemeine Relativitätstheorie nötigen mathematischen Hilfsmittel lagen fertig bereit in dem „absoluten Differentialkalkül", welcher auf den Forschungen von Gauss, Riemann und Christoffel über nichteuklidische Mannigfaltigkeiten ruht und von Ricci und Levi-Civita in ein System gebracht und bereits auf Probleme der theoretischen Physik angewendet wurde. Ich habe im Abschnitt B der vorliegenden Abhandlung alle für uns nötigen, bei dem Physiker nicht als bekannt vorauszusetzenden mathematischen Hilfsmittel in möglichst einfacher und durchsichtiger Weise entwickelt, so daß ein Studium mathematischer Literatur für das Verständnis der vorliegenden Abhandlung nicht erforderlich ist. Endlich sei an dieser Stelle dankbar meines Freundes, des Mathematikers Grossmann, gedacht, der mir durch seine Hilfe nicht nur das Studium der einschlägigen mathematischen Literatur ersparte, sondern mich auch beim Suchen nach den Feldgleichungen der Gravitation unterstützte.

Annalen der Physik. IV. Folge. 49. 50



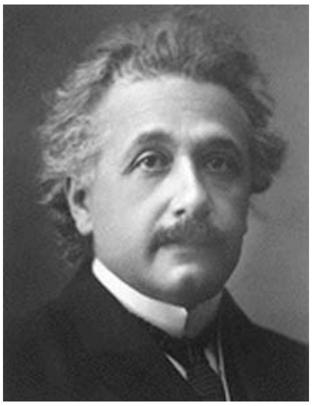
Colleague: *"Professor Eddington, you must be one of only three persons in the world who understand relativity!"*

Eddington: *" oh, I don't know..."*

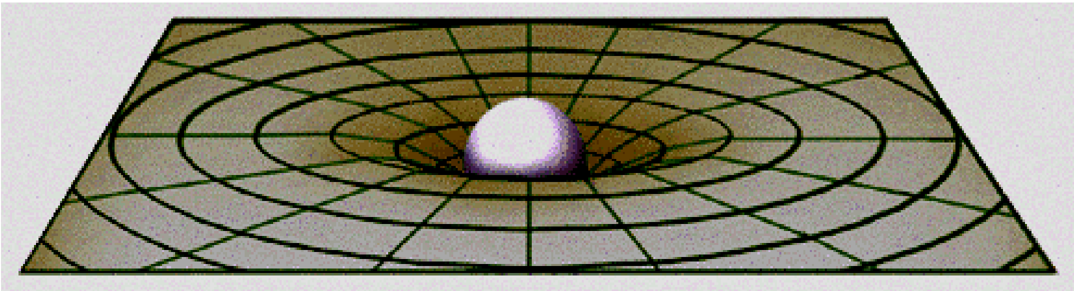
Colleague: *" Don't be modest Eddington."*

Eddington: ***" On the contrary, I am trying to think who the third person is."***

Gravity in Einstein's Universe



Spacetime tells matter how to move, and matter tells spacetime how to curve



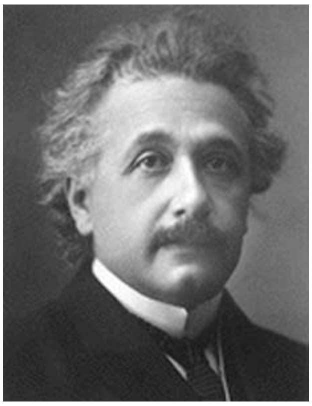
"…joy and amazement at the beauty and grandeur of this world of which man can just form a faint notion."

$$\boxed{G_{\mu\nu} = \kappa T_{\mu\nu}}$$

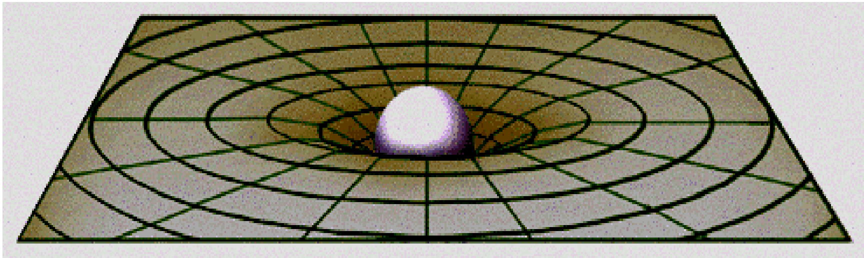
Spacetime curvature

Matter (and energy)

Gravity in Einstein's Universe



"Since the mathematicians have invaded the theory of relativity, I do not understand it myself anymore."



We are going to cram a lot of mathematics and physics into one morning.

Two-pronged approach:

- Comprehensive lecture notes, providing a 'long term' resource and reference source
- Lecture slides presenting "highlights" and some additional illustrations / examples

Copies of both available at

http://www.astro.gla.ac.uk/users/martin/teaching/vesf/

What we are going to cover

Introduction to GR

1. Foundations of general relativity
2. Introduction to geodesic deviation
3. A mathematical toolbox for GR
4. Spacetime curvature in GR
5. Einstein's equations

Gravitational Waves and detector principles

6. A wave equation for gravitational radiation
7. The Transverse Traceless gauge
8. The effect of gravitational waves on free particles
9. The production of gravitational waves

Websites of my Glasgow University Courses

Part 1: Introduction to General Relativity.

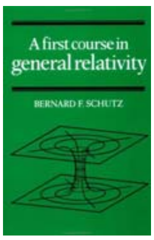
http://www.astro.gla.ac/users/martin/teaching/gr1/gr1_index.html

Part 2: Applications of General Relativity.

http://www.astro.gla.ac.uk/users/martin/teaching/gr2/gr2_index.html

Both websites are password-protected, with username and password 'honours'.

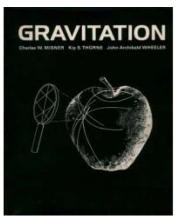
Recommended textbooks



"A First Course in General Relativity"
Bernard Schutz

ISBN: 052177035

Excellent introductory textbook.
Good discussion of gravitational wave generation, propagation and detection.



"Gravitation"
Charles Misner, Kip Thorne, John Wheeler

ISBN: 0716703440

The 'bible' for studying GR

1. Foundations of General Relativity (pgs. 6 – 12)

GR is a generalisation of Special Relativity (1905).

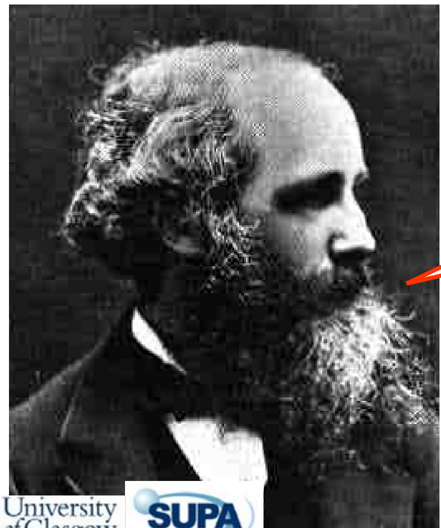
In SR Einstein formulated the laws of physics to be valid for all **inertial observers**

→ **Measurements of space and time relative to observer's motion.**

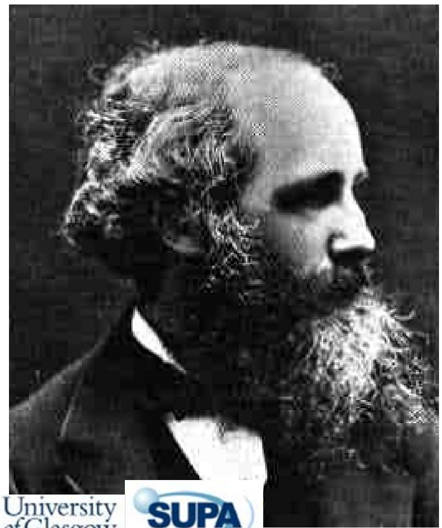
Classical Physics:

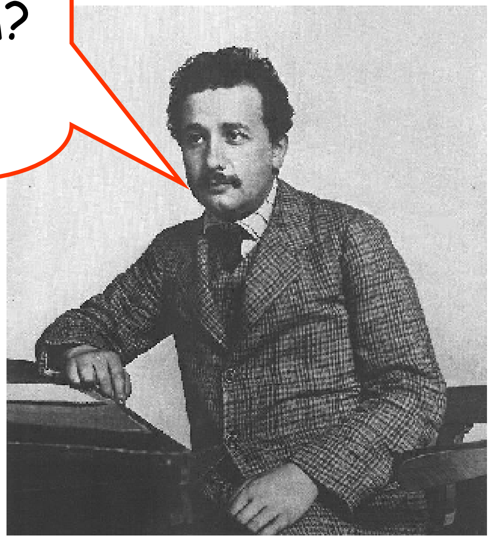
James Clerk Maxwell's theory of light

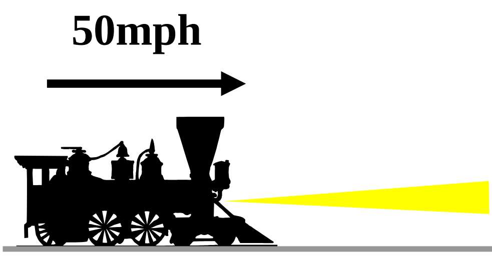
Light is a *wave* caused by varying *electric* and *magnetic* fields



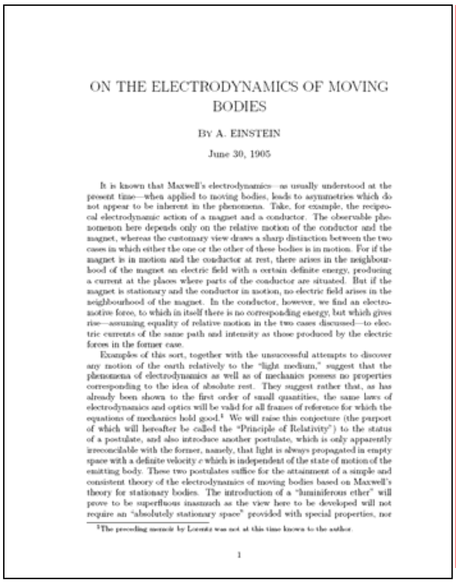
But what if I travelled *alongside* a light beam? Would it still wave?







In Special Relativity, the speed of light is *unchanged* by the motion of the train



ON THE ELECTRODYNAMICS OF MOVING BODIES

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[1] The preceding memoir by Lorentz was not at this time known to the author.

1

- Measurements of space and time are ***relative*** and depend on our motion
- Unified ***spacetime*** - only measurements of the ***spacetime*** interval are invariant
- Equivalence of **matter** and **energy**

1. Foundations of General Relativity (pgs. 6 – 12)

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In SR Einstein formulated the laws of physics to be valid for all **inertial observers**

→ **Measurements of space and time relative to observer's motion.**

$$\boxed{ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2}$$

Minkowski metric

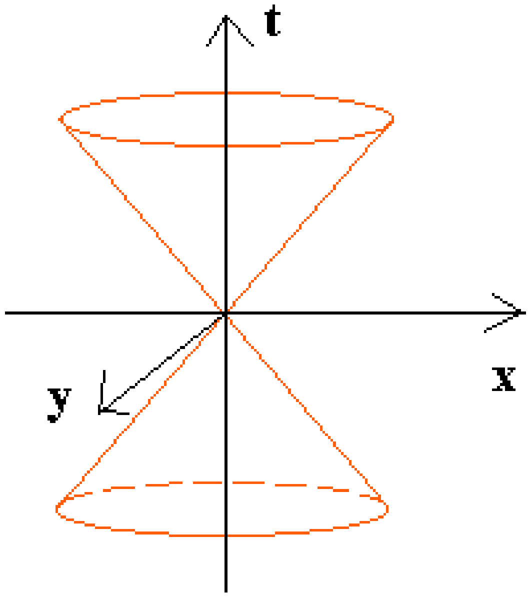
Invariant interval

Intervals between neighbouring events:

