Chapter 5

Gravitational Radiation

5.1 Non-stationarity

In Chapters 2 – 4 we have considered a static metric, which we defined as a metric for which we can find a time coordinate, t, satisfying

1. all metric components are independent of t
2. the metric is unchanged if we apply the transformation $t \to -t$

A metric which satisfies property (1) but not property (2) is known as **stationary**. An example is the metric of a spherically symmetric star which is *rotating*: reversing the time coordinate changes the sense of the rotation, even though one can find a coordinate system in which the metric components are all independent of time. In the next chapter (time permitting) we will consider the stationary metric of a rotating black hole.

In this chapter we explore some consequences of also relaxing the assumption of property (1), by considering spacetimes in which the metric components are time dependent.

This can happen when the source of the gravitational field is varying. One of the most important predictions of General Relativity was that such a time-varying gravitational field generates **gravitational radiation** – 'ripples' in spacetime, which propagate at the speed of light. These gravitational waves are vanishingly weak for all but the most extreme astrophysical situations, and their detection and measurement presents enormous technological challenges compared with electromagnetic radiation.

For several decades physicists at Glasgow University have been at the forefront of the global effort to build gravitational wave detectors. Several such detectors have recently gone 'online' and are now searching for gravitational wave signatures. Although no signals have been detected directly to date, the first detections are confidently expected within the next few years. Moreover, for several decades there has been strong *indirect* evidence for the existence of gravitational radiation because the waves carry away energy from the source of the gravitational field. Observations of the binary pulsar system PSR 1913+16 show that its orbital semi-major axis is 'shrinking' (see Figure 9) and the rate of decrease in the semi-major axis is in excellent agreement with the calculated prediction based on the energy loss from gravitational radiation.

Although the study of gravitational radiation is one of the most exciting and active fields of research in astrophysics and cosmology today, unfortunately it is also an extremely technically difficult subject and mainly lies well beyond the scope of this course. Nevertheless, in this chapter we will try to highlight some of the important characteristics of gravitational radiation, beginning with a discussion of how plane gravitational waves arise as the free-space solutions to Einstein's equations in the 'linearised theory' of a weak gravitational field.

5.2 Weak gravitational fields

5.2.1 'Nearly' flat spacetimes

Since spacetime is flat in the absence of a gravitational field, a weak gravitational field is one in which spacetime is 'nearly' flat. What we mean by 'nearly' here is that we can find a coordinate system in which the metric has components

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{5.1}$$

where

$$\eta_{\alpha\beta} = \text{diag}\ (-1, 1, 1, 1) \tag{5.2}$$

is the Minkowski metric of Special Relativity, and $|h_{\alpha\beta}| << 1$ for all α and β.

A coordinate system which satisfies equations (5.1) and (5.2) is referred to as a 'Nearly Lorentz' coordinate system. Notice that we say that we can find *a* coordinate system satisfying these equations. It certainly does *not* follow that for *any* choice of coordinate system we can write the metric components of the nearly flat spacetime in the form of equations (5.1) and (5.2). Indeed, even if the spacetime is precisely Minkowskian, we could adopt (somewhat perversely perhaps) a coordinate system in which the metric components were very far from the simple form of equation (5.2). This does not affect the validity of the principle of general covariance in General Relativity: remember that if we write down a tensor equation in one coordinate system, the equation remains valid in any other coordinate system, but the tensor *components* change from one coordinate system to another.

In some coordinate systems, therefore, the components may be enormously more complicated than in others. The secret to *solving* tensor equations in General Relativity is, often, to first choose a coordinate system in which the components are as simple as possible. In that sense, equations (5.1) and (5.2) represent a 'good' choice of coordinate system; just as equation (5.2) represents the simplest form we can find for the metric components in flat spacetime, so equation (5.1) represents the metric components of a nearly flat spacetime in their simplest possible form.

The coordinate system in which one may express the metric components of a nearly flat spacetime in the form of equations (5.1) and (5.2) is certainly not unique. If we have identified such a coordinate system then we can find (an infinite family of) others by carrying out particular coordinate transformations. There are two types of coordinate transformations which preserve the properties of equations (5.1) and (5.2). These are known as **Background Lorentz transformations** and **Gauge transformations**.

