HEAII   Ex Sheet 4   Model Answers

1) $r_{gyro} = \frac{c \gamma m_p}{eB} = 10 \text{ kpc} \Rightarrow \gamma = \frac{1.6\times10^{-19} \times 10^{-10} \times 3.09\times10^{20}}{3\times10^{8} \times 1.67\times10^{-27}}$

$= 3\times10^{10}$

$\therefore$ Energy of proton $= \gamma m_p c^2$

$= 4.4 \text{ J} = 2.8\times10^{19} \text{ eV}$

Suppose mass of tennis ball $= 50\text{g} = 0.05 \text{ kg}$

Speed $= 100 \text{ km/h} = 27.8 \text{ ms}^{-1}$

$\Rightarrow \quad KE = 0.05 \times (27.8)^2 = 38.5 \text{ J}$

i.e. the KE of tennis ball is about 9 times greater

2) $E_K = 2 \text{ keV} \Rightarrow \frac{1}{2}mv^2/mc^2 = \frac{2}{511} = 3.9\times10^{-3}$

$\therefore \frac{v}{c} = \sqrt{7.8\times10^{-3}} \Rightarrow v = 2.6\times10^{7} \text{ ms}^{-1}$

We want $\frac{1}{2}m_e\langle v^2\rangle \simeq \frac{3}{2}kT$

$\Rightarrow T = 1.5\times10^{7} \text{ K}$

After N scatterings $\left\langle \frac{\varepsilon_N}{\varepsilon_{init}} \right\rangle \sim 1 + \frac{4NkT}{m_e c^2} = 1.5$

$\Rightarrow$ We require $4NkT = 0.5 m_e c^2$

$\Leftrightarrow \quad N = \frac{m_e c^2}{8kT} \simeq 50$

(2)

3) Assuming isotropic emission,

$$L = 4\pi d^2 F \qquad F = 9.9 \times 511 \text{ keV } s^{-1} m^{-2}$$

$$= 8.1 \times 10^{-13} \text{ W m}^{-2}$$

$\Rightarrow L = 7.0 \times 10^{29}$ W, as required

This corresponds to $\sim 8.6 \times 10^{42}$ e^-e^+ pairs / second.

Ang. size = 5° = 0.087 rad.

$\Rightarrow$ radius of emitting region = 740 pc = 2.3×10^{19} m

$\Rightarrow$ volume of region = 5×10^{58} m^3

$\therefore$ we require to inject $\sim \left(\frac{8.6 \times 10^{42}}{5 \times 10^{58}}\right)$ positrons $m^{-3} s^{-1}$

$$= 1.7 \times 10^{-16} \; e^+ \, m^{-3} \, s^{-1}$$

This injection rate won't come from massive stars, type II SN because no recent SF in the Bulge.

Also can't be ISM cosmic rays because these trace disk, so why no 511 keV emission from disk ?

We typically get $\sim 2.5 \times 10^{54}$ e^+ / SNIa, of which a fraction $f \sim 0.03$ escape.

Hence we would inject 7.5×10^{52} e^+ / SNIa

We need $10^{43} \times 3 \times 10^{9}$ e^+ / century $\Rightarrow \sim 0.4$ SNIa / century

4) Classical X-rays are less energetic and are absorbed by the atmosphere, while hard X-rays can penetrate to depths accessible to balloons (~ 10 km)
100 Hard X-rays / m^2 s from Sco X-1

Assume mean energy of 50 keV => Flux $= 5 \times 10^4 \times 1.6 \times 10^{-19}$ W m^{-2}
$= 8 \times 10^{-15}$ W m^{-2}

$$L_X = 4\pi d^2 F$$
$$= 4\pi \times (700 \times 3.09 \times 10^{16})^2 \times 8 \times 10^{-15} \text{ W}$$
$$= 4.7 \times 10^{25} \text{ W}$$

5) $kT = 4.1 \times 10^{-16}$ J $= 2.6$ keV

$$L_X = 4\pi d^2 F = 4\pi \times (720 \times 3.09 \times 10^{22})^2 \times 3.1 \times 10^{-17} \text{ W}$$
$$= 1.9 \times 10^{35} \text{ W}$$

From formula for thermal bremsstrahlung,
$$L_X = 2 \times \left(\frac{2}{\pi \times 9.1 \times 10^{-31}}\right)^{1/2} (4.1 \times 10^{-16})^{1/2} \times V \times 1.54 \times 10^{-31} \times 9.1 \times 10^{-31} \times 9 \times 10^{16} \times \Delta \times n_p^2$$