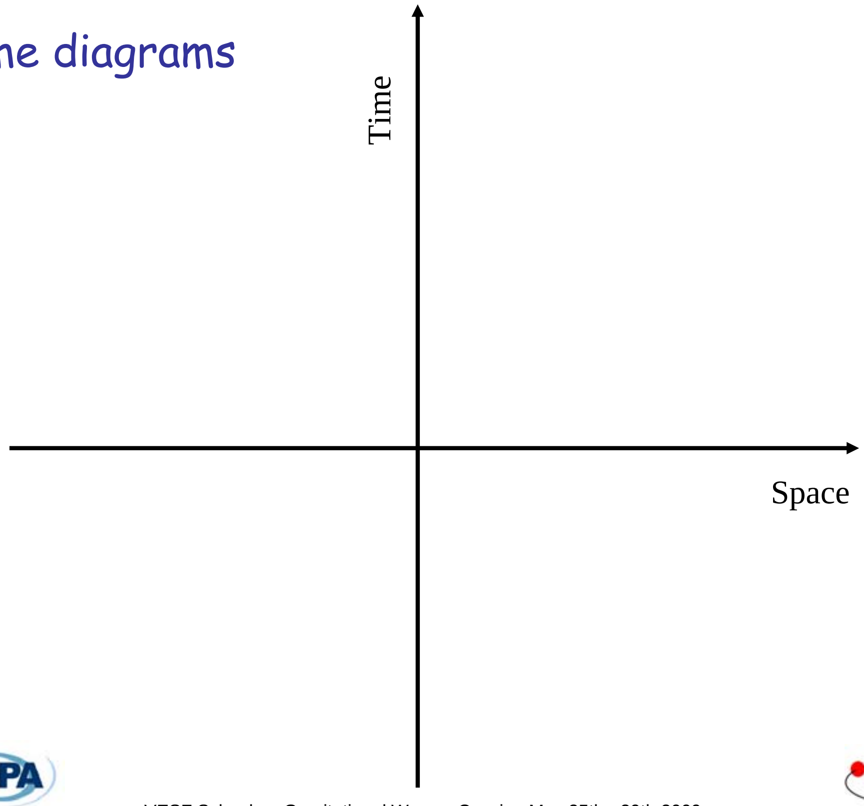
timelike $ds^2 < 0$

spacelike $ds^2 > 0$

lightlike $ds^2 = 0$



Spacetime diagrams



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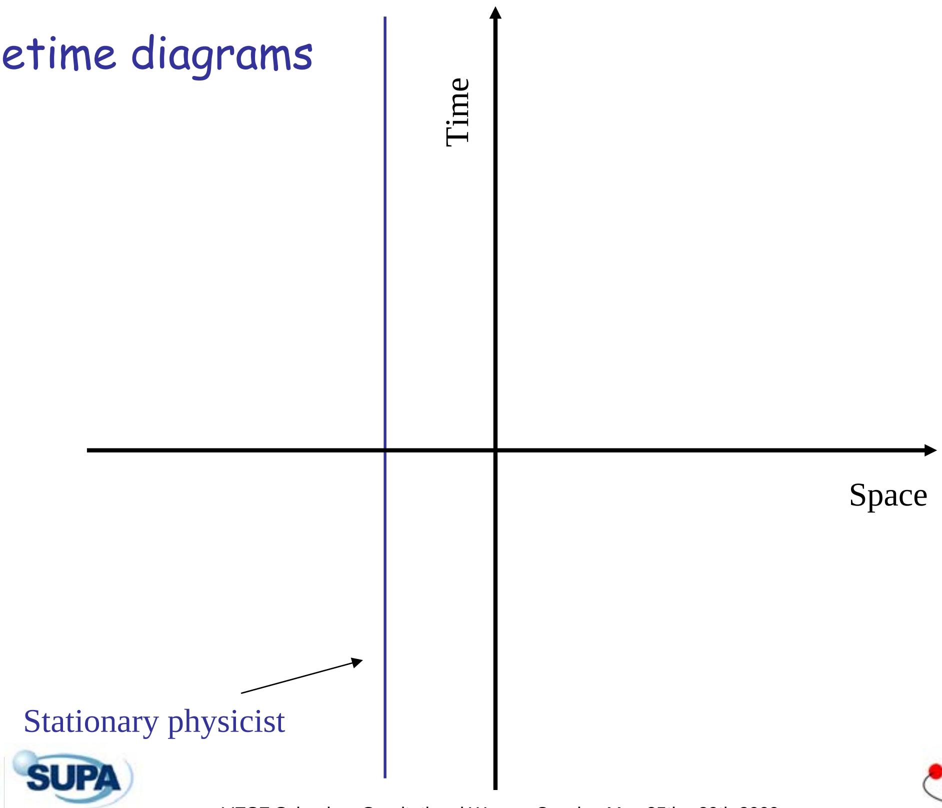
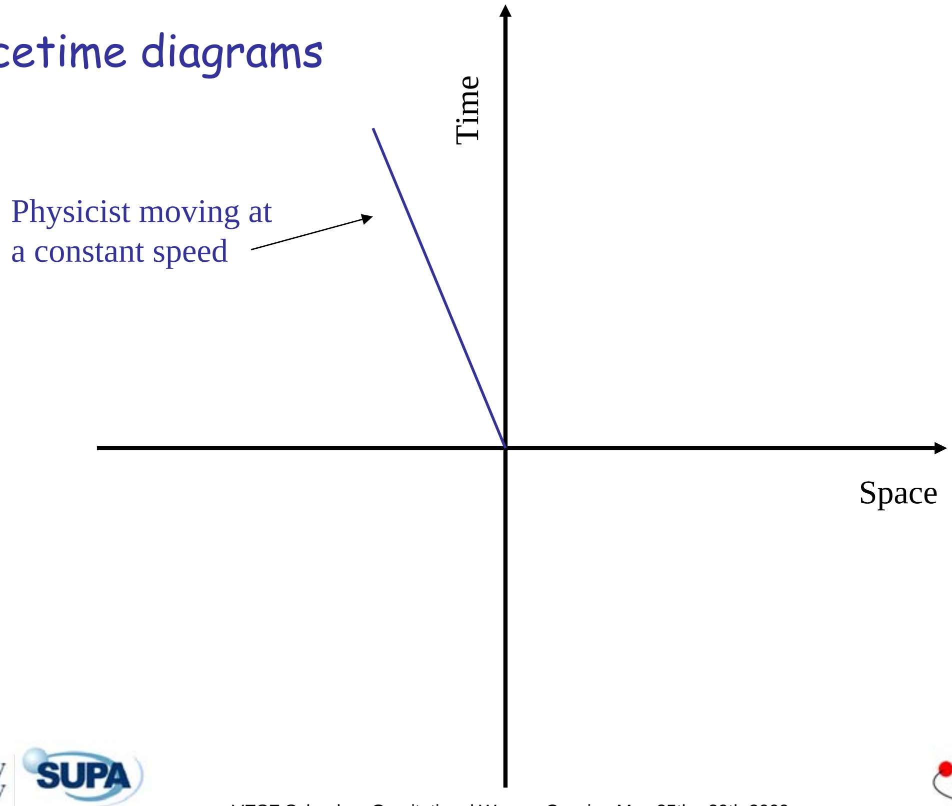


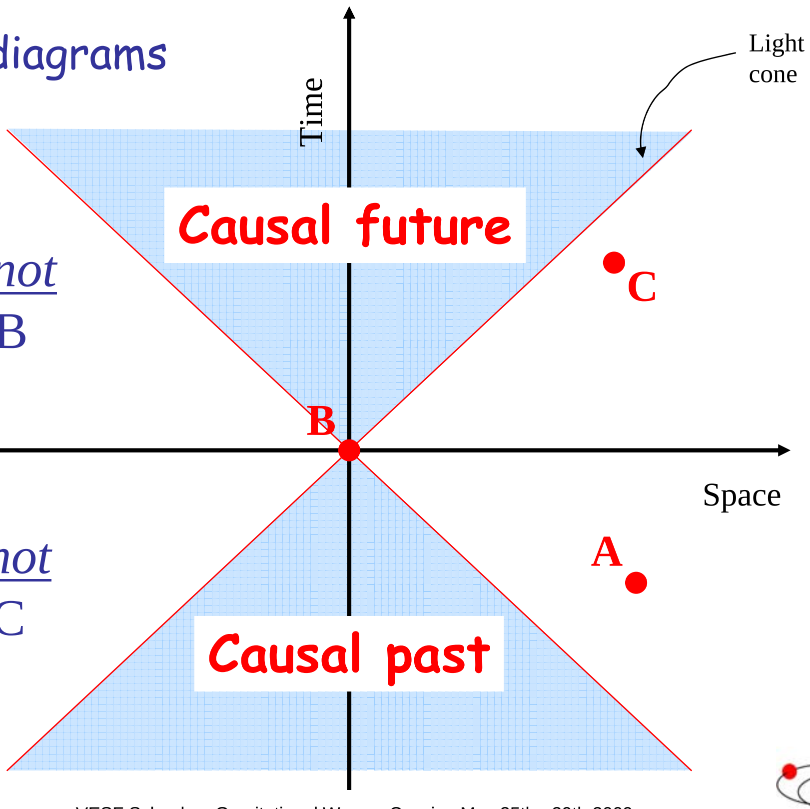
Spacetime diagrams



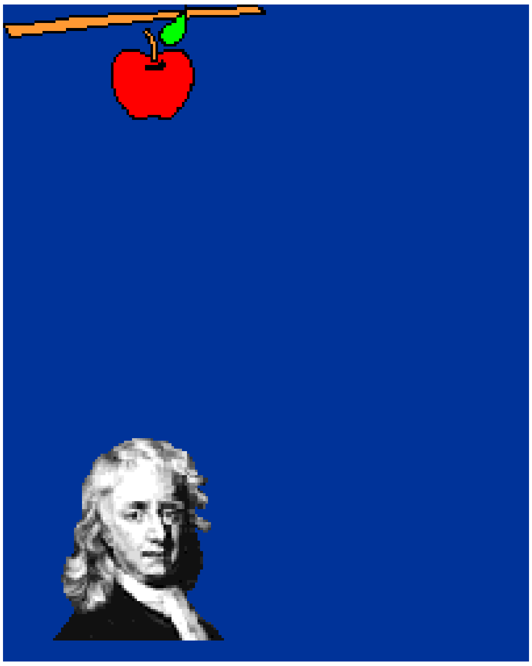
Spacetime diagrams

Event A *cannot* cause Event B

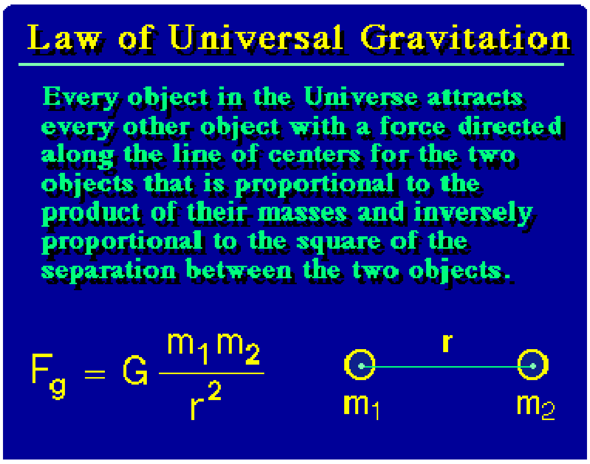
Event B *cannot* cause Event C

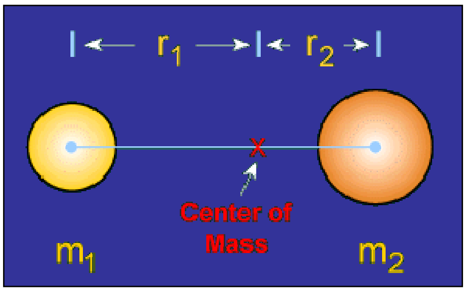


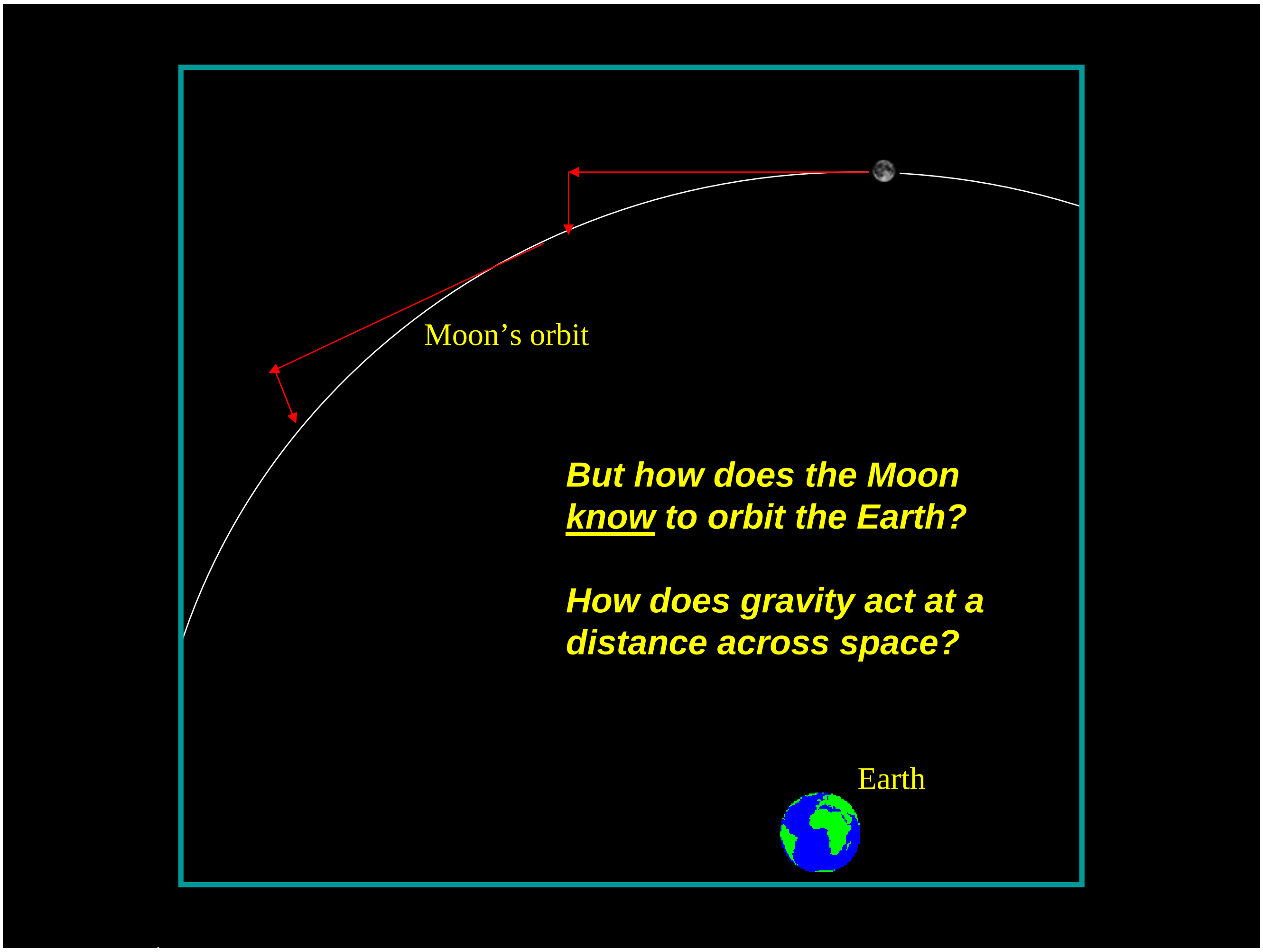
Newtonian gravity is incompatible with SR



Isaac Newton:
1642 – 1727 AD







Principles of Equivalence

Inertial Mass $$\vec{F}_I = m_I \vec{a}$$

Gravitational Mass $$\vec{F}_G = \frac{m_G M}{r^2}\hat{r} \equiv m_G \vec{g}$$

Weak Equivalence Principle $$\boxed{m_I = m_G}$$

Gravity and acceleration are ***equivalent***

The WEP implies:

A object freely-falling in a uniform gravitational field inhabits an **inertial frame** in which all gravitational forces have disappeared.

