Chapter 6

Black Holes

6.1 Introduction

In Chapters 2 and 3 we first derived, and then investigated, the Schwarzschild solution for the static, spherically symmetric spacetime exterior to a star of (Newtonian) mass, M. We found that the line element for the Schwarzschild metric was given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad (6.1)$$

We noted in passing that this metric 'misbehaves' at $r = 2M$, since the g_{rr} component becomes infinite. However, since e.g. for a solar mass star, $M \simeq 1.5$km in geometrised units, it was clear that – for any 'normal' star – one would reach the physical surface of the star (where in any case one would have to replace the Schwarzschild solution for the exterior metric by the interior solution, considered in Chapter 4, derived from solution of the Oppenheimer-Volkoff equation) before reaching the 'misbehaving' coordinate radius of $r = 2M$. Moreover, as a kind of "insurance policy" against the misbehaviour of the Schwarzschild metric, in Chapter 4 we considered **Buchdahl's Theorem**. This tells us that, for a star with finite central density, we cannot have a static solution with radius, $R < 9M/4$, since such a star would require an infinite pressure gradient at the centre to support itself against its own gravity.

Thus, provided that a star has finite central density and pressure, then the minimum allowed coordinate radius of its surface is still large enough to exceed the troublesome value of $r = 2M$ in the Schwarzschild metric.

6.2 Beyond white dwarfs and neutron stars?

Should we always assume a finite central density and pressure? In Astronomy A1Y we considered the internal structure of stellar remnants after they have left the Main Sequence. These stars are no longer supported by thermal pressure, since they are no longer producing heat from fusion reactions. Instead, they collapse and shrink under their own gravity until **quantum degeneracy pressure** becomes large enough to support them.

Simple arguments based on the **Heisenberg Uncertainty Principle** allowed us to estimate the maximum remnant mass which degeneracy pressure can support. For a **white dwarf** – where electron degeneracy pressure dominates – this maximum mass is $M \simeq 1.3M_\odot$, and is known as the **Chandrasekhar Limit**. For a stellar remnant with a mass greater than the Chandrasekhar Limit, electron degeneracy pressure cannot support its weight and the collapse continues until **neutron degeneracy pressure** becomes large enough to halt it – we then have a **neutron star**. The astrophysical properties of white dwarfs and neutron stars will be left to other honours courses to discuss further. What interests us is simply the question: is there a maximum allowable neutron star mass?

For a stellar remnant of mass greater than about 2 solar masses, even neutron degeneracy pressure is insufficient; according to current physics, *nothing* can then halt the star's collapse under its own gravity, and the star shrinks to a singular point of infinite density – a **black hole**.

Does this singularity actually occur in nature? The current answer would have to be "nobody knows for sure". Many physicists believe that a (thus far incomplete) theory of quantum gravity would provide a new source of 'pressure' (from e.g. superstrings) to halt the star's collapse before reaching infinite density as $r \to 0$. Whether the central singularity *does* exist, at the very least it seems clear that the collapse of the stellar remnant can proceed beyond the point where the coordinate radius of the star satisfies $r = 2M$: the surface known as the **Schwarzschild radius**. We, therefore, need to consider carefully what exactly is going on in the Schwarzschild metric at $r = 2M$.

6.3 The nature of the Schwarzschild surface

Specifically, we want to know whether the misbehaviour of g_{rr} at $r = 2M$ is the result of a real, physical problem with the geometry of spacetime at the Schwarzschild radius, or is simply a consequence of our choice of coordinate system – what is referred to as a 'coordinate singularity'.

A more familiar example of a coordinate singularity is the North (or South) pole of the Earth. The poles are singular points in the latitude and longitude coordinate system because their longtiude is not uniquely defined. However, there is nothing physically different about these points (apart from being rather cold!) compared with any other on the Earth's surface.

We will determine the nature of the $r = 2M$ singularity by considering a material particle falling radially (i.e. with $d\theta = d\phi = 0$) towards the Schwarzschild radius from some finite coordinate radius, $R > 2M$. Suppose that the particle is released from rest at coordinate time $t = 0$ and proper time $\tau = 0$ in the particle's frame. Recall from equation (3.11) that

$$\left(\frac{dr}{d\tau}\right)^2 = k^2 - 1 - \frac{h^2}{r^2} + \frac{2M}{r}\left(1 + \frac{h^2}{r^2}\right) \tag{6.2}$$

Also, from equation (3.9)

$$\frac{d\phi}{d\tau} = \frac{h}{r^2} \tag{6.3}$$

Hence, for a radial trajectory, $h = 0$ and

$$\left(\frac{dr}{d\tau}\right)^2 = k^2 - 1 + \frac{2M}{r} \tag{6.4}$$

Since the particle is released from rest, it must follow that

$$k^2 - 1 = -\frac{2M}{R} \tag{6.5}$$

i.e.