5.2.2 Background Lorentz transformations

Suppose we are in the Minkowski spacetime of Special Relativity, and we define the inertial frame, S, with coordinates (t, x, y, z). Suppose we then transform to another inertial frame, S', corresponding to a 'Lorentz boost' of velocity v in the direction of the positive x-axis. Under the Lorentz transformation, S' has coordinates given by, in matrix form

$$(t', x', y', z')^T \;=\; \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (t, x, y, z)^T \tag{5.3}$$

where $\gamma = (1 - v^2)^{-1/2}$. (Remember that we are taking $c = 1$). We can write this in more compact notation as

$$x'^{\alpha} = \Lambda^{\alpha'}_{\beta} x^{\beta} \quad \equiv \quad \frac{\partial x'^{\alpha}}{\partial x^{\beta}} x^{\beta} \tag{5.4}$$

The Lorentz matrix has inverse, corresponding to a boost of velocity v along the *negative* x-axis, given by

$$(t, x, y, z)^T \;=\; \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (t', x', y', z') \tag{5.5}$$

or

$$x^{\alpha} = \Lambda^{\alpha}_{\beta'} x'^{\beta} \quad \equiv \quad \frac{\partial x^{\alpha}}{\partial x'^{\beta}} x'^{\beta} \tag{5.6}$$

Now suppose we are in a nearly flat spacetime in which we have identified nearly Lorentz coordinates (t, x, y, z) satisfying equations (5.1) and (5.2). Suppose we now transform to a new coordinate system (t', x', y', z') defined such that

$$x'^{\alpha} = \Lambda^{\alpha'}_{\beta} x^{\beta} \tag{5.7}$$

i.e. where the transformation matrix is identical in form to equation (5.3) for some constant v. In this new coordinate system the metric components take the form

$$g'_{\alpha\beta} \;=\; \Lambda^{\mu}_{\alpha'} \Lambda^{\nu}_{\beta'} g_{\mu\nu} \;=\; \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} g_{\mu\nu} \tag{5.8}$$

Substituting from equation (5.1) this becomes

$$g'_{\alpha\beta} \;=\; \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}\eta_{\mu\nu} + \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}h_{\mu\nu} \tag{5.9}$$

Because of the particular form of the coordinate transformation in this case, it follows that

$$g'_{\alpha\beta} \;=\; \eta'_{\alpha\beta} + \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}h_{\mu\nu} \;=\; \eta'_{\alpha\beta} + h'_{\alpha\beta} \tag{5.10}$$

Thus, provided we consider **only** transformations of the form of equation (5.3), the components of $h_{\mu\nu}$ transform **as if** they are the components of a $(0,2)$ tensor defined on a Background flat spacetime. Our original 'Nearly Lorentz' coordinate system remains 'Nearly Lorentz' in the new coordinate system. In other words, our 'nearly flat' spacetime still looks 'nearly flat' under the Background Lorentz transformation.

5.2.3 Gauge transformations

Suppose now we make a very small change in our coordinate system by applying a coordinate transformation of the form

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x^{\beta}) \tag{5.11}$$

i.e. where the components ξ^{α} are functions of the coordinates, $\{x^{\alpha}\}$. It then follows that

$$\frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta} + \xi^{\alpha}_{,\beta} \tag{5.12}$$

From equation (5.11) we can also write

$$x^{\alpha} = x'^{\alpha} - \xi^{\alpha}(x^{\beta}) \tag{5.13}$$

If we now demand that the ξ^{α} are small, in the sense that

$$|\xi^{\alpha}{}_{,\beta}| << 1 \quad \text{for all} \quad \alpha, \beta \tag{5.14}$$

then it follows by the chain rule that

$$\frac{\partial x^\alpha}{\partial x'^\gamma} = \delta^\alpha_\gamma - \frac{\partial x^\beta}{\partial x'^\gamma}\frac{\partial \xi^\alpha}{\partial x^\beta} \simeq \delta^\alpha_\gamma - \xi^\alpha_{,\gamma} \tag{5.15}$$

where we have neglected terms higher than first order in small quantities. We have also used the fact that the components of the Kronecker delta are the same in any coordinate system (see Examples sheet I.1, Question 3).

Suppose now that the unprimed coordinate system is nearly Lorentz – i.e. the metric components satisfy equations (5.1) and (5.2). What about the metric components in the primed coordinate system?