*But only **LIF**: only local over region for which gravitational field is uniform.*

The WEP explains why gravitational acceleration of a falling body is independent of its nature, mass and composition.

c.f. Galileo

Apollo 15

Eotvos experiment

Newton's Laws of Motion and Gravitation

Aristotle's Theory:

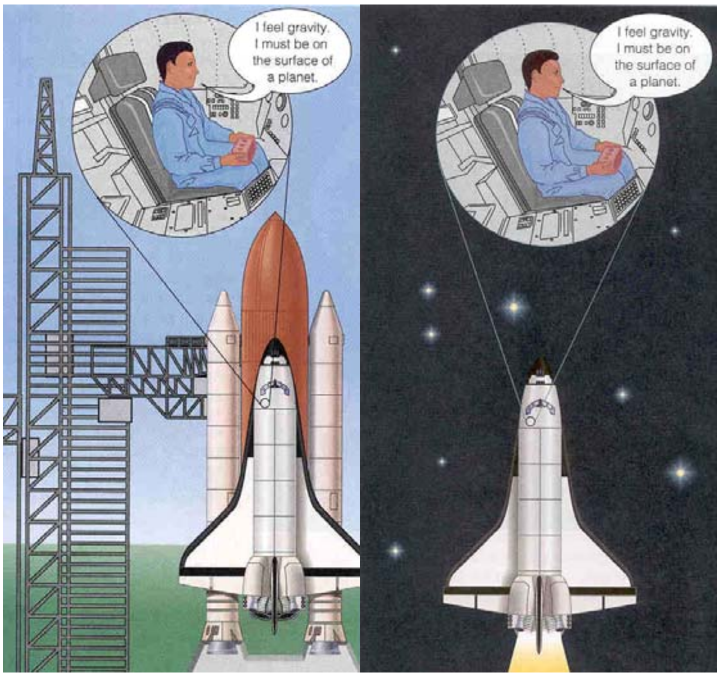
1. *Objects move only as long as we apply a force to them*
2. *Falling bodies fall at a constant rate*
3. *Heavy bodies fall faster than light ones*

Galileo's Experiment:

1. *Objects keep moving after we stop applying a force (if no friction)*
2. *Falling bodies accelerate as they fall*
3. *Heavy bodies fall at the same rate as light ones*

Strong Equivalence Principle

Locally (i.e. in a LIF) *all* laws of physics reduce to their SR form – apart from gravity, which simply disappears.

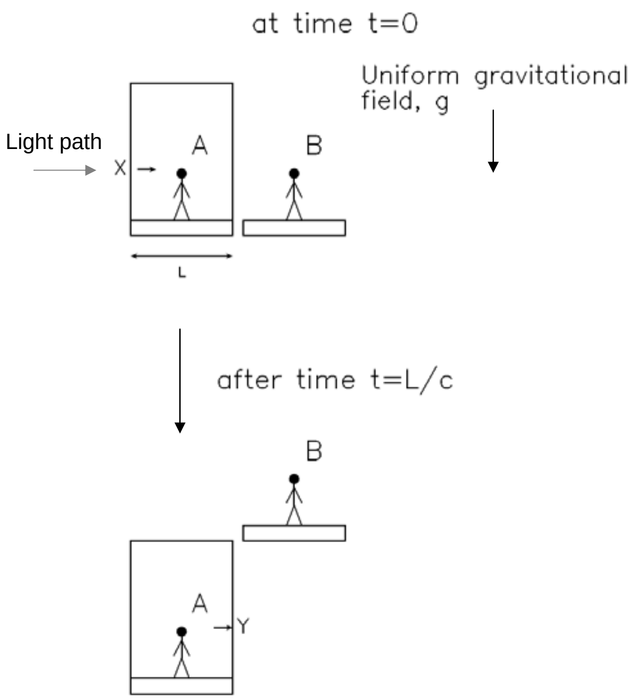


The Equivalence principles also predict gravitational light deflection…

Light enters lift horizontally at X, at instant when lift begins to free-fall.

Observer A is in LIF. Sees light reach opposite wall at Y (same height as X), in agreement with SR.

To be consistent, observer B outside lift must see light path as ***curved****, interpreting this as due to the gravitational field*

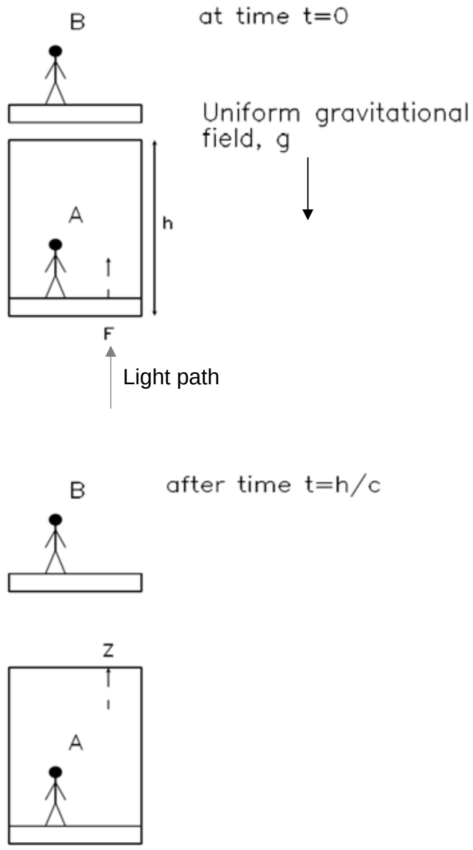


The Equivalence principles also predict gravitational redshift…

Light enters lift vertically at F, at instant when lift begins to free-fall.

Observer A is in LIF. Sees light reach ceiling at Z with unchanged frequency, in agreement with SR.

Observer B outside lift is moving away from A (and Z); sees light as ***redshifted****, interpreting this as due to gravitational field.*



The Equivalence principles also predict gravitational redshift…

$$\boxed{\frac{\Delta\lambda}{\lambda} \sim \frac{gh}{c^2}}$$

Measured in Pound-Rebka experiment

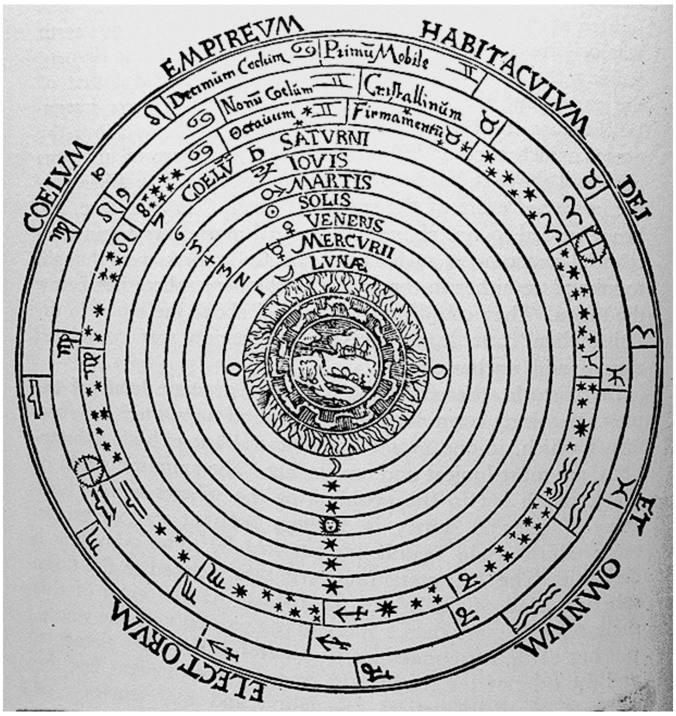
Also measured in ***white dwarf spectra***

See e.g. Barstow et al. (2005)

From SR to GR…

How do we 'stitch' all the LIFs together?

Can we find a ***covariant*** *description?*



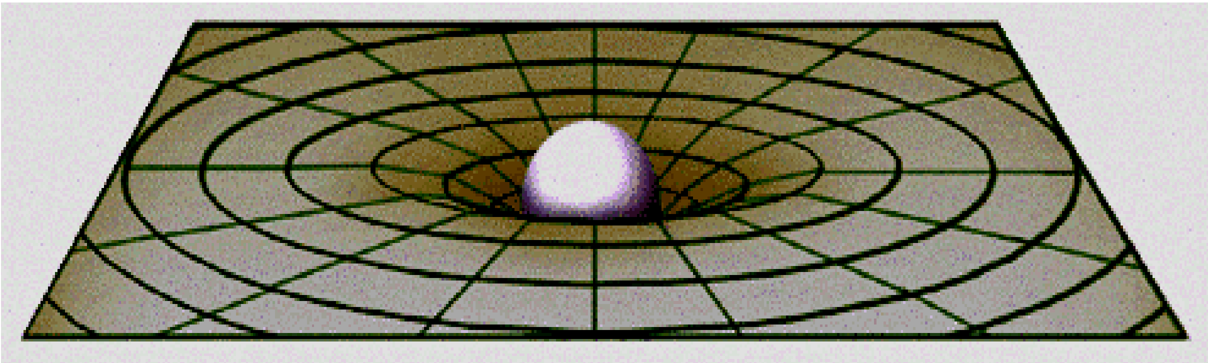
Ptolemy: 90 – 168 AD

Ptolemy proposed a model which could explain planetary motions – including retrograde loops

2. Introduction to Geodesic Deviation (pgs.13 – 17)

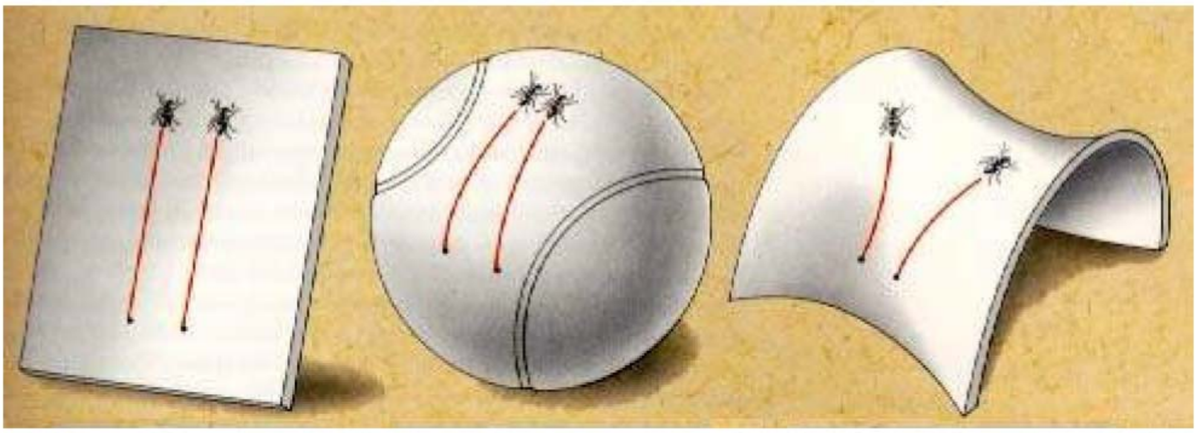
In GR trajectories of freely-falling particles are **geodesics** – the equivalent of straight lines in curved spacetime.

Analogue of Newton I: Unless acted upon by a non-gravitational force, a particle will follow a geodesic.



The curvature of spacetime is revealed by the behaviour of neighbouring geodesics.

Consider a 2-dimensional analogy.



Zero curvature: geodesic deviation unchanged.

Positive curvature: geodesics converge

Negative curvature: geodesics diverge

Non-zero curvature

$$\Leftrightarrow$$

Acceleration of geodesic deviation

$$\Leftrightarrow$$

Non-uniform gravitational field

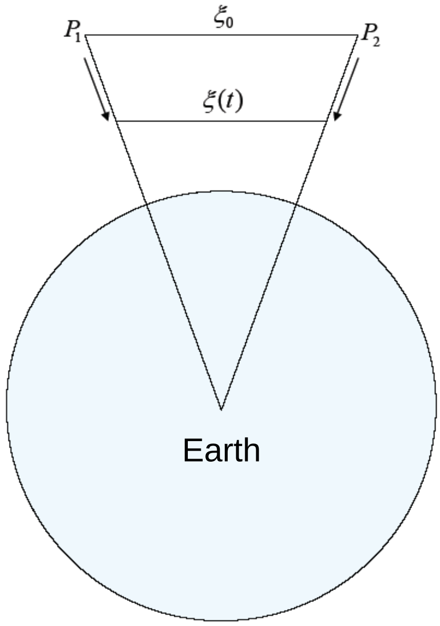
We can first think about geodesic deviation and curvature in a Newtonian context

By similar triangles

$$\frac{\xi(t)}{r(t)} = \frac{\xi_0}{r_0} = k$$

Hence

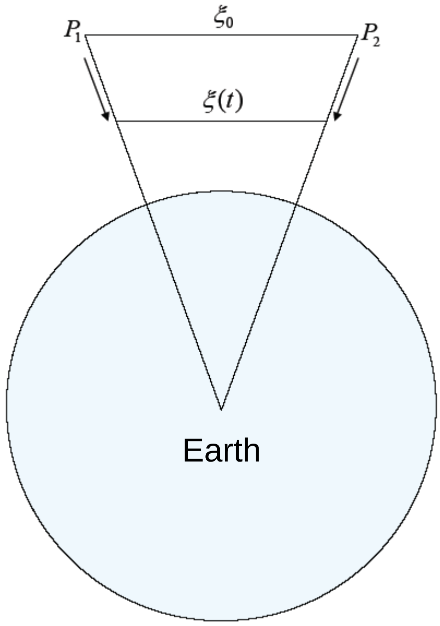
$$\ddot{\xi} = k\ddot{r} = -\frac{kGM}{r^2}$$



We can first think about geodesic deviation and curvature in a Newtonian context

or

$$\ddot{\xi} = -\frac{\xi}{r}\frac{GM}{r^2} = -\frac{GM\xi}{r^3}$$



which we can re-write as

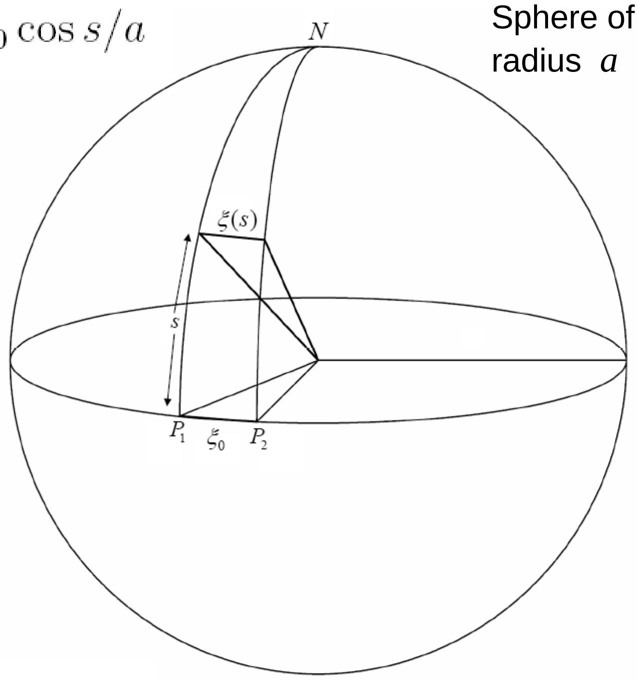
$$\frac{d^2\xi}{d(ct)^2} = -\frac{GM}{R^3c^2}\,\xi$$

At Earth's surface this equals $2\times10^{-23}\ \mathrm{m}^{-2}$

Another analogy will help us to interpret this last term

$$\xi(s) = a\cos\theta\, d\phi = \xi_0\cos\theta = \xi_0\cos s/a$$

Sphere of radius a



Differentiating: $$\frac{d^2\xi}{ds^2} = -\frac{1}{a^2}\xi$$

Comparing with previous slide:

$$\mathcal{R} = \left\{\frac{GM}{R^3c^2}\right\}^{-\frac{1}{2}}$$

represents radius of curvature of spacetime at the Earth's surface

$$\mathcal{R} \sim 2\times10^{11}\ \mathrm{m}$$

At the surface of the Earth $\mathcal{R} \sim 2 \times 10^{11}$ m

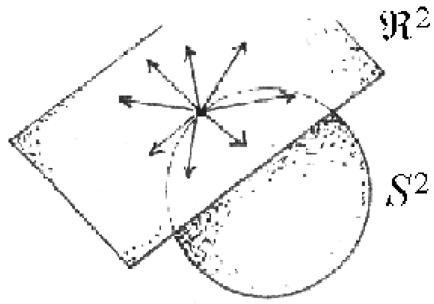
The fact that this value is so much larger than the physical radius of the Earth tells us that spacetime is 'nearly' flat in the vicinity of the Earth – i.e. the Earth's gravitational field is rather weak. (By contrast, if we evaluate $\mathcal{R}$ for e.g. a white dwarf or neutron star then we see evidence that their gravitational fields are much stronger).