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2M}{r} - \frac{2M}{R} \tag{6.6}$$

or, given that the particle is falling inwards

$$d\tau = -\frac{dr}{\sqrt{\frac{2M}{r} - \frac{2M}{R}}} \tag{6.7}$$

and the elapsed proper time experienced by the particle as it falls from $r = R$ to

$r = 2M$ is given by

$$\Delta\tau = \int_{2M}^{R} \frac{dr}{\sqrt{\frac{2M}{r} - \frac{2M}{R}}} \tag{6.8}$$

We can see that this integral is clearly finite, since $2M/r$ tends to the (perfectly well behaved) value unity as $r \to 2M$. Hence, the particle reaches the Schwarzschild radius in a finite proper time.

What about the interval of coordinate time for the particle to reach the Schwarzschild radius? This follows from equation (3.8)

$$\frac{dt}{d\tau} = \frac{k}{1 - 2M/r} \quad = \quad \frac{\sqrt{1 - 2M/R}}{1 - 2M/r} \tag{6.9}$$

or, using, equation (6.7)

$$dt = -\frac{\left(\sqrt{1 - 2M/R}\right)\, dr}{\sqrt{\frac{2M}{r} - \frac{2M}{R}}\left(1 - \frac{2M}{r}\right)} \tag{6.10}$$

i.e.

$$\Delta t = \int_{2M}^{R} \frac{\left(\sqrt{1 - 2M/R}\right)\, dr}{\sqrt{\frac{2M}{r} - \frac{2M}{R}}\left(1 - \frac{2M}{r}\right)} \tag{6.11}$$

This rather ugly-looking integral diverges as $r \to 2M$; i.e. the particle takes an infinite amount of coordinate time to reach the Schwarzschild radius. It is straightforward to show that the same result is true for a photon emitted radially at $r = R$; it appears, to a distant observer (whose clock essentially measures coordinate time) to take an infinite time to reach the Schwarzschild radius.

Thus, we have shown that the misbehaviour of the Schwarzschild metric at $r = 2M$ is only a coordinate singularity; the proper time experienced by a particle as it reaches the Schwarzschild radius is perfectly well-behaved, and the misbehaviour of g_{rr} in the Schwarzschild metric is purely a consequence of the coordinate system breaking down at $r = 2M$. Below we discuss briefly an alternative coordinate system in which the coordinate singularity at $r = 2M$ is removed.

What are the physical consequences for a material particle or photon crossing the Schwarzschild radius? Superficially, the answer is "nothing much". For example, astronauts free-falling in their spaceship towards the black hole would not be aware of any sudden, dramatic changes in the laws of physics as they reach $r = 2M$. In the astronaut's Local Inertial Frame, the results of any experiment would still agree with the predictions of Special Relativity – exactly as the strong equivalence principle requires[1]. Crossing the Schwarzschild radius has profound consequences for the ultimate fate of the astronaut, however, and is the reason why the $r = 2M$ surface is also known as the **Event Horizon** of the black hole.

Provided that the astronauts are at coordinate radius $r > 2M$, they can always fire their spaceship rockets and escape to infinity; as the spaceship approaches the Schwarzschild

[1]Depending on the *mass* of the black hole, however, the astronauts may have begun to experience the rather unpleasant phenomenon of 'spaghettification' by the time the spaceship crosses the Schwarzschild radius. This phenomenon arises because of the intense gradient in the gravity field of the black hole: if the astronauts are falling in feet first, then the gravitational pull is significantly different between their heads and feet. This tidal difference would stretch out the astronauts (and probably their spaceship too!) like a piece of spaghetti. However, this stretching is due to the tidal effect of the black hole's gravity field (which for a stellar mass black hole is very large for $r \sim 2M$) and does not directly have anything to do with crossing the Schwarzschild radius. Putting this another way, a LIF still exists at the Schwarzschild radius; it is simply that the LIF may be very small (and in particular much smaller than the astronauts!) due to the tidal gravitational field. For a supermassive black hole at the heart of a quasar, on the other hand, the Schwarzschild radius is much larger – e.g. if $M \simeq 10^6 M_\odot$, then $R_S = 3 \times 10^6$km. In this case the tidal stresses are small enough that the astronauts would cross the Schwarzschild radius without any noticeable discomfort.

radius this would require ever-increasing amounts of fuel, but it remains possible in principle. As soon as the Schwarzschild radius is crossed, however, it is **impossible** for the spaceship to escape from the black hole's clutches. **All** trajectories must inevitably carry the spaceship to smaller coordinate radii – i.e. it cannot even remain at fixed r, no matter how much fuel it uses up trying to do so. The spaceship proceeds inexorably to $r = 0$ where (at least in classical General Relativity) there is a true, physical singularity of infinite density which will crush the astronauts and spaceship out of existence.