Here $\Delta = \left[e^{-y_1} - e^{-y_2}\right]$ where $y_1 = 0.1/2.6 = 0.038$
$y_2 = 2.4/2.6 = 0.92$

$$V = \frac{4}{3}\pi R^3$$
$$= \frac{4}{3}\pi \left(22 \times 3.09 \times 10^{19}\right)^3 = 1.3 \times 10^{63} \text{ m}^3$$

$\Delta = 0.564 \longrightarrow \ldots\ldots \longrightarrow n_p = 2.5 \times 10^4$ m^{-3}

6) The energy of an ultra-relativistic electron, $E \propto \gamma$

We are given that $\frac{dE}{dt} \propto \gamma^2$

$$\Rightarrow \quad \frac{d\gamma}{dt} = A\gamma^2$$

i.e. $$\frac{d\gamma}{\gamma^2} = A\,dt$$

$$\Leftrightarrow \quad -\frac{1}{\gamma} = At + C'$$

Let $\gamma = \gamma_0$ at $t = 0$

$$\Rightarrow \quad C' = -\frac{1}{\gamma_0}$$

$$\Leftrightarrow \quad \frac{1}{\gamma} = \frac{1}{\gamma_0} - At = \frac{(1 - A\gamma_0 t)}{\gamma_0}$$

$$\Leftrightarrow \quad \gamma = \gamma_0 \left(1 + C\gamma_0 t\right)^{-1} \qquad (\text{where } C = -A)$$

7) $$L_x = 4\pi d^2 F \text{ W}$$

$$= 4\pi \times 3 \times 10^{-16} d^2 \text{ W} \qquad (d \text{ in m})$$

$$= 4\pi \times 3 \times 10^{-16} \times (3.09 \times 10^{19})^2 d^2 \text{ W} \qquad (d \text{ in kpc})$$

$$= 3.6 \times 10^{24} d^2 \text{ W}$$

For a black-body, assuming X-ray emission dominates, $T = \frac{4.8 \times 10^{-17}}{1.38 \times 10^{-23}}$ K

$T = 3.5 \times 10^6 \, K$

$\Rightarrow \quad L = 4\pi R^2 \sigma T^4$

$\quad = 4\pi (10^4)^2 \times 5.67 \times 10^{-8} \times (3.5 \times 10^6)^4$

$\quad = 10^{28} \, W$

$\Rightarrow \quad d^2 = \left(10^{28} / 3.6 \times 10^{24}\right)$

$\quad \Rightarrow \quad d = 54 \text{ kpc}$

8) Thomson scattering: classical scattering of a photon by a free electron

Compton scattering: high energy photon collides with low-energy electron, electron gains energy at expense of photon

Inverse Compton scattering: reverse case.

IC cross-section $\simeq$ Thomson cross-section if $h\nu \ll m_e c^2$

$$L_{IC} = \frac{4}{3} Q_{IC} \, c \, \gamma^2 U_{\nu_0} \qquad \text{per electron}$$

$$\tau \simeq E/L = \frac{\gamma m_e c^2}{\frac{32}{9}\pi r_e^2 c \gamma^2 U_{\nu_0}} = \frac{9 m_e c}{32\pi r_e^2 \gamma U_{\nu_0}}$$

Substituting $r_e^2 = \dfrac{e^4}{16\pi^2 \varepsilon_0^2 m_e^2 c^4}$

$$\Rightarrow \quad \tau = \frac{9 m_e c}{32\pi \gamma U_{\nu_0}} \, \frac{16\pi^2 \varepsilon_0^2 m_e^2 c^4}{e^4} = \frac{9\pi m_e^3 \varepsilon_0^2 c^5}{2 \gamma U_{\nu_0} e^4}$$

(6)

For a 1 GeV electron $E = \gamma m_e c^2$

$$\Rightarrow \quad \gamma = 1.96 \times 10^3$$

For the CMBR $T = 2.73\,K \Rightarrow U_{\nu_0} = 7.53 \times 10^{-16} \times (2.73)^4$

$$= 4.2 \times 10^{-14}\ J m^{-3}$$

$$\tau = \frac{9\pi \times (9.1 \times 10^{-31})^3 \times (8.85 \times 10^{-12})^2 \times (3 \times 10^8)^5}{2 \times 1.96 \times 10^3 \times 4.2 \times 10^{-14} \times (1.6 \times 10^{-19})^4}$$

$$= 3.8 \times 10^{16}\ s = 1.2\ Gyr$$