3. A Mathematical Toolbox for GR (pgs.18 – 32)

Riemannian Manifold

A continuous, differentiable space which is locally **flat** and on which a distance, or **metric**, function is defined.

(e.g. the surface of a sphere)



The tangent space in a generic point of an $\mathbb{S}^2$ sphere

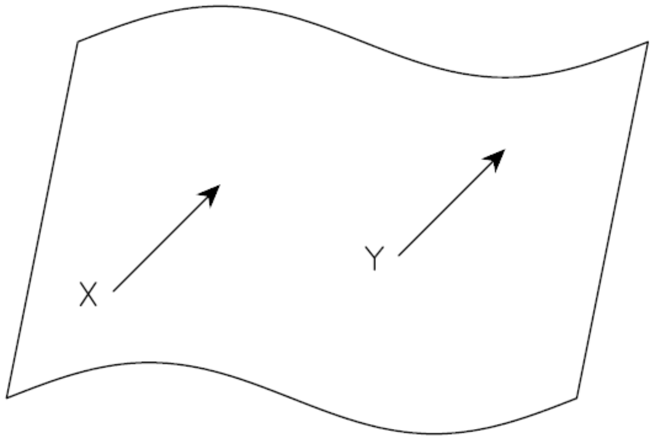
The mathematical properties of a Riemannian manifold match the physical assumptions of the strong equivalence principle

Vectors on a curved manifold

We think of a vector as an arrow representing a displacement.

$$\Delta\vec{x} = \Delta x^{\alpha}\,\vec{e}_{\alpha}$$

components   basis vectors

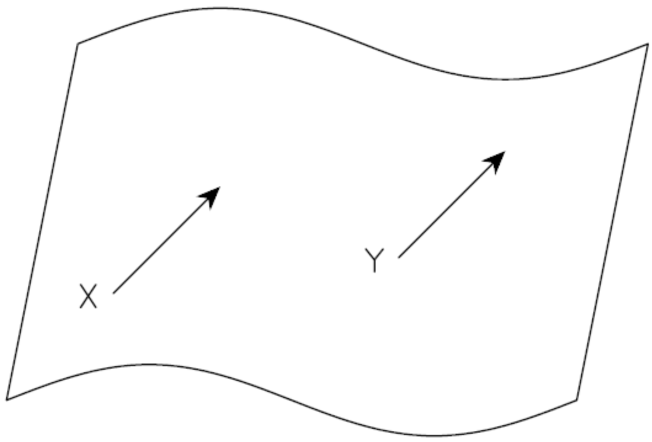


In general, components of vector different at X and Y, even if the vector is the same at both points.

We need rules to tell us how to express the components of a vector in a different coordinate system, and at different points in our manifold.

e.g. in new, dashed, coordinate system, by the chain rule

$$\boxed{\Delta x'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\Delta x^{\alpha}}$$



We need to think more carefully about what we mean by a vector.

Tangent vectors

We can generalise the concept of vectors to curved manifolds.

Suppose we have a scalar function, ϕ, defined at a point, P, of a Riemannian manifold, where P has coordinates $\{x^1, x^2, ..., x^n\}$ in some coordinate system. Since our manifold is differentiable we can evaluate the derivative of ϕ with respect to each of the coordinates, x^i, for $i = 1, ..., n$.

Tangent vectors

We can think of the derivatives as a set of n 'operators', denoted by

$$\frac{\partial}{\partial x^i}$$

These operators can act on any scalar function, ϕ, and yield the rate of change of the function with respect to the x^i.

We can now define a **tangent vector** at point, P, as a linear operator of the form

$$a^\mu \frac{\partial}{\partial x^\mu} \equiv a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + ... + a^n \frac{\partial}{\partial x^n}$$

This tangent vector operates on any function, ϕ, and essentially gives the rate of change of the function – or the *directional derivative* – in a direction which is defined by the numbers $(a^1, a^2, ..., a^n)$.

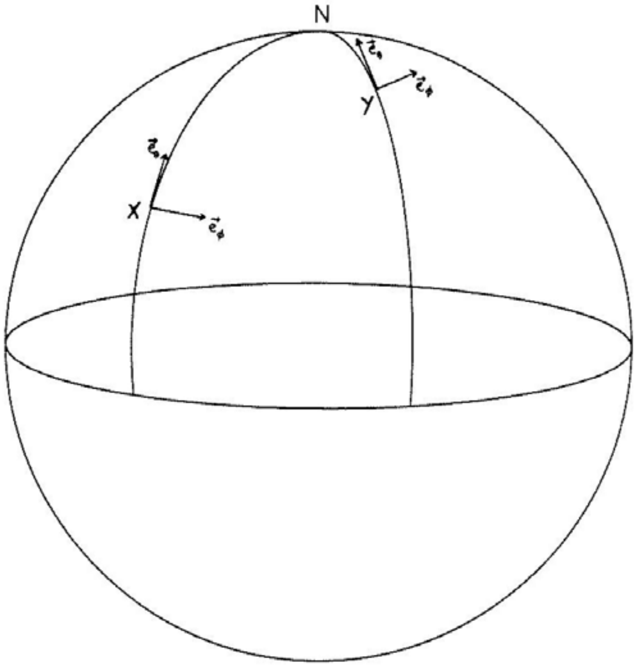
The n operators $\frac{\partial}{\partial x^\mu}$ can be thought of as forming a set of basis vectors, $\{\vec{e_\mu}\}$, spanning the vector space of tangent vectors at P.

Simple example: 2-D sphere.

Set of curves parametrised by ***coordinates***

$$\vec{e_i} \equiv \frac{\partial}{\partial x^i}$$ ***tangent*** to i[th] curve

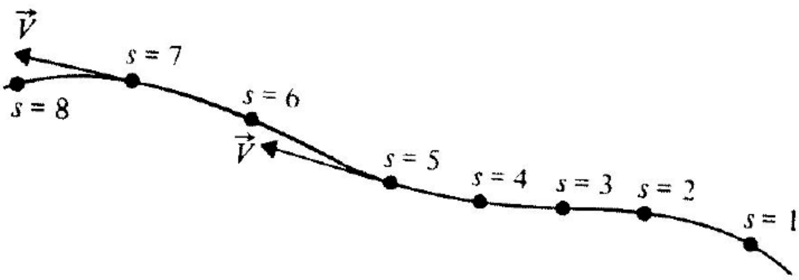
Basis vectors different at X and Y.



Summary

To sum up, we can represent vectors as tangent vectors of curves in our manifold. Once we have specified our coordinate system, we can write down the components of a vector defined at any point of the manifold with respect to the natural basis generated by the derivative operators $\{\frac{\partial}{\partial x^\mu}\}$ at that point. A vector **field** can then be defined by assigning a tangent vector at *every* point of the manifold.

Extends easily to more general curves, manifolds



Transformation of vectors

Suppose we change to a new coordinate system $\{x'^1, x'^2, ..., x'^n\}$. Our basis vectors are now

$$\vec{e'_\mu} \equiv \frac{\partial}{\partial x'^\mu}.$$

How do the components, $\{a^1, a^2, ..., a^n\}$, transform in our new coordinate system?

Let the vector $\vec{a}$ operate on an arbitrary scalar function, ϕ. Then

$$\vec{a}(\phi) = a^\nu \frac{\partial \phi}{\partial x^\nu}$$

By the chain rule for differentiation we may write this as

$$\vec{a}(\phi) = a^\nu \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial \phi}{\partial x'^\mu}$$

However, if we write $\vec{a}$ directly in terms of coordinate basis $\{\vec{e'_\mu}\} = \{\frac{\partial}{\partial x'^\mu}\}$, we have

$$\vec{a}(\phi) = a'^\mu \frac{\partial \phi}{\partial x'^\mu}$$

Hence we see that

$$\boxed{a'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} a^\nu}$$

This is the transformation law for a contravariant vector.

Any set of components which transform according to this law, we call a contravariant vector.

Transformation of basis vectors

What is the relationship between the basis vectors $\vec{e'_\mu}$ and $\vec{e_\mu}$ in the primed and unprimed coordinate systems?

$$\boxed{\vec{e'_\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\vec{e_\nu}}$$

Thus we see that the basis vectors do *not* transform in the same way as the components of a contravariant vector. This should not be too surprising, since the transformation of a basis and the transformation of components are different things: the former is the expression of *new* vectors in terms of *old* vectors; the latter is the expression of the *same* vector in terms of a new basis.

$$\boxed{A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}A_\nu}$$

This is the transformation law for a **one-form** or covariant vector.

Any set of components which transform according to this law, we call a one-form.

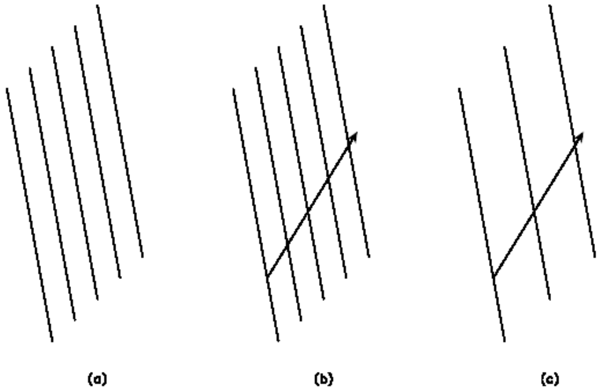
A one-form, operating on a vector, produces a real number (and vice-versa)

Picture of a one-form

Not a vector, but a way of ‘slicing up’ the manifold.

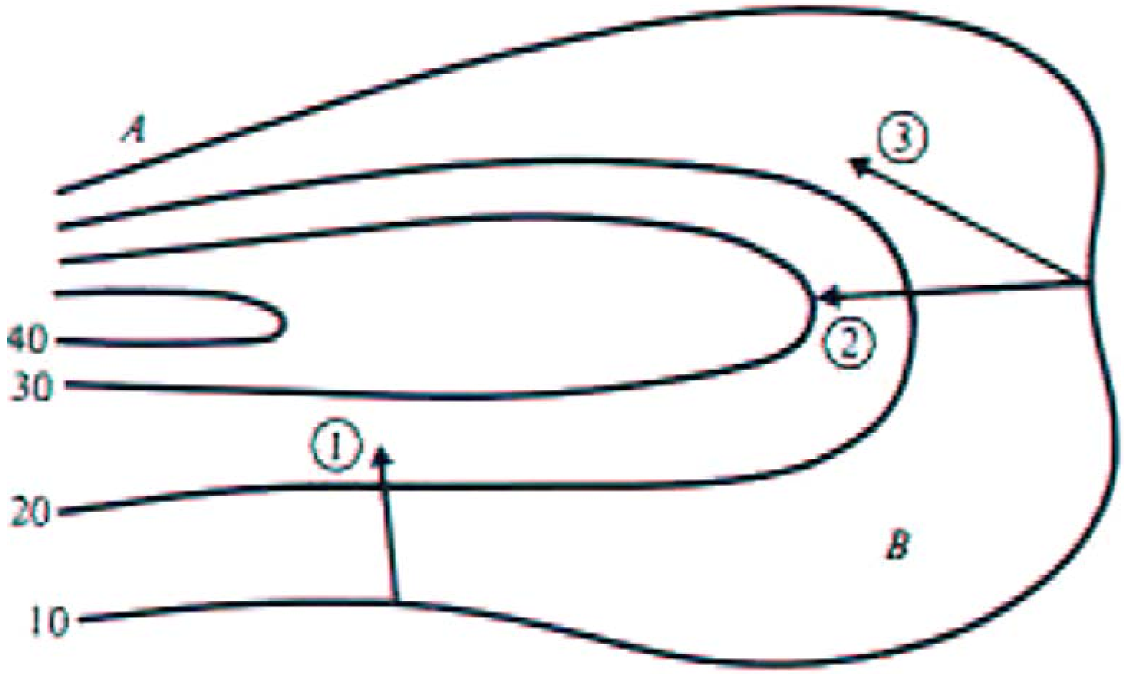
The smaller the spacing, the larger the magnitude of the one-form.

When one-form shown acts on the vector, it produces a real number: the number of ‘slices’ that the vector crosses.



Example: the gradient operator (c.f. a topographical map)

Picture of a one-form



Extension to tensors

An (l,m) tensor is a linear operator that maps l one-forms and n vectors to a real number.

Transformation law

$$\boxed{A'^{u_1\, u_2\, \ldots\, u_l}_{r_1\, r_2\, \ldots\, r_m} = \frac{\partial x'^{u_1}}{\partial x^{t_1}} \cdots \frac{\partial x'^{u_l}}{\partial x^{t_l}} \frac{\partial x^{q_1}}{\partial x'^{r_1}} \cdots \frac{\partial x^{q_m}}{\partial x'^{r_m}} A^{t_1\, t_2\, \ldots\, t_l}_{q_1\, q_2\, \ldots\, q_m}}$$

If a tensor equation can be shown to be valid in a particular coordinate system, it must be valid in *any* coordinate system.

Specific cases

(2,0) tensor

$$\boxed{T'^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} T^{kl}}$$

(1,1) tensor

$$\boxed{D'^i_j = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} D'^k_l}$$

(0,2) tensor

$$\boxed{B'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} B_{kl}}$$

Example:

metric tensor

$$\boxed{g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}}$$

which justifies

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Invariant interval
(scalar)

Contravariant vectors
or (1,0) tensors

We can use the metric tensor to convert contravariant vectors to one-forms, and vice versa.