Once inside the Schwarzschild radius, the astronauts cannot even send a message – e.g. a single photon – to explain their fate to anxious friends waiting outside; even light emitted inside the Schwarzschild radius must follow a trajectory that inevitably leads inwards to the singularity[2].

The astronauts can, however, receive messages – and even food parcels! – from outside the Schwarzschild radius, although as seen by a distant observer they would appear to take an infinite time to reach the Schwarzschild radius. Thus, the astronauts are truly 'beyond the horizon' of any observer outside the Schwarzschild radius, which justifies the use of the terminology 'Event Horizon'.

6.4 Inside the event horizon

How can we prove these remarkable claims about crossing the Event Horizon? In fact they follow fairly straightforwardly from the Schwarzschild metric for $r < 2M$.

[2] We will later see that, when quantum effects are included, it is not strictly true that light cannot escape from inside the Schwarzschild radius, but in purely classical terms this statement *is* true

(Although we have seen that a coordinate singularity exists at $r = 2M$, we can still use the Schwarzschild metric to describe the spacetime interior to the Event Horizon).

First, recall that the interval, ds^2, between any two neighbouring events – which we label (t, x, y, z) and $(t + dt, x + dx, y + dy, z + dz)$ in some coordinate system – in spacetime can be *null* (in which case $ds^2 = 0$), *spacelike*, ($ds^2 > 0$), or *timelike*, ($ds^2 < 0$). If the interval is spacelike, then one can find a Lorentz frame (S', say) in which the events occur at the same coordinate time – i.e. $dt = 0$ and $\sqrt{ds^2}$ is the proper distance between them. Thus, if $ds^2 > 0$, the two events cannot lie on the worldline of a material particle, since an observer in S' would then see the particle in two places at the same time – violating causality.

Now let us suppose that a particle is at rest inside the Event Horizon of a Schwarzschild black hole, and consider neighbouring events with (in spherical polar coordinates) $dr = d\theta = d\phi = 0$ and $dt \neq 0$. From equation (6.1),

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 \tag{6.12}$$

Since $r < 2M$, it follows that $ds^2 > 0$ – i.e. the interval between the events is **positive**, and so **cannot** lie on the worldline of the particle.

Thus, no particle can be stationary inside the Event Horizon. This is essentially because, for $r < 2M$ the roles of our coordinate labels r and t are in the opposite sense to their familiar interpretation far outside the Schwarzschild radius. Intervals with only $dt \neq 0$ behave as spacelike, instead of timelike, intervals, with the reverse being the case for intervals with only $dr \neq 0$. Consequently, just as in the exterior spacetime, far from the black hole, we can move freely through *space* but not through *time*, inside the Event Horizon we lose our ability to move freely in coordinate radius.

6.4.1 Removing the coordinate singularity

To proceed further we need to get round the problem of the misbehaving coordinate radius at $r = 2M$. We do this by introducing a new time coordinate, $\tilde{t}$, defined by[3]

$$\tilde{t} = t + 2M \ln \left| \frac{r}{2M} - 1 \right| \tag{6.13}$$

We can show (see Examples sheet II.5) that, in the coordinate system $(r, \tilde{t})$, and for simplicity taking $d\theta = d\phi = 0$,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 + \frac{4M}{r} dr d\tilde{t} + \left(1 + \frac{2M}{r}\right) dr^2 \tag{6.14}$$

which has **no** coordinate singularity at $r = 2M$. Note, however, that there is *still* a singularity at $r = 0$, as there should be as this is a true, physical singularity. A more general coordinate transformation which is similarly well-behaved at $r = 2M$ is discussed in some detail in Green Schutz, pg 292: it is known as **Kruskal – Szekeres** coordinates. We do not consider Kruskal – Szekeres coordinates further here, however.

We obtain the equations of the **null cones** – i.e. the trajectories in spacetime which are the geodesics of photons – by setting $ds^2 = 0$ in equation (6.14), dividing through by dr^2 and solving for $d\tilde{t}/dr$. This gives us a quadratic equation, with roots (see Examples

[3]Remember that the coordinate time is only a convenient label and not of itself a meaningful, invariant physical quantity, so we are not altering any fundamental physical property of the spacetime geometry – only providing an easier way to visualise it

sheet II.5)

$$\frac{d\tilde{t}}{dr} = -1 \qquad \text{or} \qquad \frac{d\tilde{t}}{dr} = \frac{1 + 2M/r}{1 - 2M/r} \tag{6.15}$$