Since the metric is a tensor, we know that

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} \tag{5.16}$$

Substituting from equations (5.1) and (5.15) this becomes, to first order

$$g'_{\alpha\beta} = \left(\delta^\mu_\alpha \delta^\nu_\beta - \xi^\mu_{,\alpha}\delta^\nu_\beta - \xi^\nu_{,\beta}\delta^\mu_\alpha\right)\eta_{\mu\nu} + \delta^\mu_\alpha \delta^\nu_\beta h_{\mu\nu} \tag{5.17}$$

This simplifies to

$$g'_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{5.18}$$

Note that in equation (5.18) we have defined

$$\xi_\alpha = \eta_{\alpha\nu}\xi^\nu \tag{5.19}$$

and we have also used the fact that all the partial derivatives of $\eta_{\alpha\nu}$ are zero.

Thus, equation (5.19) has the same form as equation (5.1) provided

$$h'_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{5.20}$$

Note that if $|\xi^\alpha{}_{,\beta}|$ are small, then so too are $|\xi_{\alpha,\beta}|$, and hence $h'_{\alpha\beta}$. Thus, our new primed coordinate system is *still* nearly Lorentz.

The above results tell us that – given that we have identified a coordinate system which is nearly Lorentz – we can add an arbitrary small 'vector' ξ^α to the coordinates, x^α, without altering the validity of our assumption that spacetime is nearly flat. We can, therefore, choose the components ξ^α to make Einstein's equations as simple as possible. We call this step choosing a **gauge** for the problem – a name which has resonance with a similar procedure in electromagnetism – and the coordinate transformation given by equation (5.20) is known as a **gauge transformation**.

5.3 Einstein's equations for a weak gravitational field

If we can work in a nearly Lorentz coordinate system for a nearly flat spacetime this simplifies Einstein's equations considerably, and leads us to spot that the deviations from the metric of Minkowski spacetime – the components $h_{\alpha\beta}$ in equation (5.1) – obey a **wave equation**.

Before we arrive at this key result, however, we have some algebraic work to do first. We begin by deriving an expression for the Riemann-Christoffel tensor in a weak gravitational field.

5.3.1 Riemann-Christoffel tensor for a weak gravitational field

In its fully covariant form the Riemann-Christoffel tensor is given by

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\mu} R^{\mu}_{\beta\gamma\delta} = g_{\alpha\mu} \left[\Gamma^{\sigma}_{\beta\delta} \Gamma^{\mu}_{\sigma\gamma} - \Gamma^{\sigma}_{\beta\gamma} \Gamma^{\mu}_{\sigma\delta} + \Gamma^{\mu}_{\beta\delta,\gamma} - \Gamma^{\mu}_{\beta\gamma,\delta} \right] \tag{5.21}$$

Recall from the previous section that, if we are considering Background Lorentz transformations – i.e. if we restrict our attention only to the class of coordinate transformations which obey equation (5.7) – then the metric perturbations, $h_{\alpha\beta}$, transform as if they are the components of a $(0,2)$ tensor defined on flat, Minkowski spacetime. In this case the Christoffel symbols of the first two bracketed terms on the right hand side of equation (5.21) are equal to zero. It is then easy to show (see Examples sheet II.2) that, to first order in small quantities, the Riemann-Christoffel tensor reduces to

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}\right) \tag{5.22}$$

Moreover, it can also be shown (see Examples sheet II.2) that, to first order, equation (5.22) is invariant under gauge transformations – i.e. the components of the Riemann-Christoffel tensor are independent of the choice of gauge.

5.3.2 Einstein's tensor for a weak gravitational field

From equations (5.21) and (5.22) we can contract the Riemann-Christoffel tensor and thus obtain an expression for the Ricci tensor in linearised form. This can be shown (see Supplementary Notes) to take the form

$$R_{\mu\nu} = \frac{1}{2}\left(h^{\alpha}_{\mu,\nu\alpha} + h^{\alpha}_{\nu,\mu\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu}\right) \tag{5.23}$$

where we have written

$$h \equiv h^{\alpha}_{\alpha} = \eta^{\alpha\beta} h_{\alpha\beta} \tag{5.24}$$

Note that we have raised the indices of the components $h_{\alpha\beta}$ using $\eta^{\alpha\beta}$ since $h_{\alpha\beta}$ behaves like a $(0,2)$ tensor defined on a flat spacetime, for which the metric is $\eta^{\alpha\beta}$. The derivation of equation (5.23) also uses the fact that all partial derivatives of $\eta^{\alpha\nu}$ are zero.