Lowering the index

$$A_i = g_{ik} A^k$$

Raising the index

$$B^i = g^{ij} B_j$$

Can generalise to tensors of arbitrary rank.

(this also explains why we generally think of gradient as a vector operator. In flat, Cartesian space components of vectors and one-forms are identical)

We are going to cram a lot of mathematics and physics into (less than) 4 hours.

Two-pronged approach:

- Comprehensive lecture notes, providing a 'long term' resource and reference source
- Lecture slides presenting "highlights" and some additional illustrations / examples

Copies of both available at

http://www.astro.gla.ac.uk/users/martin/teaching/vesf/

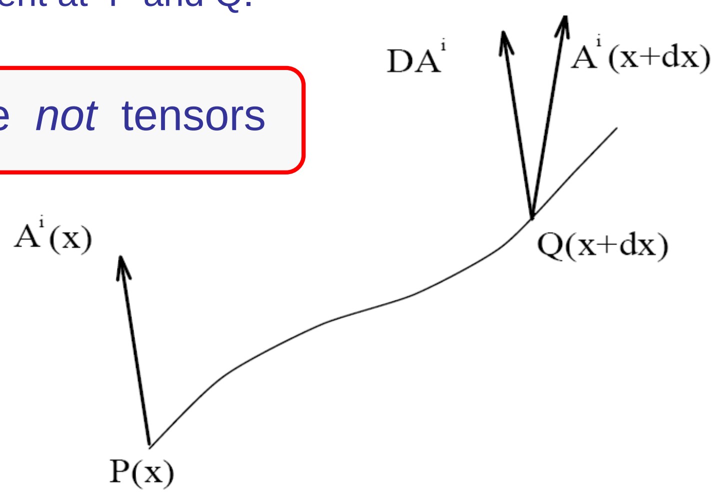
Covariant differentiation

Differentiation of e.g. a **vector field** involves subtracting vector components at two neighbouring points.

This is a problem because the transformation law for the components of A will in general be different at P and Q.

→ Partial derivatives are *not* tensors

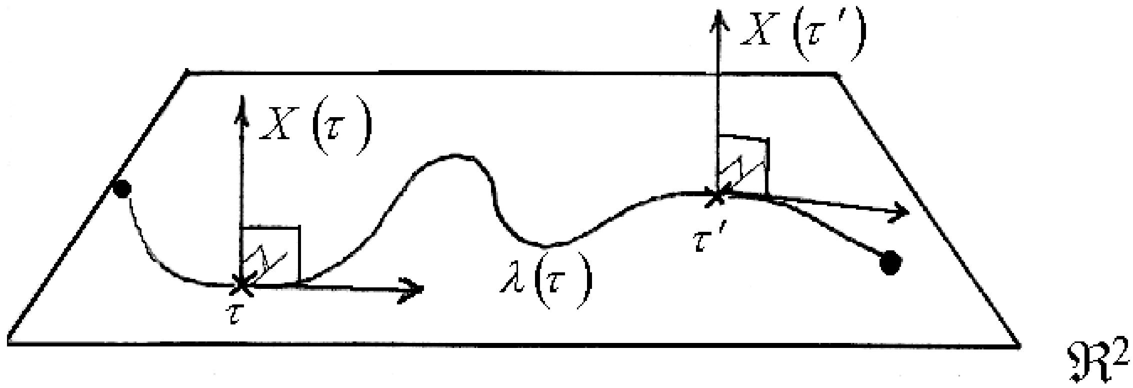
To fix this problem,
we need a procedure for
transporting the components
of A to point Q.



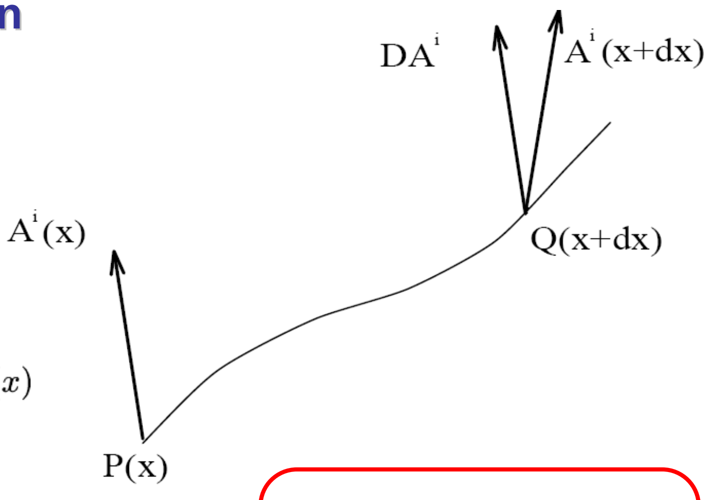
Covariant differentiation

We call this procedure **Parallel Transport**

A vector field is parallel transported along a curve, when it mantains a constant angle with the tangent vector to the curve



Covariant differentiation



We can write

$$DA^i(x+dx) = A^i(x) + \delta A^i(x)$$

where

$$\boxed{\delta A^i(x) = -\Gamma^i_{jk} A^j dx^k}$$

$$\boxed{\frac{\partial \vec{e}_i}{\partial x^k} = \Gamma^j_{ik}\ \vec{e}_j}$$

Christoffel symbols, connecting the basis vectors at Q to those at P

Covariant differentiation

We can now define the **covariant derivative** (which *does* transform as a tensor)

Vector $$A^i_{;k} = A^i_{,k} + \Gamma^i_{jk} A^j$$

One-form $$B_{i;k} = B_{i,k} - \Gamma^j_{ik} B_j$$

(with the obvious generalisation to arbitrary tensors)

Covariant differentiation

We can show that the covariant derivatives of the metric tensor are identically zero, i.e.

$$\boxed{g_{\alpha\beta;\gamma} = 0 \quad \text{and} \quad g^{\alpha\beta}_{;\gamma} = 0}$$

From which it follows that

$$\boxed{\Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l})}$$

Geodesics

We can now provide a more mathematical basis for the phrase "spacetime tells matter how to move".

One can define a geodesic as a **curve along which the tangent vector to the curve is parallel-transported**. In other words, if one parallel transports a tangent vector along a geodesic, it remains a tangent vector.

The covariant derivative of a tangent vector, along the geodesic is identically zero, i.e.

$$\boxed{\vec{\nabla}_{\vec{U}}\vec{U} = 0}$$

Geodesics

Suppose we parametrise the geodesic by the proper time, τ, along it (fine for a material particle). Then

$$\frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

i.e.

$$\boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0}$$

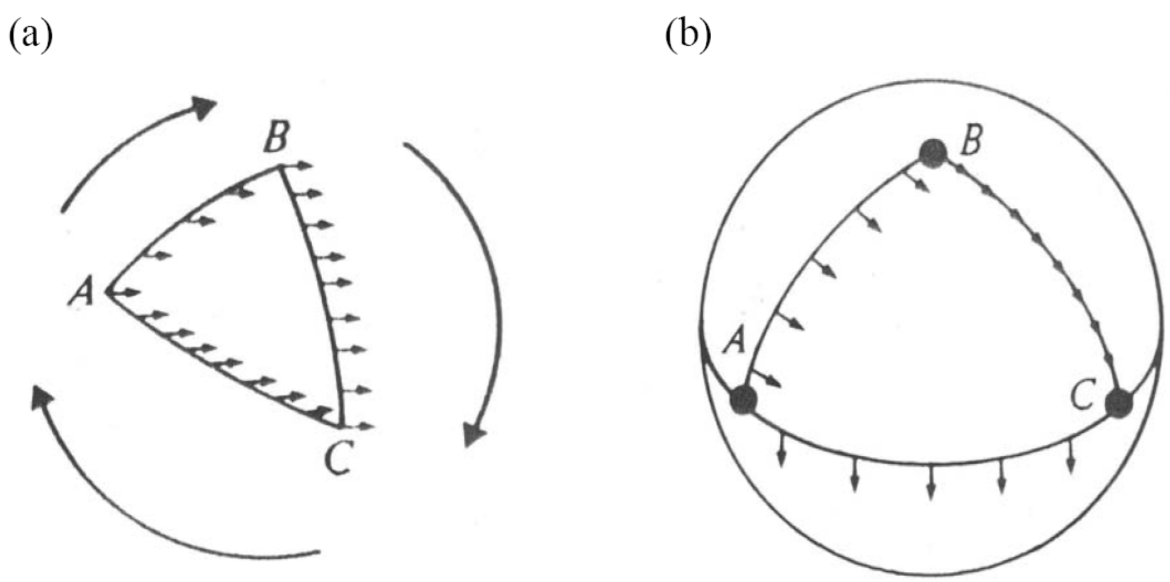
with the equivalent expression for a photon (replacing τ with λ)

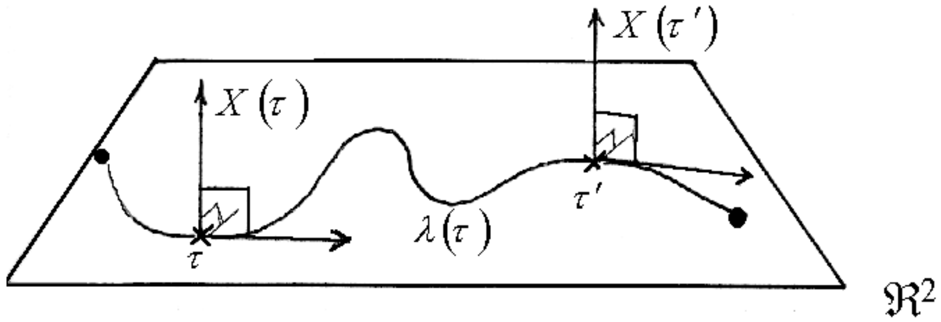
4. Spacetime curvature in GR (pgs.33 – 37)

This is described by the **Riemann-Christoffel tensor**, which depends on the metric and its first and second derivatives.

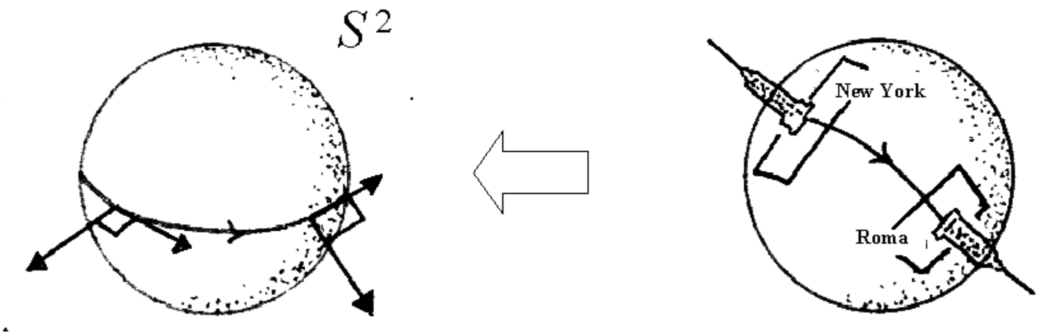
We can derive the form of the R-C tensor in several ways

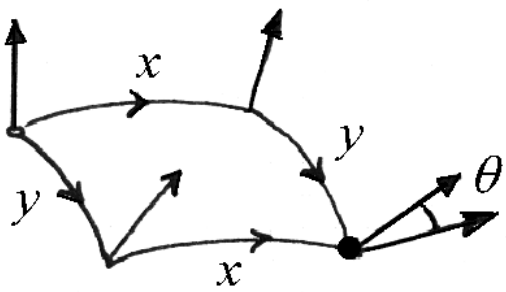
1. by parallel transporting of a vector around a closed loop in our manifold
2. by considering the commutator of the second order covariant derivative of a vector field
3. by computing the deviation of two neighbouring geodesics in our manifold





In a fat manifold, parallel transport does not rotate vectors, while on a curved manifold it *does*.





After parallel transport around a closed loop on a curved manifold, the vector does not come back to its original orientation but it is rotated through some angle.

The R-C tensor is related to this angle.

$$\boxed{R^{\mu}{}_{\alpha\beta\gamma} = \Gamma^{\sigma}_{\alpha\gamma}\Gamma^{\mu}_{\sigma\beta} - \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\mu}_{\sigma\gamma} + \Gamma^{\mu}_{\alpha\gamma,\beta} - \Gamma^{\mu}_{\alpha\beta,\gamma}}$$

If spacetime is flat then, for all indices $R^{\mu}{}_{\alpha\beta\gamma} = 0$

5. Einstein's Equations (pgs.38 – 45)

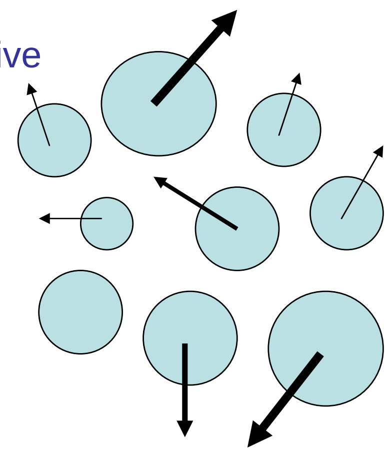
What about "matter tells spacetime how to curve"?...

The source of spacetime curvature is the **Energy-momentum tensor** which describes the presence and motion of gravitating matter (and energy).

We define the E-M tensor for a **perfect fluid**

In a fluid description we treat our physical system as a smooth continuum, and describe its behaviour in terms of locally averaged properties in each ***fluid element****.*

Each fluid element may possess a **bulk motion** with respect to the rest of the fluid, and this relative motion may be non-uniform.

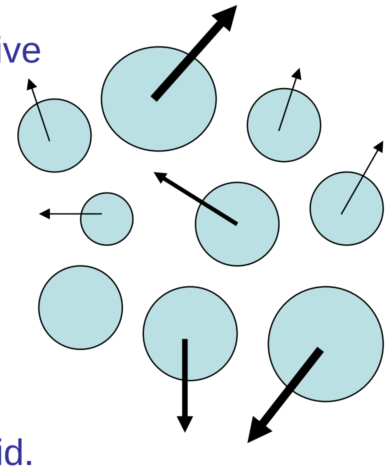


At any instant we can define **Momentarily comoving rest frame (MCRF)** of the fluid element – Lorentz Frame in which the fluid element as a whole is instantaneously at rest.

Particles in the fluid element will not be at rest:

1. Pressure (c.f. molecules in an ideal gas)
2. Heat conduction (energy exchange with neighbours)
3. Viscous forces (shearing of fluid)

Each fluid element may possess a **bulk motion** with respect to the rest of the fluid, and this relative motion may be non-uniform.



Perfect Fluid if, in MCRF, each fluid element has no heat conduction or viscous forces, only pressure.