Figure 11 (following Berry – Principles of Cosmology and Gravitation) shows how these null cones change with coordinate radius. For $r >> R_S$, the null cones have an approximate slope of ± 1, i.e. as for flat Minkowski spacetime. As r approaches R_S, however, the null cones begin to 'tip over' and lean inwards. For $r = R_S$ one side of the null cone is vertical, while the other side (as at all other coordinate radii) has slope of exactly -1. Hence, at $r = R_S$ the timelike geodesics (which are the possible worldlines of material particles, and lie within the forward null cone) all point inwards towards $r = 0$. Moreover, for $r < R_S$ the null cones continue to tip over, so that even photons are inevitably carried to smaller coordinate radius. Thus, both material particles *and* photons, once inside the Event Horizon, **cannot** escape again, nor indeed can they avoid the fate of being pulled towards the singularity at $r = 0$. We can see from Figure 12, however, that our intrepid astronauts could indeed still receive messages and food parcels as they head towards their doom, inside the Event Horizon.

Figure 11

Null cones close to the Event Horizon of
a Schwarzschild black hole

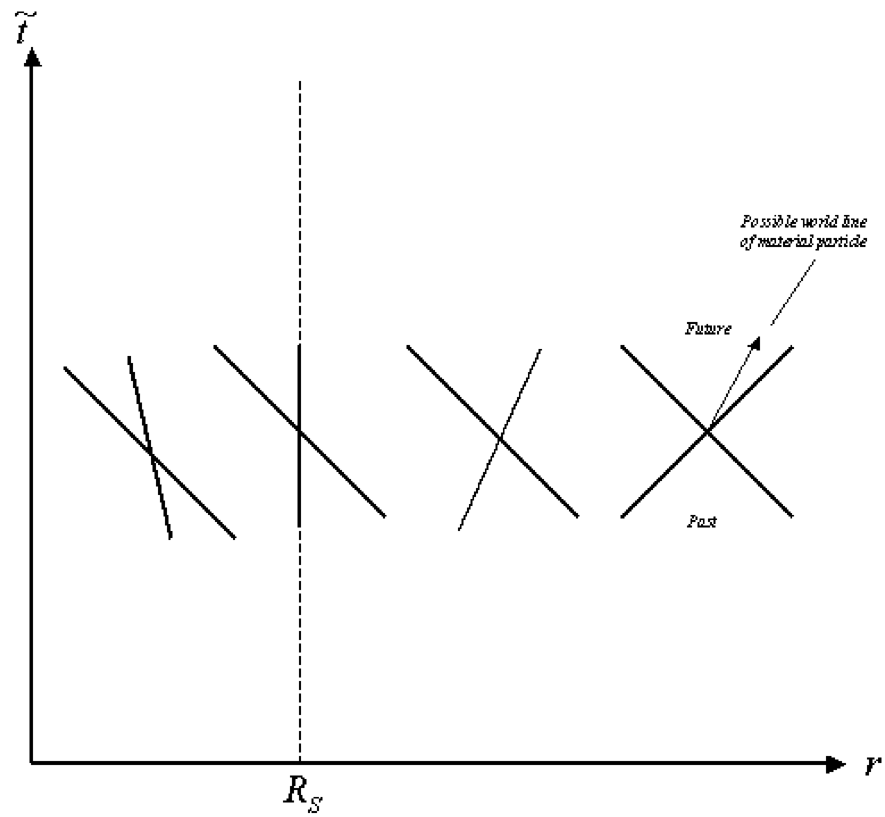
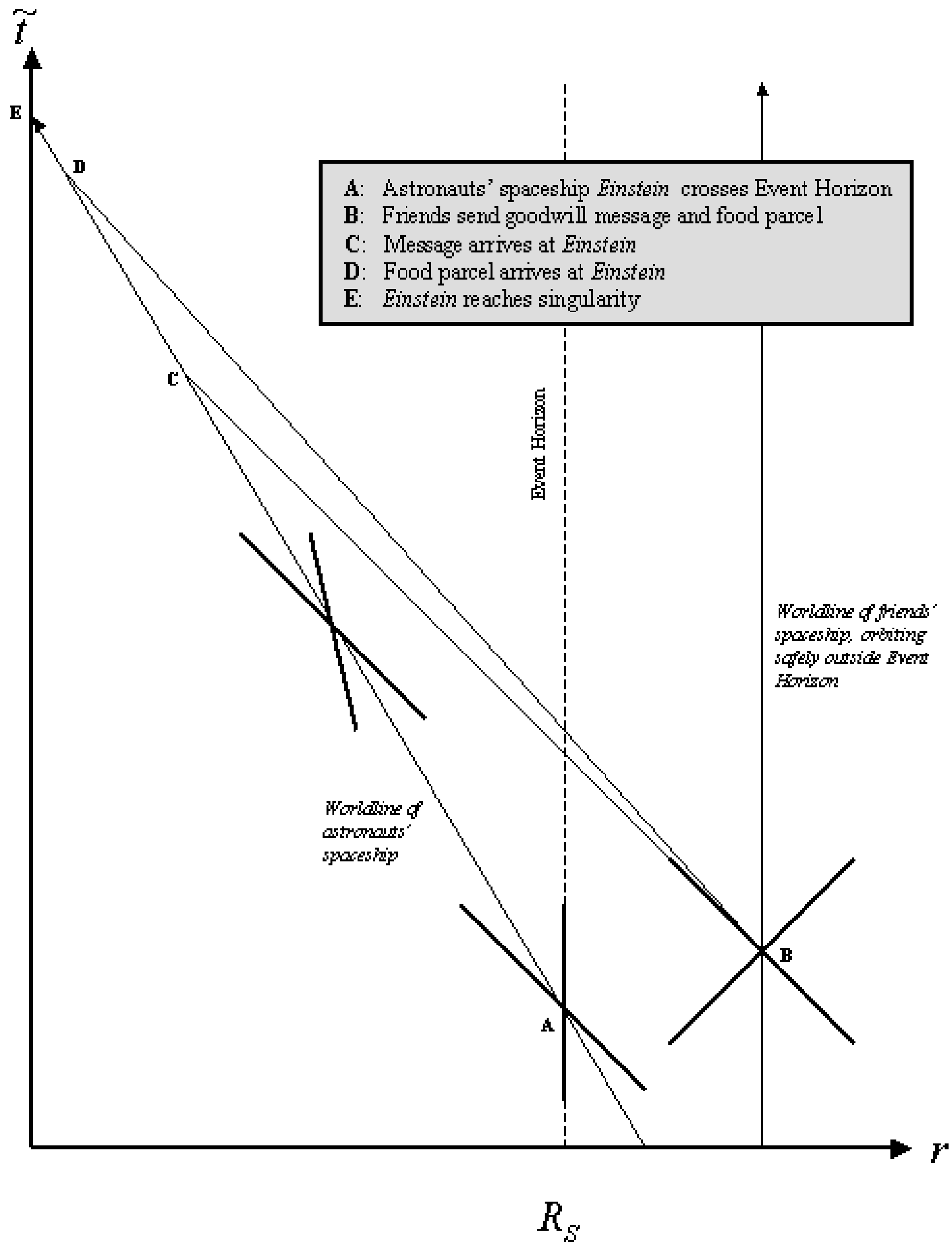