Note also, that we have introduced the notation, generalising the definition of equation

(5.19)

$$f^{\alpha} = \eta^{\alpha\nu} f_{\nu} \tag{5.25}$$

where f^{α} are the components of a 'vector'. We can also extend this notation for raising and lowering indices to the components of more general geometrical objects, and to their partial derivatives. For example, in equation (5.23)

$$h_{\mu\nu,\alpha}{}^{,\alpha} \;=\; \eta^{\alpha\sigma} \left(h_{\mu\nu,\alpha}\right)_{\sigma} \;=\; \eta^{\alpha\sigma}\, h_{\mu\nu,\alpha\sigma} \tag{5.26}$$

After a further contraction of the Ricci tensor, to obtain the curvature scalar, R, where

$$R = \eta^{\alpha\beta} R_{\alpha\beta} \tag{5.27}$$

and substitution into the equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R \tag{5.28}$$

we obtain, after considerable further algebraic manipulation, an expression for the Einstein tensor, $G_{\mu\nu}$, in linearised, fully covariant form

$$G_{\mu\nu} = \frac{1}{2}\left[h_{\mu\alpha,\nu}{}^{,\alpha} + h_{\nu\alpha,\mu}{}^{,\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{,\beta}\right)\right] \tag{5.29}$$

(The details of this derivation are worked through in the Supplementary Notes, and should not be regarded as examinable). This rather messy expression can be simplified a little by introducing a new form (rather like a 're-scaling') for the metric perturbations

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{5.30}$$

after which (see Supplementary notes) equation (5.29) becomes

$$G_{\mu\nu} = -\frac{1}{2}\left[\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} + \eta_{\mu\nu}\overline{h}_{\alpha\beta}{}^{,\alpha\beta} - \overline{h}_{\mu\alpha,\nu}{}^{,\alpha} - \overline{h}_{\nu\alpha,\mu}{}^{,\alpha}\right] \tag{5.31}$$

5.3.3 Einstein's equations for a weak gravitational field

Having ploughed our way through all of the above algebra, we can now write down Einstein's equations in their linearised, fully covariant form for a weak gravitational field, in terms of the (re-scaled) metric perturbations, $\overline{h}_{\mu\nu}$. Since

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{5.32}$$

it follows that

$$-\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} - \eta_{\mu\nu}\overline{h}_{\alpha\beta}{}^{,\alpha\beta} + \overline{h}_{\mu\alpha,\nu}{}^{,\alpha} + \overline{h}_{\nu\alpha,\mu}{}^{,\alpha} = 16\pi T_{\mu\nu} \tag{5.33}$$

It can be shown (see Supplementary Notes) that we can *always* find a gauge transformation which sets the last three terms on the left hand side of equation (5.33) equal to zero. This transformation is the **Lorentz gauge** referred to earlier, and is equivalent to adopting a coordinate system in which

$$\overline{h}^{\mu\alpha}{}_{,\alpha} = 0 \tag{5.34}$$

i.e. the **divergence** of the metric perturbations is equal to zero. Thus, in the Lorentz gauge, the linearised Einstein field equations reduce to the somewhat simpler form

$$-\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} = 16\pi T_{\mu\nu} \tag{5.35}$$

5.3.4 Solution to Einstein's equations in free space

The free space solutions of equation (5.35) are solutions of the equation

$$\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} = 0 \tag{5.36}$$

or, using equation (5.26)

$$\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} \equiv \eta^{\alpha\alpha}\overline{h}_{\mu\nu,\alpha\alpha} \tag{5.37}$$

In fact, when we write out equation (5.37) explicitly, it takes the form

$$\left(-\frac{\partial^2}{\partial t^2} \; + \; \nabla^2 \right) \overline{h}_{\mu\nu} \quad = \quad 0 \tag{5.38}$$

Remembering that we are taking $c = 1$, if instead we write

$$\eta^{00} = -\frac{1}{c^2} \tag{5.39}$$

then equation (5.38) can be re-written as

$$\left(-\frac{\partial^2}{\partial t^2} \; + \; c^2 \, \nabla^2 \right) \overline{h}_{\mu\nu} \quad = \quad 0 \tag{5.40}$$

This is a key result. Equation (5.40) has the mathematical form of a wave equation, propagating with speed c. Thus, we have shown that the metric perturbations – the 'ripples' in spacetime produced by disturbing the metric – propagate at the speed of light as waves in free space.