Dust = special case of pressure-free perfect fluid.

Definition of E-M tensor

We can define the energy momentum tensor, $\mathbf{T}$, in terms of its components in some coordinate system, $\{x^1, x^2, ..., x^n\}$, for each fluid element. Thus we define $T^{\alpha\beta}$ for a fluid element to be equal to the **flux of the α component of four momentum of all gravitating matter**[2] **across a surface of constant** x^β.

[2] By 'gravitating matter' we mean here all material particles, plus (from the equivalence of matter and energy) any electromagnetic fields and particle fields which may be present

Components of T in the MCRF for dust

only non-zero component is $T^{00} = \rho$, the energy density of the fluid element.

Components of T in the MCRF for a general perfect fluid

$$\mathbf{T} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Pressure due to random motion of particles in fluid element

Components of T in a general Lorentz frame

Extending our expression for $T^{\alpha\beta}$ from the MCRF to a general Lorentz frame is fairly straightforward, but the interested reader is referred e.g. to Schutz for the details and here we just state the result. If $\vec{u} = \{u^\alpha\}$ is the *four* velocity of a fluid element in some Lorentz frame, then

$$\boxed{T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + P\eta^{\alpha\beta},}$$

where $\eta^{\alpha\beta}$ is the Minkowski metric of SR.

Conservation of energy and momentum requires that

$$\boxed{T^{\alpha\beta}_{\ \ ,\beta} = 0.}$$

Extending to GR

In Section 1 we introduced the strong principle of equivalence which stated that, in a LIF, all physical phenomena are in agreement with special relativity. In the light of our discussion of tensors, we can write down an immediate consequence of the strong principle of equivalence as follows

Any physical law which can be expressed as a tensor equation in SR has exactly the same form in a local inertial frame of a curved spacetime

How is this extension justified? From the principle of covariance a tensorial description of physical laws must be equally valid in any reference frame. Thus, if a tensor equation holds in one frame it must hold in any frame. In particular, a tensor equation derived in a LIF (i.e. assuming SR) remains valid in an arbitrary reference frame (i.e. assuming GR).

Hence

$$\boxed{T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}}$$

and

$$\boxed{T^{\mu\nu}_{;\nu} = 0}$$

Covariant expression of energy conservation in a curved spacetime.

So how does "matter tell spacetime how to curve"?...

Einstein's Equations

BUT the E-M tensor is of rank 2, whereas the R-C tensor is of rank 4.

Einstein's equations involve contractions of the R-C tensor.

Define the **Ricci tensor** by $$R_{\alpha\gamma} = R^{\mu}_{\alpha\mu\gamma}$$

and the **curvature scalar** by $$R = g^{\alpha\beta} R_{\alpha\beta}$$

We can raise indices via $$R^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta}$$

and define the Einstein tensor $$\boxed{G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R}$$

We can show that $$G^{\mu\nu}_{\ \ ;\nu} = 0$$

so that $$\boxed{T^{\mu\nu}_{\ \ ;\nu} = G^{\mu\nu}_{\ \ ;\nu}}$$

Einstein took as solution the form

$$\boxed{G^{\mu\nu} = kT^{\mu\nu}}$$

where we can determine the constant k by requiring that we should recover the laws of Newtonian gravity and dynamics in the limit of a weak gravitational field and non-relativistic motion. In fact k turns out to equal $8\pi G/c^4$.

Solving Einstein's equations

Given the metric, we can compute the Chirstoffel symbols, then the geodesics of 'test' particles.

We can also compute the R-C tensor, Einstein tensor and E-M tensor.

What about the other way around?...

Highly non-trivial problem, in general intractable, but given E-M tensor can solve for metric in some special cases.

e.g. Schwarzschild solution, for the spherically symmetric static spacetime exterior to a mass M

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}$$

Coordinate singularity at $r=2M$

Geodesics for the Schwarzschild metric

Radial geodesic

$$\left(\frac{dr}{d\tau}\right)^2 = k^2 - 1 - \frac{h^2}{r^2} + \frac{2M}{r}\left(1 + \frac{h^2}{r^2}\right)$$

Changing the dependent variable from r to u and the independent variable from τ to ϕ, our radial geodesic equation reduces to

$$h^2\left(\frac{du}{d\phi}\right)^2 = \left(k^2 - 1\right) - h^2u^2 + 2Mu\left(1 + h^2u^2\right)$$

or

$$\boxed{\frac{d^2u}{d\phi^2} = -u + \frac{M}{h^2} + 3Mu^2}$$

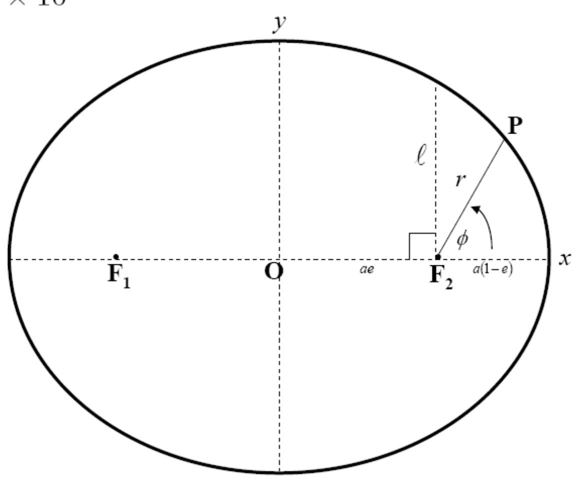
Extra term, only in GR

e.g. for the Earth's orbit the ratio

$$\frac{3Mu^2}{M/h^2} \simeq 3 \times 10^{-8}$$

Newtonian solution:

Elliptical orbit



Foci at $\mathbf{F_1}$ and $\mathbf{F_2}$

$\mathbf{O}x \;=\; a \;=\;$ semi-major axis

$\mathbf{O}y \;=\; b \;=\;$ semi-minor axis

$$b^2 = a^2\left(1 - e^2\right)$$

$e \;=\;$ eccentricity

$\mathbf{P}$ defined by $\boxed{r = \dfrac{\ell}{1 + e\cos\phi}}$

$\ell \;=\;$ semi-latus rectum

$\;=\; b^2/a$

GR solution:

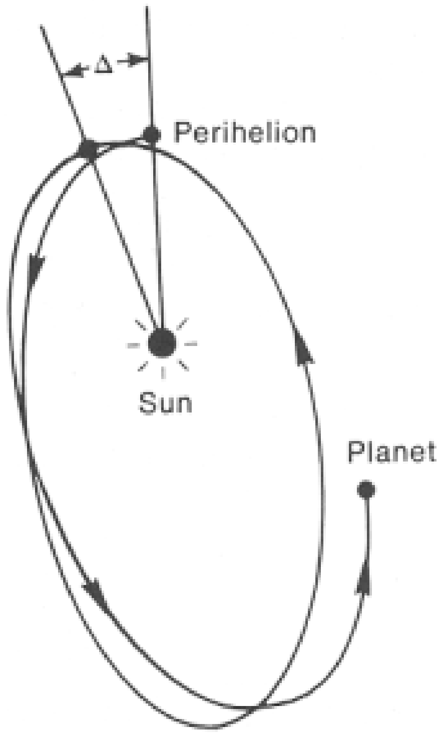
Precessing ellipse

$$u = \frac{M}{h^2}\left[1 + e\cos\left(1 - \frac{3M^2}{h^2}\right)\phi\right]$$

Here

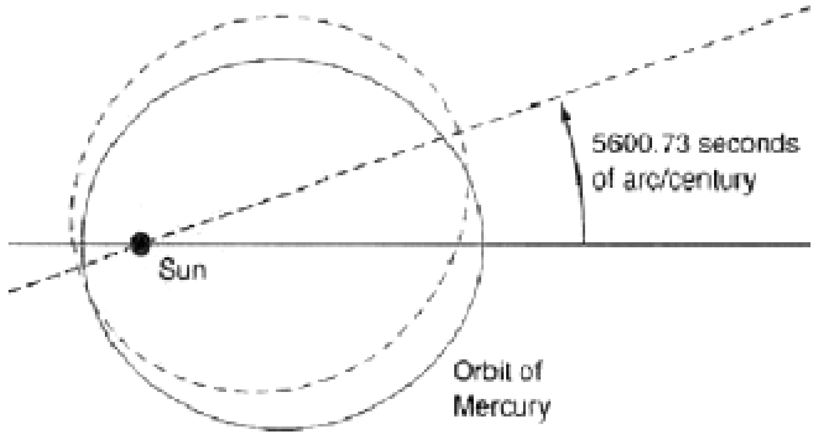
$$P = \frac{2\pi}{1 - 3M^2/h^2} > 2\pi$$

$$\Delta = \frac{6\pi M}{a(1-e^2)}$$



GR solution:

Precessing ellipse



$$\Delta = \frac{6\pi M}{a(1-e^2)}$$

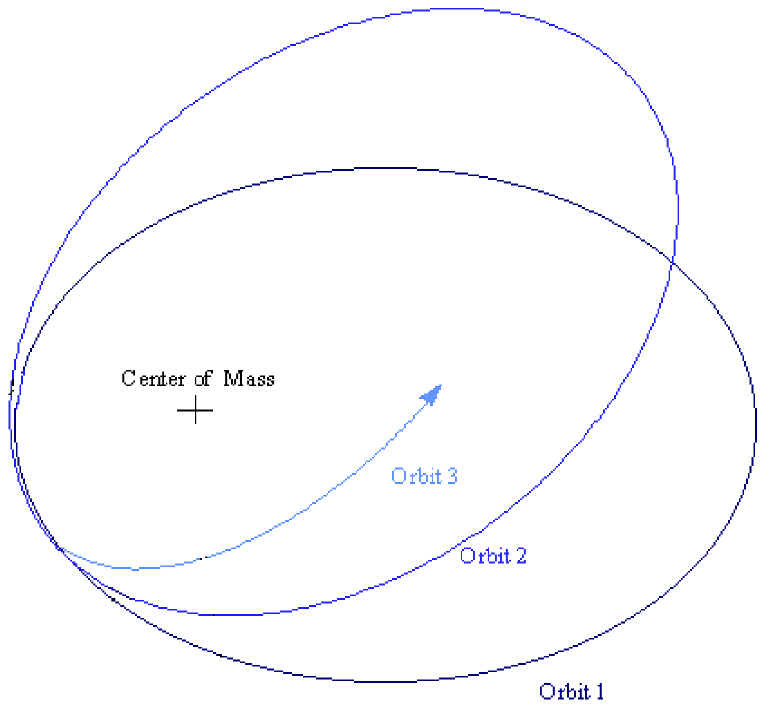
If we apply this equation to the orbit of Mercury, we obtain a perihelion advance which builds up to about 43 seconds of arc per century.

GR solution:

Precessing ellipse

Seen much more dramatically in the **binary pulsar** PSR 1913+16.

Periastron is advancing at a rate of ~4 degrees per year!



Gravitational light deflection in GR

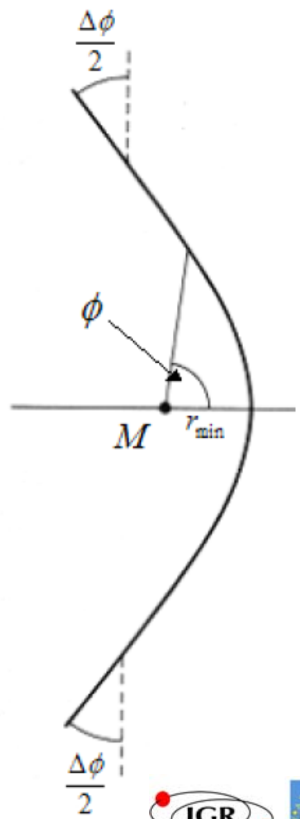
Radial geodesic for a photon

$$\left(\frac{dr}{d\lambda}\right)^2 = k^2 - \frac{h^2}{r^2} + \frac{2Mh^2}{r^3}$$

or $$\frac{d^2u}{d\phi^2} + u = 3Mu^2$$

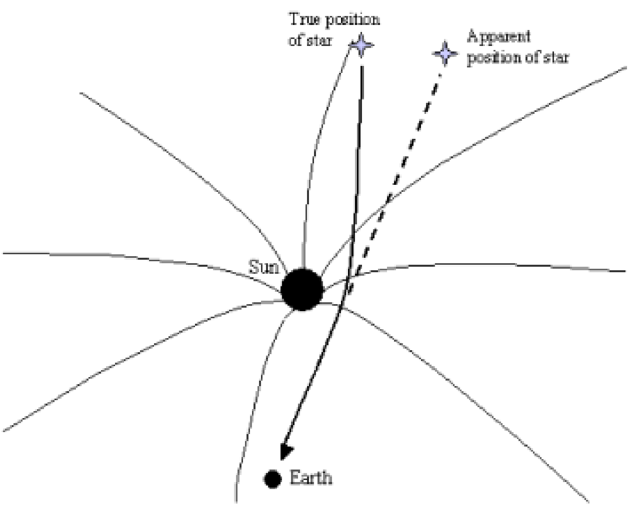
Solution reduces to $$u = -\frac{\Delta\phi}{2r_{\min}} + \frac{2M}{r_{\min}^2}$$

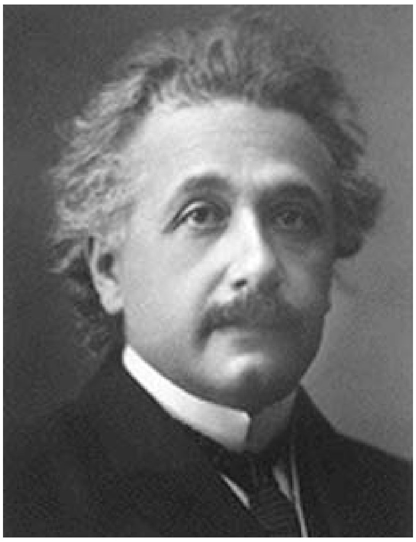
So that asymptotically $$\boxed{\Delta\phi = \frac{4M}{r_{\min}} \equiv \frac{4GM}{c^2 r_{\min}}}$$



This is exactly twice the deflection angle predicted by a Newtonian treatment. If we take $r_{\min}$ to be the radius of the Sun (which would correspond to a light ray grazing the limb of the Sun from a background star observed during a total solar eclipse) then we find that

$$\Delta\phi = \frac{4 \times 1.5 \times 10^3}{6.95 \times 10^8} = 8.62 \times 10^{-6} \text{ radians} \quad = \quad 1.77 \text{ arcsec}$$





1919 expedition, led by Arthur Eddington, to observe total solar eclipse, and measure light deflection.

GR passed the test!