Figure 12

Possible world lines of photons and 'food parcels' sent from outside the Event Horizon of a Schwarzschild black hole.



6.5 How black are black holes?

Since the region inside the Event Horizon of a Schwarzschild black hole is cut off from the outside Universe, we see the justification of the term 'hole': the interior of the Event Horizon really is like a 'hole' in spacetime. Is the name 'black hole' also justified, however? Does a collapsing simply star 'switch off' and become completely black?

At first we might think that the black hole would *not* switch off in this manner. We saw earlier that a material particle infalling radially towards the Schwarzschild radius takes an infinite amount of coordinate time to get there. Doesn't this mean, therefore, that as a massive star undergoes spherical collapse, a distant observer would see the light emitted from the star just before it collapses through the Schwarzschild radius, forever 'frozen' at the Event Horizon?

An answer to this question is provided when we realise that the light from the collapsing star is also redshifted, as it 'climbs out' of the star's gravity field. In Chapter 3 we derived equation (3.75) for the redshift, z, of light emitted at coordinate radius, r_e, and observed at coordinate radius, r_o.

$$z \equiv \frac{\lambda_o - \lambda_e}{\lambda_e} = \sqrt{\frac{1-2M/r_o}{1-2M/r_e}} - 1 = \sqrt{\frac{r_e(r_o - R_S)}{r_o(r_e - R_S)}} - 1 \qquad (6.16)$$

(Note that $z \to \infty$ as $r_e \to R_S$). Consider the bolometric luminosity of the star as it collapses, compared with the (constant) luminosity, L_C, which the star *would* have if we ignored General Relativistic effects. Suppose the light from the star is emitted radially at coordinates (t_e, r_e), and is then observed by a distant observer who is at coordinates (t_o, r_o). Then

$$L(t_o) = \frac{L_C}{(1+z)^2} \tag{6.17}$$

We can understand equation (6.17) as follows. Firstly the energy of each photon received by the observer is redshifted by a factor of $(1+z)$; secondly the arrival times of the photons are also increased by the same factor, so that the luminosity, L, is reduced by a factor of $(1+z)^2$.