5.4 Plane wave solutions for the metric perturbations

We now explore a little further the properties of solutions to equation (5.38). The simplest solutions are **plane waves**

$$\overline{h}_{\mu\nu} = \mathrm{Re} \left[A_{\mu\nu} \, \exp \left(i k_\alpha x^\alpha \right) \right] \tag{5.41}$$

where 'Re' denotes the real part, and the constant components $A_{\mu\nu}$ and k_α are known as the **wave amplitude** and **wave vector** respectively. (Note that, as it appears in equation (5.41), the k_α are the components of a one-form. However, since we are considering the weak field limit of a background Minkowski spacetime, converting between covariant and contravariant components is very straightforward).

Equation (5.41) may appear to restrict the metric perturbations to a particular mathematical form, but *any* $\overline{h}_{\mu\nu}$ can be Fourier-expanded as a superposition of plane waves.

The wave amplitude and wave vector components are not completely arbitrary. Firstly, $A_{\mu\nu}$ is symmetric, since $\overline{h}_{\mu\nu}$ is symmetric. This immediately reduces the number of indpendent components from 16 to 10. Next, given that

$$\overline{h}_{\mu\nu,\alpha}{}^{,\alpha} = \eta^{\alpha\sigma}\,\overline{h}_{\mu\nu,\alpha\sigma} = 0 \tag{5.42}$$

it is easy to show (see Examples sheet II.3) that

$$k_\alpha\, k^\alpha = 0 \tag{5.43}$$

i.e. the wave vector is a **null vector**.

Thus, equation (5.41) describes a plane wave of frequency

$$\omega = k^t = \left(k_x^2 + k_y^2 + k_z^2\right)^{1/2} \tag{5.44}$$

propagating in direction $(1/k^t)\,(k_x, k_y, k_z)$.

Also, it follows from the Lorentz gauge condition

$$\overline{h}^{\mu\alpha}{}_{,\alpha} = 0 \tag{5.45}$$

that

$$\left(\overline{h}^{\alpha}_{\mu}\right)_{,\alpha} = 0 \tag{5.46}$$

(see Examples sheet II.3), from which condition it follows that

$$A_{\mu\alpha}\, k^\alpha = 0 \tag{5.47}$$

i.e. the wave amplitude components must be orthogonal to the wave vector, **k**.

Equation (5.47) is, in fact, four linear equations – one for each coordinate. This further reduces the number of independent components of $A_{\mu\nu}$ from 10 to 6. Can we restrict the components of the wave amplitude further still? The answer is 'yes', since we have sufficient freedom in our choice of gauge transformation.

It is shown in the Supplementary Notes that, if we begin with arbitrary metric perturbation components, $h^{(\text{old})}_{\mu\nu}$ (defined on a background Minkowski spacetime), we can transform these components to

$$h^{(\text{new})}_{\mu\nu} \;=\; h^{(\text{old})}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \tag{5.48}$$

defined such that

$$G_{\mu\nu} \;=\; h^{(\text{new}),\alpha}_{\mu\nu,\alpha} \tag{5.49}$$

This transformation to the Lorentz gauge requires identification of 'vector' components ξ^{μ} which satisfy

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) \xi^{\mu} \quad = \quad \overline{h}^{(\text{old})\,\mu\nu}{}_{,\nu} \tag{5.50}$$

Note that the transformation defined by equations (5.48) and (5.50) does **not** determine ξ^{μ} uniquely. To any set of components ξ^{μ} which satisfy equation (5.50), we could add the components, ζ^{μ}, to define a new transformation

$$\xi'^{\mu} = \xi^{\mu} + \zeta^{\mu} \tag{5.51}$$

and provided the ζ^{μ} satisfy

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) \zeta^{\mu} \quad = \quad 0 \tag{5.52}$$

then ξ'^{μ} will still satisfy equation (5.50), and thus still express the Einstein tensor in the simplified, Lorentz gauge form of equation (5.49).