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6. Wave Equation for Gravitational Radiation (pgs.46 – 57)

Weak gravitational fields

In the absence of a gravitational field, spacetime is flat. We define a weak gravitational field as one is which spacetime is 'nearly flat'

i.e. we can find a coord system such that

$$\boxed{g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}}$$

This is known as a Nearly Lorentz coordinate system.

where $\eta_{\alpha\beta} = \text{diag}\,(-1, 1, 1, 1)$

$|h_{\alpha\beta}| << 1 \;\text{ for all }\; \alpha \;\text{ and }\; \beta$

If we find a coordinate system in which spacetime looks nearly flat, we can carry out certain coordinate transformations after which spacetime will *still* look nearly flat:

1) Background Lorentz transformations

$$(t', x', y', z')^T = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (t, x, y, z)^T$$

i.e. **Lorentz boost** of speed v

If we find a coordinate system in which spacetime looks nearly flat, we can carry out certain coordinate transformations after which spacetime will *still* look nearly flat:

1) Background Lorentz transformations

Under this transformation

$$\boxed{g'_{\alpha\beta} = \eta'_{\alpha\beta} + \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}} h_{\mu\nu} = \eta_{\alpha\beta} + h'_{\alpha\beta}}$$

provided $v << 1$, then if $|h_{\alpha\beta}| << 1$

for all α and β, then $|h'_{\alpha\beta}| << 1$ also.

If we find a coordinate system in which spacetime looks nearly flat, we can carry out certain coordinate transformations after which spacetime will *still* look nearly flat:

1) Background Lorentz transformations

Hence, our original nearly Lorentz coordinate system remains nearly Lorentz in the new coordinate system. In other words, a spacetime which looks nearly flat to one observer still looks nearly flat to any other observer in uniform relative motion with respect to the first observer.

If we find a coordinate system in which spacetime looks nearly flat, we can carry out certain coordinate transformations after which spacetime will *still* look nearly flat:

2) Gauge transformations

Suppose now we make a very small change in our coordinate system by applying a coordinate transformation of the form

$$\boxed{x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x^{\beta})}$$

we now demand that the ξ^{α} are small, in the sense that

$$|\xi^{\alpha}{}_{,\beta}| << 1 \quad \text{for all} \quad \alpha, \beta$$

If we find a coordinate system in which spacetime looks nearly flat, we can carry out certain coordinate transformations after which spacetime will *still* look nearly flat:

2) Gauge transformations

Suppose now that the unprimed coordinate system is nearly Lorentz

Then $$g'_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$$

and we can write $$h'_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$$

Note that if $|\xi^{\alpha}{}_{,\beta}|$ are small, then so too are $|\xi_{\alpha,\beta}|$, and hence $h'_{\alpha\beta}$

If we find a coordinate system in which spacetime looks nearly flat, we can carry out certain coordinate transformations after which spacetime will *still* look nearly flat:

2) Gauge transformations

The above results tell us that – once we have identified a coordinate system which is nearly Lorentz – we can add an arbitrary small vector ξ^α to the coordinates x^α without altering the validity of our assumption that spacetime is nearly flat. We can, therefore, choose the components ξ^α to make Einstein's equations as simple as possible. We call this step choosing a **gauge** for the problem – a name which has resonance with a similar procedure in electromagnetism – and coordinate transformations of this type given by equation are known as **gauge transformation**. We will consider below specific choices of gauge which are particularly useful.

Einstein's equations for a weak gravitational field

To first order, the R-C tensor for a weak field reduces to

$$\boxed{R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}\right)}$$

and is invariant under gauge transformations.

Similarly, the Ricci tensor is $$R_{\mu\nu} = \frac{1}{2}\left(h^{\alpha}_{\mu,\nu\alpha} + h^{\alpha}_{\nu,\mu\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu}\right)$$

where
$$h \equiv h^\alpha_\alpha = \eta^{\alpha\beta} h_{\alpha\beta}$$

$$h_{\mu\nu,\alpha}{}^{,\alpha} = \eta^{\alpha\sigma}\left(h_{\mu\nu,\alpha}\right)_{,\sigma} = \eta^{\alpha\sigma}\, h_{\mu\nu,\alpha\sigma}$$

The Einstein tensor is the (rather messy) expression

$$G_{\mu\nu} = \frac{1}{2}\left[h_{\mu\alpha,\nu}{}^{,\alpha} + h_{\nu\alpha,\mu}{}^{,\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{,\beta}\right)\right]$$

but we can simplify this by introducing $\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$

So that

$$G_{\mu\nu} = -\frac{1}{2}\left[\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} + \eta_{\mu\nu}\overline{h}_{\alpha\beta}{}^{,\alpha\beta} - \overline{h}_{\mu\alpha,\nu}{}^{,\alpha} - \overline{h}_{\nu\alpha,\mu}{}^{,\alpha}\right]$$

And we can choose the **Lorentz gauge** to eliminate the last 3 terms

In the Lorentz gauge, then Einstein's equations are simply

$$-\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} = 16\pi T_{\mu\nu}$$

And in free space this gives

$$\boxed{\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} = 0}$$

Writing $\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} \equiv \eta^{\alpha\alpha}\overline{h}_{\mu\nu,\alpha\alpha}$

or

$$\boxed{\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\overline{h}_{\mu\nu} = 0}$$

Remembering that we are taking $c = 1$, if instead we write

$$\eta^{00} = -\frac{1}{c^2}$$

then $$\boxed{\left(-\frac{\partial^2}{\partial t^2} + c^2 \nabla^2 \right) \bar{h}_{\mu\nu} = 0}$$

This is a key result. It has the mathematical form of a wave equation, propagating with speed c.
We have shown that the metric perturbations – the 'ripples' in spacetime produced by disturbing the metric – propagate at the speed of light as waves in free space.

7. The Transverse Traceless Gauge (pgs.57 – 62)

Simplest solutions of our wave equation are **plane waves**

$$\bar{h}_{\mu\nu} = \mathrm{Re}\left[\underbrace{A_{\mu\nu}}_{\text{Wave amplitude}} \exp\left(i \underbrace{k_\alpha}_{\text{Wave vector}} x^\alpha\right)\right]$$

Note the wave amplitude is symmetric → 10 independent components.

Also, easy to show that $$\boxed{k_\alpha k^\alpha = 0}$$

i.e. the wave vector is a **null** vector

Thus $$\omega = k^t = \left(k_x^2 + k_y^2 + k_z^2\right)^{1/2}$$

Also, from the Lorentz gauge condition $$\overline{h}^{\mu\alpha}{}_{,\alpha} = 0$$

which implies that $$\boxed{A_{\mu\alpha}\, k^{\alpha} = 0}$$

i.e. the wave amplitude components must be orthogonal to the wave vector $\mathbf{k}$.

But this is 4 equations, one for each value of the index μ.

Hence, we can eliminate 4 more of the wave amplitude components,

Can we do better? **Yes**

Our choice of Lorentz gauge, chosen to simplify Einstein's equations, was not unique. We can make small adjustments to our original Lorentz gauge transformation and still satisfy the Lorentz condition.

We can choose adjustments that will make our wave amplitude components even simpler – we call this choice the **Transverse Traceless** gauge:

$$A^{\mu}_{\mu} = \eta^{\mu\nu} A_{\mu\nu} = 0 \qquad \text{(traceless)}$$

$$A_{\alpha t} = 0 \quad \text{for all} \quad \alpha$$

Suppose we orient our coordinate axes so that the plane wave is travelling in the positive z direction. Then

$$k^t = \omega\,, \quad k^x = k^y = 0\,, \quad k^z = \omega$$

and

$$A_{\alpha z} = 0 \quad \text{for all } \alpha$$

i.e. there is no component of the metric perturbation in the direction of propagation of the wave. This explains the origin of the 'Transverse' part

So in the transverse traceless gauge,

$$\overline{h}_{\mu\nu}^{(\mathrm{TT})} = A_{\mu\nu}^{(\mathrm{TT})} \cos\left[\omega(t-z)\right]$$

where

$$A_{\mu\nu}^{(\mathrm{TT})} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx}^{(\mathrm{TT})} & A_{xy}^{(\mathrm{TT})} & 0 \\ 0 & A_{xy}^{(\mathrm{TT})} & -A_{xx}^{(\mathrm{TT})} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Also, since the perturbation is traceless $\quad \overline{h}_{\alpha\beta}^{(\mathrm{TT})} = h_{\alpha\beta}^{(\mathrm{TT})}$

8. Effect of Gravitational Waves on Free Particles (pgs.63 – 75)

Choose Background Lorentz frame in which test particle initially at rest. Set up coordinate system according to the TT gauge.

Initial acceleration satisfies $\left(\dfrac{dU^\beta}{d\tau}\right)_0 = 0$

i.e. coordinates do not change, but adjust themselves as wave passes so that particles remain 'attached' to initial positions.

Coordinates are frame-dependent labels.

What about **proper distance** between neighbouring particles?

Consider two test particles, both initially at rest, one at origin and the other at $x = \epsilon\,,\; y = z = 0$

$$\Delta\ell = \int \left| g_{\alpha\beta} dx^\alpha dx^\beta \right|^{1/2}$$

i.e. $$\Delta\ell = \int_0^\epsilon |g_{xx}|^{1/2} \quad \simeq \quad \sqrt{g_{xx}(x=0)}\ \epsilon$$

Now $$g_{xx}(x=0) = \eta_{xx} + h_{xx}^{(\mathrm{TT})}(x=0)$$

so $$\boxed{\Delta\ell \simeq \left[1 + \frac{1}{2} h_{xx}^{(\mathrm{TT})}(x=0)\right]\epsilon}$$

In general, this is time-varying

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More formally, consider geodesic deviation ξ^α between two particles, initially at rest

i.e. initially with $U^\mu = (1,0,0,0)^T \quad \xi^\beta = (0,\epsilon,0,0)^T$

Then
$$\frac{\partial^2 \xi^\alpha}{\partial t^2} = \epsilon R^\alpha_{ttx} = -\epsilon R^\alpha_{txt}$$

and
$$R^x_{txt} = \eta^{xx} R_{xtxt} = -\frac{1}{2} h^{(\mathrm{TT})}_{xx,tt}$$
$$R^y_{txt} = \eta^{yy} R_{ytxt} = -\frac{1}{2} h^{(\mathrm{TT})}_{xy,tt}$$

Hence
$$\boxed{\frac{\partial^2}{\partial t^2}\xi^x = \frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{(\mathrm{TT})}_{xx} \qquad \frac{\partial^2}{\partial t^2}\xi^y = \frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{(\mathrm{TT})}_{xy}}$$

Similarly, two test particles initially separated by ϵ in the y-direction satisfy

$$\boxed{\frac{\partial^2}{\partial t^2}\xi^x = \frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{(\mathrm{TT})}_{xy} \qquad \frac{\partial^2}{\partial t^2}\xi^y = -\frac{1}{2}\epsilon \frac{\partial^2}{\partial t^2} h^{(\mathrm{TT})}_{xx}}$$

We can further generalise to a ring of test particles: one at origin, the other initially a $x = \epsilon\cos\theta \quad y = \epsilon\sin\theta \quad z = 0$:

$$\frac{\partial^2}{\partial t^2}\xi^x = \frac{1}{2}\epsilon\cos\theta \frac{\partial^2}{\partial t^2} h^{(\mathrm{TT})}_{xx} + \frac{1}{2}\epsilon\sin\theta \frac{\partial^2}{\partial t^2} h^{(\mathrm{TT})}_{xy}$$

$$\frac{\partial^2}{\partial t^2}\xi^y = \frac{1}{2}\epsilon\cos\theta \frac{\partial^2}{\partial t^2} h^{(\mathrm{TT})}_{xy} - \frac{1}{2}\epsilon\sin\theta \frac{\partial^2}{\partial t^2} h^{(\mathrm{TT})}_{xx}$$

So in the transverse traceless gauge,

$$\overline{h}_{\mu\nu}^{(\mathrm{TT})} = A_{\mu\nu}^{(\mathrm{TT})} \cos\left[\omega(t-z)\right]$$

where

$$A_{\mu\nu}^{(\mathrm{TT})} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx}^{(\mathrm{TT})} & A_{xy}^{(\mathrm{TT})} & 0 \\ 0 & A_{xy}^{(\mathrm{TT})} & -A_{xx}^{(\mathrm{TT})} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Also, since the perturbation is traceless $\overline{h}_{\alpha\beta}^{(\mathrm{TT})} = h_{\alpha\beta}^{(\mathrm{TT})}$

Solutions are:

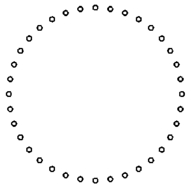
$$\xi^x = \epsilon\cos\theta + \frac{1}{2}\epsilon\cos\theta\, A_{xx}^{(\mathrm{TT})}\cos\omega t + \frac{1}{2}\epsilon\sin\theta\, A_{xy}^{(\mathrm{TT})}\cos\omega t$$

$$\xi^y = \epsilon\sin\theta + \frac{1}{2}\epsilon\cos\theta\, A_{xy}^{(\mathrm{TT})}\cos\omega t - \frac{1}{2}\epsilon\sin\theta\, A_{xx}^{(\mathrm{TT})}\cos\omega t$$

Suppose we now vary θ between 0 and 2π, so that we are considering an initially circular ring of test particles in the x-y plane, initially equidistant from the origin.