The light ray, being a null geodesic, satisfies

$$\int_{t_e}^{t_o} dt \quad = \quad \int_{r_e}^{r_o} \frac{dr}{1-2M/r} \quad \equiv \quad \int_{r_e}^{r_o} \frac{dr}{1-R_S/r} \tag{6.18}$$

It is easy to verify that the integrand of the right hand side has integral $r + R_S \ln(r - R_S)$, so that

$$t_o - t_e \quad = \quad r_o - r_e + R_S \ln\left(\frac{r_o - R_S}{r_e - R_S}\right) \tag{6.19}$$

Taking $t_e = 0$ and re-arranging

$$\ln\left(\frac{r_e - R_S}{r_o - R_S}\right) \quad = \quad -\left[\frac{t_o - (r_o - r_e)}{R_S}\right] \tag{6.20}$$

i.e.

$$\frac{r_e - R_S}{r_o - R_S} \quad \propto \quad \exp\left[-t_o/R_S\right] \tag{6.21}$$

However, from equation (6.16)

$$\frac{r_e - R_S}{r_o - R_S} \quad = \quad \frac{r_o}{r_e}\frac{1}{(1+z)^2} \tag{6.22}$$

so, from equation (6.17), it follows that

$$\frac{L(t_o)}{L_C} \quad \propto \quad \exp\left[-t_o/R_S\right] \tag{6.23}$$

or, re-introducing the speed of light, c

$$\frac{L(t_o)}{L_C} \quad \propto \quad \exp\left[-ct_o/R_S\right] \tag{6.24}$$

Thus, we see that the luminosity of the star falls off exponentially, and on a very short timescale – equal to the time for light to cross the Schwarzschild radius of the collapsing star[4].

6.6 Rotating black holes

So far in this chapter we have considered only static black holes. The treatment of more general situations is considerably more difficult, and in fact can usually only be handled numerically for specific cases. However, we will now briefly discuss one particular example of a more general black hole for which analytic results have been derived: a **Kerr black hole**.

If the collapse of a star is nearly spherical then it can be shown that all of the non-spherical parts of the star's mass-energy distribution are radiated away as gravitational waves – except for some **angular momentum**. This residual angular momentum results in a **rotating black hole**, which can be described by the **Kerr metric**.

6.6.1 The Kerr metric

This metric is characterised by two constants, M and J, which we can identify by requiring that in the weak-field limit the behaviour of a test particle reduces to the predictions of Newtonian dynamics: M is the Newtonian mass of the star and J is the magnitude of its total angular momentum. Writing $a \equiv J/M$, the line element for the

[4]Our analysis here is a little simplistic, since the Schwarzschild metric is a *static* metric, and yet we are considering the behaviour of the metric as the star itself is collapsing. However, a more rigorous treatment would essentially give us the same result: that the luminosity of the collapsing star decreases exponentially

Kerr metric takes the (rather complicated) form

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2}\, dt^2 - 4a\frac{Mr\sin^2\theta}{\rho^2}\, dtd\phi + \frac{\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\, d\phi^2 + \frac{\rho^2}{\Delta}\, dr^2 + \rho^2\, d\theta^2 \tag{6.25}$$

where

$$\Delta = r^2 - 2Mr + a^2 \tag{6.26}$$

and

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{6.27}$$

A detailed discussion of the Kerr metric lies well beyond the scope of this course, but we note that the metric is **not** diagonal, because $g_{t\phi} \neq 0$. The presence of this term produces a quite remarkable effect known as 'frame dragging'. Before we can demonstrate this effect, however, we first consider an important result on conserved quantities along geodesics.

6.6.2 Conservation of four momentum along geodesics

Recall from Chapter 1, equation (1.25) the geodesic equation for a material particle

$$\frac{dv^\alpha}{d\tau} + \Gamma^\alpha_{\beta\delta} v^\beta v^\delta = 0 \tag{6.28}$$

We can rewrite this equation as

$$\left(\frac{\partial v^\alpha}{\partial x^\beta} v^\beta + \Gamma^\alpha_{\beta\delta} v^\beta v^\delta\right) = 0 \tag{6.29}$$

or

$$v^\beta\, v^\alpha{}_{;\beta} = 0 \tag{6.30}$$

(This equation basically *defines* a geodesic as a curve along which the tangent vector to the curve is parallel-transported.)

