XRA I   Ex Sheet 3   Model Answers

From notes: $$\frac{dL}{d\varepsilon} \propto \int_0^\infty \xi(T) \frac{e^{-\varepsilon/kT}}{T^{1/2}} \, dT$$

We need to work out $\xi(T)$

$$\xi(T) = \frac{n_p^2(r(T)) \times 4\pi r^2(T)}{\left|\frac{dT}{dr}\right|}$$

$n_p \propto r^{-a}$ and $T \propto r^{-b}$ => $r \propto T^{-1/b}$

Hence $n_p \propto T^{(a/b)}$ $\frac{dT}{dr} = \left|\frac{dr}{dT}\right|^{-1}$ $\left|\frac{dr}{dT}\right| \propto T^{-\frac{1}{b}-1}$

$$\xi(T) \propto T^{(2a/b)} \, T^{-2/b} \, T^{-\frac{1}{b}-1}$$

$$\propto T^{\frac{2a-3-b}{b}}$$

$$\frac{dL}{d\varepsilon} \propto \int_0^\infty T^{\left(\frac{2a-3-b}{b}\right)-\frac{1}{2}} e^{-\varepsilon/kT} \, dT$$

$$= \int_0^\infty T^{\left(\frac{4a-6-3b}{2b}\right)} e^{-\varepsilon/kT} \, dT = \int_0^\infty T^{-\left(\frac{6+b-4a}{2b}\right)} e^{-\varepsilon/kT} \frac{dT}{T}$$

Put $x = \frac{\varepsilon}{kT}$ => $dx = \frac{-\varepsilon}{kT^2} dT = -x \frac{dT}{T}$

$$\frac{dL}{d\varepsilon} \propto \int_0^\infty \left(\frac{\varepsilon}{kx}\right)^{-\frac{(6+b-4a)}{2b}} e^{-x} \frac{dx}{x}$$

$$\propto \varepsilon^{-\left(\frac{6+b-4a}{2b}\right)}$$ as required

2) $z = 0.037 \Rightarrow cz = v_{rad} = 11100 \text{ kms}^{-1}$

$\Rightarrow d = 11100/70 = 158 \text{ Mpc}$

$= 4.89 \times 10^{24} \text{ m}$

X ray lum $= 4\pi d^2 F$

$= 4\pi \times (4.89 \times 10^{24})^2 \times 1.3 \times 10^{-15} \text{ W}$

$= 3.9 \times 10^{35} \text{ W}$

$$L = \int \frac{dL}{d\varepsilon} d\varepsilon$$

$$\int \frac{dL}{d\varepsilon} d\varepsilon = 2\left(\frac{2}{\pi m_e}\right)^{1/2} \frac{Q_0 m_e c^2}{(kT)^{1/2}} n_p^2 V \int_{0.1 \text{ keV}}^{2.4 \text{ keV}} e^{-\varepsilon/kT} d\varepsilon$$

$$\int_{\varepsilon_1}^{\varepsilon_2} e^{-\varepsilon/kT} d\varepsilon = \left[-kT e^{-\varepsilon/kT}\right]_{\varepsilon_1}^{\varepsilon_2}$$

$$= kT\left[e^{-\varepsilon_1/kT} - e^{-\varepsilon_2/kT}\right]$$

$$\Rightarrow L = 2\left(\frac{2}{\pi m_e}\right)^{1/2} Q_0 m_e c^2 n_p^2 V (kT)^{1/2} \left[e^{-\varepsilon_1/kT} - e^{-\varepsilon_2/kT}\right]$$

$\Leftrightarrow n_p^2 =$ etc ....