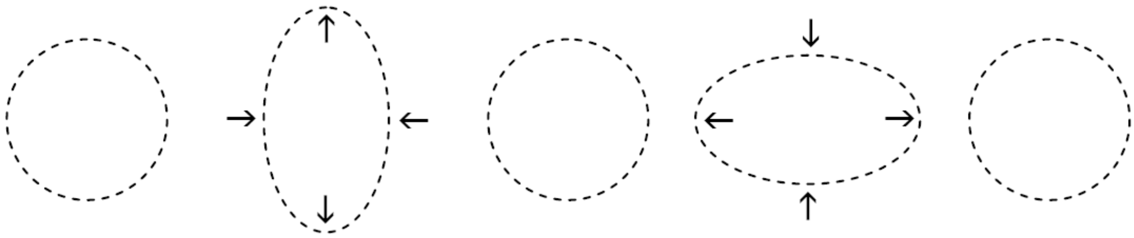
$$A_{xx}^{(\mathrm{TT})} \neq 0 \qquad A_{xy}^{(\mathrm{TT})} = 0$$

$$\xi^x = \epsilon \cos\theta \left(1 + \frac{1}{2} A_{xx}^{(\mathrm{TT})} \cos\omega t \right)$$



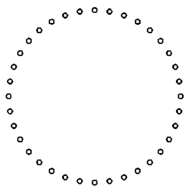
$$\xi^y = \epsilon \sin\theta \left(1 - \frac{1}{2} A_{xx}^{(\mathrm{TT})} \cos\omega t \right)$$

$A_{xx}^{(\mathrm{TT})} \neq 0$ $\quad +$ **Polarisation**



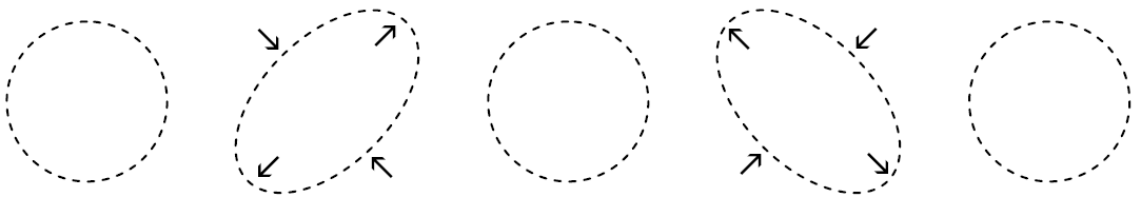
$$A_{xy}^{(\mathrm{TT})} \neq 0 \qquad A_{xx}^{(\mathrm{TT})} = 0$$

$$\xi^x = \epsilon \cos\theta + \frac{1}{2} \epsilon \sin\theta \, A_{xy}^{(\mathrm{TT})} \cos\omega t$$



$$\xi^y = \epsilon \sin\theta + \frac{1}{2} \epsilon \cos\theta \, A_{xy}^{(\mathrm{TT})} \cos\omega t$$

$A_{xy}^{(\mathrm{TT})} \neq 0$ $\quad \times$ **Polarisation**



Rotating axes through an angle of $-\pi/4$ to define $x' = \frac{1}{\sqrt{2}}(x-y)$

We find that $y' = \frac{1}{\sqrt{2}}(x+y)$

$$\xi'^{x} = \epsilon \cos\left(\theta + \frac{\pi}{4}\right) + \frac{1}{2}\epsilon \sin\left(\theta + \frac{\pi}{4}\right) A_{xy}^{(\mathrm{TT})} \cos\omega t$$

$$\xi'^{y} = \epsilon \sin\left(\theta + \frac{\pi}{4}\right) + \frac{1}{2}\epsilon \cos\left(\theta + \frac{\pi}{4}\right) A_{xy}^{(\mathrm{TT})} \cos\omega t$$

These are identical to earlier solution, apart from rotation.

- The two solutions, for $A_{xx}^{(\mathrm{TT})} \neq 0$ and $A_{xy}^{(\mathrm{TT})} \neq 0$ represent two independent gravitational wave **polarisation states**, and these states are usually denoted by '+' and '×' respectively. In general any gravitational wave propagating along the z-axis can be expressed as a linear combination of the '+' and '×' polarisations, i.e. we can write the wave as

$$\mathbf{h} = a\,\mathbf{e}_{+} + b\,\mathbf{e}_{\times}$$

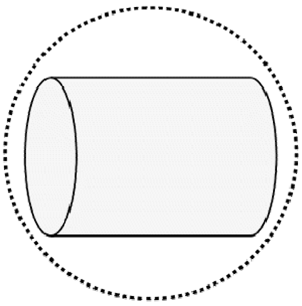
where a and b are scalar constants and the *polarisation tensors* $\mathbf{e}_{+}$ and $\mathbf{e}_{\times}$ are

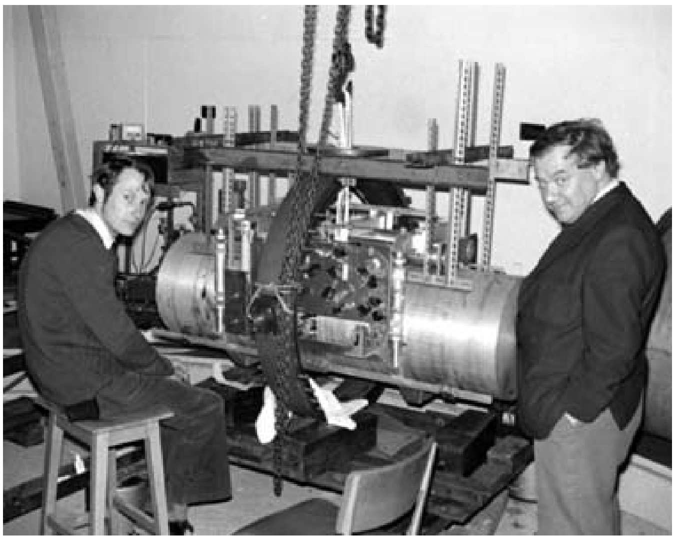
$$\mathbf{e}_{+} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{e}_{\times} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Distortions are **quadrupolar** - consequence of fact that acceleration of geodesic deviation non-zero only for tidal gravitational field.

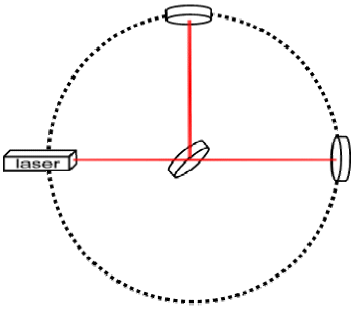
- At any instant, a gravitational wave is invariant under a rotation of 180 degrees about its direction of propagation.
  (c.f. spin states of gauge bosons; graviton must be S=2, tensor field)

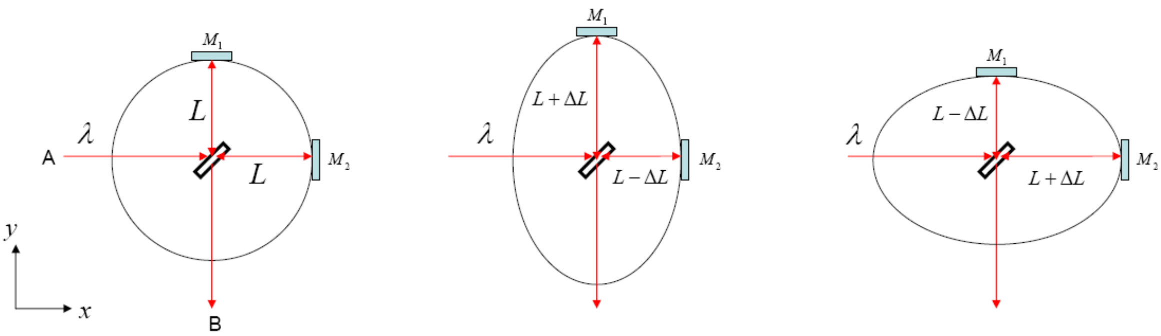
Design of gravitational wave detectors





30 yrs on - Interferometric ground-based detectors





Gravitational wave $\mathbf{h} = h\mathbf{e}_+$ propagating along z axis.

Fractional change in proper separation $\boxed{\dfrac{\Delta L}{L} = \dfrac{h}{2}}$

More generally, for $\mathbf{h} = h\,\mathbf{e}_+$

Detector 'sees'

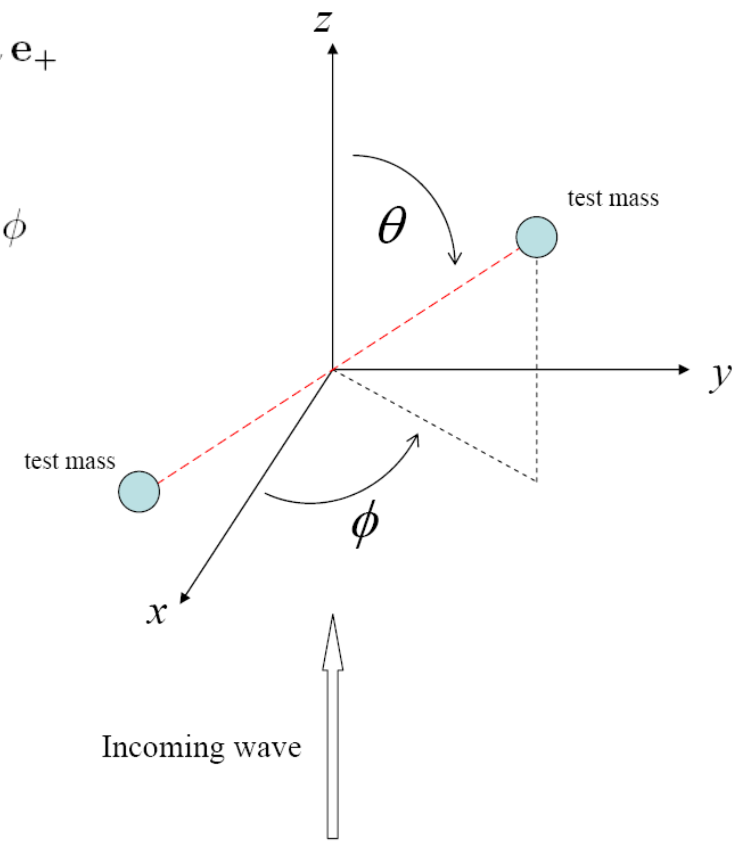
$$h_+ = h \sin^2\theta \cos 2\phi$$

Maximum response for

$$\theta = \pi/2 \qquad \phi = 0$$

Null response for

$$\theta = 0 \qquad \phi = \pi/4$$



More generally, for $\mathbf{h} = h\,\mathbf{e}_\times$

Detector 'sees'

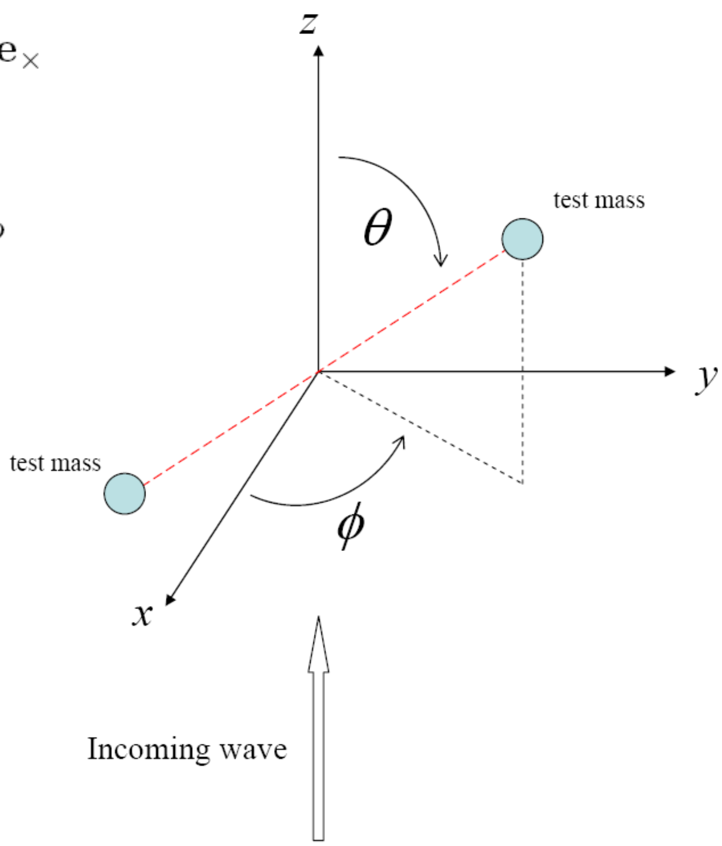
$$h_\times = h \sin^2\theta \sin 2\phi$$

Maximum response for

$$\theta = \pi/2 \qquad \phi = \pi/4$$

Null response for

$$\theta = 0 \qquad \phi = 0$$



9. The Production of Gravitational Waves (pgs 76 – 80)

We can understand something important about the nature of gravitational radiation by drawing analogies with the formulae that describe electromagnetic radiation. This approach is crude at best since the electromagnetic field is a vector field while the gravitational field is a tensor field, but it is good enough for our present purposes. Essentially, we will take familiar electromagnetic radiation formulae and simply replace the terms which involve the Coulomb force by their gravitational analogues from Newtonian theory.

Net electric dipole moment

$$L_{\text{electric dipole}} \propto e^2\, \ddot{\mathbf{d}}^2$$

$$L_{\text{magnetic dipole}} \propto \ddot{\mu}$$

$$\mu = \sum_{q_i} (\text{position of } q_i) \times (\text{current due to } q_i)$$

Gravitational analogues?...

Mass dipole moment: $\mathbf{d} = \sum_{A_i} m_i \mathbf{x}_i$

But $\dot{\mathbf{d}} = \sum_{A_i} m_i \dot{\mathbf{x}}_i \equiv \mathbf{p}$

Conservation of linear momentum implies no mass dipole radiation

$$L_{\text{magnetic dipole}} \propto \ddot{\mu}$$

$$\mu = \sum_{q_i} (\text{position of } q_i) \times (\text{current due to } q_i)$$

Gravitational analogues?...

$$\mu = \sum_{A_i} (\mathbf{x}_i) \times (m_i \mathbf{v}_i) \quad \equiv \quad \mathbf{J}$$

Conservation of angular momentum implies no mass dipole radiation

Also, the quadrupole of a **spherically symmetric mass distribution** is zero.

Metric perturbations which are spherically symmetric don't produce gravitational radiation.

Example: binary neutron star system.

$$\boxed{h_{\mu\nu} = \frac{2G}{c^4 r} \ddot{I}_{\mu\nu}}$$

where $I_{\mu\nu}$ is the **reduced quadrupole moment** defined as

$$I_{\mu\nu} = \int \rho(\vec{r}) \left(x_\mu x_\nu - \frac{1}{3}\delta_{\mu\nu} r^2 \right) dV$$

Consider a binary neutron star system consisting of two stars both of Schwarzschild mass M, in a circular orbit of coordinate radius R and orbital frequency f.

$$I_{xx} = 2MR^2 \left[\cos^2(2\pi f t) - \frac{1}{3}\right]$$

$$I_{yy} = 2MR^2 \left[\sin^2(2\pi f t) - \frac{1}{3}\right]$$

$$I_{xy} = I_{yx} = 2MR^2 \left[\cos(2\pi f t)\sin(2\pi f t)\right]$$

Thus

$$h_{xx} = -h_{yy} = h\cos(4\pi f t)$$

$$h_{xy} = h_{yx} = -h\sin(4\pi f t)$$

where

$$h = \frac{32\pi^2 G M R^2 f^2}{c^4 r}$$

So the binary system emits gravitational waves at **twice** the orbital frequency of the neutron stars.

Also

$$h = 2.3 \times 10^{-28} \; \frac{R^2[\mathrm{km}] f^2[\mathrm{Hz}]}{r[\mathrm{Mpc}]}$$

Huge Challenge!