We can obtain another version of equation (6.30) in covariant form using

$$v^{\alpha} = g^{\alpha\sigma} v_{\sigma} \tag{6.31}$$

from which it is straightforward to show (see Examples sheet II.5) that

$$v^{\alpha}\, v_{\beta;\alpha} = 0 \tag{6.32}$$

If we introduce the contravariant and covariant components of the **four momentum**, defined by

$$p^{\alpha} = mv^{\alpha} \qquad \text{and} \qquad p_{\alpha} = mv_{\alpha} \tag{6.33}$$

where m is the rest mass of the particle, then it follows from equation (6.32) that

$$p^{\alpha}\, p_{\beta;\alpha} = 0 \tag{6.34}$$

or

$$p^{\alpha}\, p_{\beta,\alpha} \;=\; \Gamma^{\gamma}_{\beta\alpha} p^{\alpha} p_{\gamma} \;=\; \frac{1}{2} g^{\gamma\nu} \left(g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu} \right) p^{\alpha} p_{\gamma} \tag{6.35}$$

After contraction and some index permutation this reduces to

$$p^{\alpha}\, p_{\beta,\alpha} \;=\; \frac{1}{2} g_{\nu\alpha,\beta}\, p^{\nu} p^{\alpha} \tag{6.36}$$

or

$$mv^{\alpha} \frac{\partial p_{\beta}}{\partial x^{\alpha}} \;=\; m \frac{dx^{\alpha}}{d\tau} \frac{\partial p_{\beta}}{\partial x^{\alpha}} \;=\; m \frac{dp_{\beta}}{d\tau} \;=\; \frac{1}{2} g_{\nu\alpha,\beta}\, p^{\nu} p^{\alpha} \tag{6.37}$$

We can see an important result from equation (6.37). If all the components of the metric are independent of coordinate x^{β}, then the right hand side of the equation is zero, which then implies that p_{β} is a constant along the geodesic.

6.6.3 Frame dragging in the Kerr Metric

If we look again at the Kerr metric interval, equation (6.25), we see that the components are independent of ϕ. The results of the previous section then imply that a material particle, moving along a geodesic in the Kerr metric, conserves the p_ϕ component of its four momentum.

The contravariant component, p^ϕ, is given by

$$p^\phi = g^{\phi\alpha} p_\alpha = g^{\phi\phi} p_\phi + g^{\phi t} p_t \tag{6.38}$$

The second term on the right hand side would be zero for an orthogonal metric, but the Kerr metric is *not* orthogonal. Similarly, for p^t,

$$p^t = g^{t\alpha} p_\alpha = g^{tt} p_t + g^{t\phi} p_\phi \tag{6.39}$$

Consider now a material particle with zero angular momentum – i.e. with $p_\phi = 0$. Now

$$p^t = m\frac{dr}{d\tau} \qquad \text{and} \qquad p^\phi = m\frac{d\phi}{d\tau} \tag{6.40}$$

so that

$$\frac{d\phi}{dt} = \frac{p^\phi}{p^t} = \frac{g^{\phi t}}{g^{tt}} \neq 0 \tag{6.41}$$

However, $d\phi/dt$ defines the **angular velocity** of our material particle, as measured by a distant observer. Thus, we have the remarkable result that a zero angular momentum particle, dropped in free fall (i.e. in a locally inertial frame) radially inwards from infinity onto a rotating black hole, nevertheless acquires a non-zero angular velocity.

What is going on here? We can think of the intense gravity of the black hole 'dragging' the particle around in the same sense as its own rotation. This effect is known as the **dragging of inertial frames**. It will also occur for the metric of *any* rotating mass, although in most situations (e.g. the spacetime in the vicinity of the Earth) it is a tiny effect. Nevertheless, a satellite mission – **Gravity Probe B** – was recently launched by Stanford University to measure frame dragging from low Earth orbit. For more details see http://einstein.stanford.edu.

6.7 Hawking radiation

In 1974 the Cambridge physicist Stephen Hawking proved the startling result that black holes are not completely black! When one includes the effects of quantum mechanics, applied to electromagnetic fields near to a black hole, the classical General Relativistic result that *nothing* – not even light – can escape from inside the Event Horizon is no longer true: black holes leak photons continuously in a process that is now known as **Hawking radiation**.

To derive Hawking's result rigorously requires quantum field theory. However, we can get a feel for the main features of Hawking radiation by applying the Heisenberg Uncertainty Principle: whereas up until now we have treated photons (and indeed material particles) as precise points along a trajectory in spacetime, the Uncertainty Principle implies that, in fact, photons and material particles are not 'localisable' – their position and velocity cannot be determined to arbitrary precision.

One form of the Uncertainty Principle states that

$$\Delta E \Delta t \sim \hbar = \frac{h}{2\pi} \tag{6.42}$$

where ΔE is the minimum uncertainty in the energy of a particle which exists in a particular quantum state for time Δt. According to quantum field theory, the vacuum of empty space is not empty at all, but is filled with quantum fluctuations in the fields which it contains. For electromagnetic fields, these fluctuations consist of pairs of 'virtual photons', the existence of which violates energy conservation, but provided the photons exist for less than $\Delta t \simeq \Delta E/\hbar$ this is permissible; the energy which they borrow from the vacuum is effectively 'repaid' before the vacuum has noticed.