Equation (5.52) gives us four additional equations with which we can adjust the components of our gauge transformation, in order to choose a coordinate system which makes $h_{\mu\nu}$ – and hence $A_{\mu\nu}$ – as simple as possible. We can, in this way, reduce the number of independent components of $A_{\mu\nu}$ to only 2.

In fact, it can be shown (see e.g. Green Schutz, pg. 216) that the freedom we retain in our choice of ξ^μ, while still satisfying the Lorentz gauge conditions, allows us to restrict further $A_{\mu\nu}$ to satisfy

$$A^\mu_\mu = \eta^{\mu\nu} A_{\mu\nu} = 0 \tag{5.53}$$

and

$$A_{\alpha\beta} u^\beta = 0 \tag{5.54}$$

where u^β are the components of some constant unit four-vector. (The usefulness of introducing this restriction on $A_{\mu\nu}$ will become clear shortly). This choice of gauge transformation is known as the **Transverse – Traceless gauge**.

Suppose we now take ourselves to the background Lorentz frame in which u^β has components $(1, 0, 0, 0)$ – i.e.

$$u^\beta = \delta^\beta_t \tag{5.55}$$

Equations (5.54) and (5.55) then imply that

$$A_{\alpha t} = 0 \quad \text{for all } \alpha \tag{5.56}$$

Next we orient our spatial coordinate axes so that the wave is travelling in the positive z-direction, i.e.

$$k^t = \omega\,, \quad k^x = k^y = 0\,, \quad k^z = \omega \tag{5.57}$$

and

$$k_t = -\omega\,, \quad k_x = k_y = 0\,, \quad k_z = \omega \tag{5.58}$$

It then follows from equation (5.47) that

$$A_{\alpha z} = 0 \quad \text{for all } \alpha \tag{5.59}$$

In this coordinate frame, equation (5.41) simplifies to become

$$\overline{h}^{TT}_{\mu\nu} = A^{TT}_{\mu\nu} \cos\left[\omega(t-z)\right] \tag{5.60}$$

It is also straightforward to show (see Examples sheet II.3), using equations (5.24) and (5.30), that $h_{\mu\nu}$ can be written in the form

$$h^{TT}_{\mu\nu} = B^{TT}_{\mu\nu} \cos\left[\omega(t-z)\right] \tag{5.61}$$

where the components $B^{TT}_{\mu\nu}$ are constant.

Equations (5.56) and (5.59), combined with the symmetry of $A_{\mu\nu}$, imply that the only non-zero components of $A_{\mu\nu}$ are A_{xx}, A_{yy} and $A_{xy} = A_{yx}$. Moreover, the traceless condition, equation (5.53), implies that $A_{xx} = -A_{yy}$. Hence, the components of $A_{\mu\nu}$ in the Transverse – Traceless gauge are

$$A^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.62}$$

It then follows from equation (5.62) – see Examples sheet II.3 – that

$$\overline{h}^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \overline{h}^{TT}_{xx} & \overline{h}^{TT}_{xy} & 0 \\ 0 & \overline{h}^{TT}_{xy} & -\overline{h}^{TT}_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.63}$$

and

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^{TT}_{xx} & h^{TT}_{xy} & 0 \\ 0 & h^{TT}_{xy} & -h^{TT}_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.64}$$

5.5 The effect of gravitational waves on free particles

We see from equations (5.62) – (5.64) that the amplitude of the metric perturbation is described by just two independent constants, A_{xx} and A_{xy}. We can understand the physical significance of these constants by examining the effect of the gravitational wave on a free particle, initially in a wave-free region of spacetime.

Choose a background Lorentz frame in which the particle is initially at rest – i.e. the initial four-velocity of the particle is given by equation (5.55) – and choose the Transverse – Traceless Lorentz gauge so that the components $A^{TT}_{\mu\nu}$ in equation (5.41) are given by equation (5.62).