$\vdots$

$n_p = 2.8 \times 10^4 \text{ m}^{-3}$

$Q_0 = 1.6 \times 10^{-31} \text{ m}^2$

$V = \frac{4}{3}\pi \times (25 \times 3.086 \times 10^{19})^3 = 1.92 \times 10^{63} \text{ m}^3$

$m_e = 9.1 \times 10^{-31}$ kg $\quad T = 3 \times 10^7$ K

$k = 1.381 \times 10^{-23} \text{ JK}^{-1}$

$\varepsilon_1 = 0.1 \text{ keV} = 1.602 \times 10^{-17}$ J

$\varepsilon_2 = 2.4 \text{ keV} = 3.84 \times 10^{-16}$ J

3) XRA-1 Ex. Sheet 5

$$\frac{dJ}{d\varepsilon} = 2\left(\frac{2}{\pi m_e}\right)^{\frac{1}{2}} \frac{Q_o m_e c^2 V n_p^2}{(kT)^{1/2}} \frac{e^{-\varepsilon/kT}}{\varepsilon}$$

= differential emissivity at the source

number flux at the Earth $= 10^{12}\ m^{-2}\ s^{-1}\ keV^{-1} = N_F$

assuming isotropic emission $\Rightarrow$ $\frac{dJ}{d\varepsilon} = 4\pi d^2 N_F$ $\quad d = 1$ au $= 1.5 \times 10^{11}$ m

$= 2.827 \times 10^{35}\ s^{-1}\ keV^{-1}$

Take $\varepsilon = 1$ keV $= 1.602 \times 10^{-16}$ J

$$n_p^2 = \frac{\varepsilon \times (kT)^{1/2} \left(\frac{dJ}{d\varepsilon}\right) e^{\varepsilon/kT} (\pi m_e)^{1/2}}{2\sqrt{2}\, Q_o m_e c^2 V}$$

$k = 1.381 \times 10^{-23}\ J K^{-1}$

$Q_o = 1.6 \times 10^{-31}\ m^2$

$T = 2.5 \times 10^7$

$m_e = 9.1 \times 10^{-31}$ kg

$V = \frac{4}{3}\pi \times (1.33 \times 10^7)^3 = 9.85 \times 10^{21}\ m^3$

$$= \frac{\left[(1.602 \times 10^{-16}) \times (1.858 \times 10^{-8}) \times (2.827 \times 10^{35}) \times (1.786) \times (1.691 \times 10^{-15})\right]}{(2.828) \times 1.6 \times 10^{-31} \times 9.1 \times 10^{-31} \times 9 \times 10^{16} \times 9.85 \times 10^{21}}$$

$= 6.962 \times 10^{18}$

$\Rightarrow$ $n_p \simeq 2.6 \times 10^9$ protons / m^3

$$L = \int_1^{10} \frac{dL}{d\varepsilon} d\varepsilon = 2\left(\frac{2}{\pi m_e}\right)^{\frac{1}{2}} \frac{Q_o m_e c^2 V n_p^2}{(kT)^{1/2}} \left[e^{-\varepsilon_1/kT} - e^{-\varepsilon_2/kT}\right](kT)$$

$\varepsilon_1 = 1$ keV $= 1.6 \times 10^{-16}$ J

$\varepsilon_2 = 10$ keV $= 1.6 \times 10^{-15}$ J $\quad$ ...... $\Rightarrow$ $L = \cancel{5 \times 10^{19}\ J}\ \ 1.73 \times 10^4$ W

(4) We know that $J_{IC} = N F Q$

N = number of target particles $= n_{\gamma_0} V$

$F = n v = n_e c$ (since electrons are ultra relativistic)

$$\Rightarrow L_{IC} = J_{IC} \times \underbrace{\left(h\nu_0 \times \tfrac{4}{3}\gamma^2\right)}_{\text{average energy of IC photon}}$$

$$= n_{\gamma_0} V n_e c \times Q_{IC} \times h\nu_0 \times \tfrac{4}{3}\gamma^2$$