Even close to the Event Horizon of a black hole, spacetime is locally flat and will therefore be filled with these quantum fluctuations. Consider a fluctuation which produces two photons: one of energy, E, and the other of energy, $-E$. In the flat spacetime of Special Relativity, the negative energy photon cannot propagate freely, since it would be moving backwards in time (see Green Schutz, pgs 303-304), and so it must recombine with the positive energy photon within a time $\Delta t \leq \hbar/E$.

Suppose, however, the pair of virtual photons is produced in the locally flat spacetime just outside the Event Horizon of a black hole. The negative energy member of the pair therefore has a chance of crossing the horizon before a time $\hbar/E$ has elapsed. Once inside the horizon, the negative energy photon *can* propagate freely because the timelike and spacelike roles of the t and r coordinates are interchanged. The negative energy photon **must** move radially inwards, and therefore cannot escape again from the Event Horizon, but that still leaves the positive energy photon outside the horizon, free to propagate and to escape to infinity.

Suppose we measure the energy, $\mathcal{E}$, of the photon in a locally inertial frame just outside the Event Horizon, at coordinate radius $r = 2M + \epsilon$. Any particle in this locally inertial frame will begin free-falling towards the Event Horizon, reaching it after a proper time, $\Delta\tau$, given by, from equation (6.8)

$$\Delta\tau = \int_{2M+\epsilon}^{2M} \frac{dr}{\sqrt{\frac{2M}{r} - \frac{2M}{2M+\epsilon}}} \tag{6.43}$$

For $\epsilon << 2M$ this reduces to

$$\Delta\tau = 2\,(2M\epsilon)^{1/2} \tag{6.44}$$

If we set $\mathcal{E} = \Delta E$ and $\Delta\tau = \Delta t$ in the Uncertainty Principle, we find that

$$\mathcal{E} = \frac{1}{2}\hbar\,(2M\epsilon)^{-1/2} \tag{6.45}$$

The energy of the outgoing photon when it reaches infinity can be shown to be

$$E_\infty = \mathcal{E}\left(\frac{\epsilon}{2M}\right)^{1/2} = \frac{\hbar}{4M} = \frac{h}{8\pi M} \tag{6.46}$$

Hawking derived the more rigorous result that the outgoing photons have a **black-body** spectrum with temperature $T = \hbar/8\pi kM$ and typical energy $E = kT = \hbar/8\pi M$, which differs from our result only by a factor of 2π. (The fact that Hawking radiation is black-body makes sense, even without formal proof, since classically a black hole is a perfect absorber of radiation).

6.7.1 Lifetime of a black hole

We see that the temperature of a black hole is inversely proportional to its mass. For, e.g., a black hole of 10 solar masses, we find that

$$T = \frac{\hbar}{8\pi kM} = 1.5\times 10^{-44}\,\mathrm{K} \tag{6.47}$$

which is an incredibly small, but still formally non-zero, temperature.

The luminosity of the Hawking radiation from the black hole is, according to the Stefan-Boltzmann law, proportional to $A\,T^4$, where A is the area of the Horizon. Since

$$A \quad = \quad 4\pi R^2 \quad = \quad 4\pi(2M)^2 \quad = \quad 16\pi M^2 \tag{6.48}$$

for a Schwarzschild black hole, this means that

$$L \quad \propto \quad M^{-2} \tag{6.49}$$

This luminosity must come at the expense of a decrease in the mass of the black hole (remember, the escape to infinity of the positive energy photon is accompanied by the capture of a *negative* energy photon, which reduces the mass of the black hole). Hence

$$\frac{dM}{dt} \quad \propto \quad M^{-2} \tag{6.50}$$

from which it follows that the lifetime, τ, of a black hole satisfies

$$\tau \quad \propto \quad M^{3} \tag{6.51}$$

i.e. the more massive the black hole, the longer it lives, and the cooler its temperature. Determining the constant of proportionality, it is found that

$$\left(\frac{\tau}{10^{10}\,\mathrm{yr}} \right) \quad = \quad \left(\frac{M}{10^{12}\,\mathrm{kg}} \right)^{3} \tag{6.52}$$

So for a stellar mass black hole, Hawking radiation is completely negligible. In some cosmological theories, however, **primordial black holes** can form in the very early Universe; equation (6.52) says that a primordial black hole of mass around 10^{12} kg would have 'evaporated' on a timescale comparable to the age of the Universe. Since the black-body temperature of the black hole increases as its mass decreases, the evaporation would end with a very short and violent burst of gamma rays. Detection of such a gamma ray background would provide a useful observational test for the existence of primordial black holes.