We can write this as:-

$$L_{IC} = (n_{\gamma_0} h\nu_0) \times (n_e V) \times c \times Q_{IC} \times \tfrac{4}{3}\gamma^2$$

$$= \frac{4}{3} Q_T c N_e \gamma^2 U_{rad}$$

taking $Q_{IC} = Q_T$
and $U_{rad} = n_{\gamma_0} h\nu_0$

For a single synchrotron-emitting electron:-

$$L_s = \frac{e^4 \gamma^2 v^2 B^2 \sin^2\theta}{6\pi \varepsilon_0 c^3 m_e^2}$$

Averaging over all directions, $\langle \sin^2\theta \rangle = \frac{2}{3}$ and taking $v \simeq c$

$$\Rightarrow L_s = \frac{2 e^4 \gamma^2 B^2}{3 \times 6\pi \varepsilon_0 c\, m_e^2}$$

Now $r_e = \dfrac{e^2}{4\pi \varepsilon_0 m_e c^2}$ $\Rightarrow$ $Q_T = \dfrac{8\pi}{3} r_e^2 = \dfrac{8\pi e^4}{3 \cdot 16\pi^2 \varepsilon_0^2 m_e^2 c^4} = \dfrac{e^4}{6\pi \varepsilon_0^2 m_e^2 c^4}$

$$\therefore L_s = \frac{2}{3} Q_T \gamma^2 B^2 \varepsilon_0 c^3$$

$$= \frac{2}{3} Q_T \gamma^2 \frac{B^2}{2\mu_0} \times 2\mu_0 \varepsilon_0 c^3$$

$$= \frac{4}{3} Q_T \gamma^2 U_{mag} c$$

using $\varepsilon_0 \mu_0 = \dfrac{1}{c^2}$
$U_{mag} = \dfrac{B^2}{2\mu_0}$

Hence, for N_e electrons,

$$L_s = \frac{4}{3} Q_T c N_e \gamma^2 U_{mag}$$

$L_s = L_{IC}$ when $U_{mag} = U_{rad}$

i.e. when $\frac{B^2}{2\mu_0} = \frac{4\sigma}{c} T^4 \Rightarrow B = \left(\frac{8\sigma T^4 \mu_0}{c}\right)^{1/2}$

$$= \left(\frac{8 \times (5.671 \times 10^{-8}) \times (2.7)^4 \times 4\pi}{3 \times 10^8}\right)^{1/2}$$

$$= 1 \times 10^{-6}\ T$$

(5) $\frac{dE}{dt} = -\frac{4}{3} Q_T c^2 \gamma^2 \frac{B^2}{2\mu_0}$ (−ve since electron is losing energy)

$E = \gamma m_e c^2$

$\Rightarrow \frac{dE}{dt} = -\frac{4}{3} \frac{Q_T c}{(m_e c^2)^2} \frac{B^2}{2\mu_0} E^2$

$\Leftrightarrow -\int \frac{dE}{E^2} = \frac{4}{3} \frac{Q_T c}{(m_e c^2)^2} \frac{B^2}{2\mu_0} \int dt$

$\Leftrightarrow \frac{1}{E} = At + C'$ where $A = \frac{4}{3} \frac{Q_T c}{(m_e c^2)^2} \frac{B^2}{2\mu_0}$ = constant

At $t = t_0 = 0$, $C' = \frac{1}{\gamma_0 m_e c^2}$

$\Leftrightarrow \frac{1}{\gamma m_e c^2} = \left(A \gamma_0 m_e c^2 t + 1\right) \frac{1}{\gamma_0 m_e c^2}$

$\Leftrightarrow \gamma(t) = \gamma_0 \left(1 + A \gamma_0 m_e c^2 t\right)^{-1}$

$= \gamma_0 \left(1 + C \gamma_0 t\right)^{-1}$ where $C = \frac{4}{3} \frac{Q_T c B^2}{m_e c^2 \, 2\mu_0} = \frac{2 Q_T c B^2}{3 m_e c^2 \mu_0}$