The particle's trajectory satisfies the geodesic equation

$$\frac{du^\beta}{d\tau} + \Gamma^\beta_{\mu\nu} u^\mu u^\nu = 0 \tag{5.65}$$

where τ is the proper time. Thus, the initial acceleration of the particle is

$$\left(\frac{du^\beta}{d\tau}\right)_0 = -\Gamma^\beta_{tt} = -\frac{1}{2}\eta^{\alpha\beta}\left(h_{\alpha t,t} + h_{t\alpha,t} - h_{tt,\alpha}\right) \tag{5.66}$$

However, from equation (5.56)

$$A_{\alpha t} = 0 \quad \Rightarrow \quad \overline{h}_{\alpha t} = 0 \tag{5.67}$$

Also, from equation (5.53)

$$A^\mu_\mu = 0 \quad \Rightarrow \quad \overline{h} = \overline{h}^\mu_\mu = 0 \tag{5.68}$$

Together, equations (5.67) and (5.68) imply that

$$h_{\alpha t} = 0 \quad \text{for all } \alpha \tag{5.69}$$

from which it follows immediately that

$$\left(\frac{du^{\beta}}{d\tau} \right)_0 = 0 \tag{5.70}$$

Hence a free particle, initially at rest, will remain at rest indefinitely. However, 'being at rest' in this context simply means that the *coordinates* of the particle do *not* change. This is simply a consequence of our judicious choice of coordinate system, via the adoption of the Transverse – Traceless Lorentz gauge. As the gravitational wave passes, the coordinate system adjusts itself to the ripples in the spacetime, so that any particles remain 'attached' to their initial coordinate positions. Coordinates are merely frame-dependent labels, however, and do not directly convey any invariant geometrical information about the spacetime.

Suppose instead we consider the proper distance between two nearby particles, both initially at rest, in this coordinate system: one at the origin and the other at spatial coordinates $x = \epsilon$, $y = z = 0$. The proper distance between the particles is then given by

$$\Delta\ell = \int \left| g_{\alpha\beta} dx^{\alpha} dx^{\beta} \right|^{1/2} \tag{5.71}$$

i.e.

$$\Delta\ell = \int_0^{\epsilon} \left| g_{xx} \right|^{1/2} \quad \simeq \quad \sqrt{g_{xx}(x=0)}\, \epsilon \tag{5.72}$$

Now

$$g_{xx}(x=0) = \eta_{xx} + h_{xx}^{TT}(x=0) \tag{5.73}$$

so

$$\Delta\ell \simeq \left[1 + \frac{1}{2}h_{xx}^{TT}(x=0)\right]\epsilon \tag{5.74}$$

Since $h_{xx}^{TT}(x=0)$ is general is **not** constant, it follows that the proper distance between the particles will change as the gravitational wave passes. It is this change in the proper distance between 'test' particles which gravitational wave detectors attempt to measure.

5.5.1 The geodesic deviation of test particles

We can study the behaviour of test particles more formally using the idea of **geodesic deviation**, first introduced in GRG-I. Let us define the vector ξ^α which connects the two particles introduced above. Then, for a weak gravitational field, equation (3.45) of Norman's GRG-I notes becomes

$$\frac{\partial^2 \xi^\alpha}{\partial t^2} = R^\alpha_{\mu\nu\beta} u^\mu u^\nu \xi^\beta \tag{5.75}$$

where u^μ are the components of the four-velocity of the two particles. Since the particles are initially at rest, then

$$u^\mu \quad = \quad (1,0,0,0) \tag{5.76}$$

and

$$\xi^\beta \quad = \quad (0,\epsilon,0,0) \tag{5.77}$$

Equation (5.75) then simplifies to

$$\frac{\partial^2 \xi^\alpha}{\partial t^2} = \epsilon R^\alpha_{ttx} = -\epsilon R^\alpha_{txt} \tag{5.78}$$

Substituting from equation (5.24) for a weak gravitational field, we can write down the relevant components of the Riemann-Christoffel tensor in terms of the non-zero components of the metric perturbation

$$R^x_{txt} = \eta^{xx} R_{xtxt} = -\frac{1}{2} h^{TT}_{xx,tt} \tag{5.79}$$

$$R^y_{txt} = \eta^{yy} R_{ytxt} = -\frac{1}{2} h^{TT}_{xy,tt} \tag{5.80}$$

Hence, two particles initially separated by ϵ in the x-direction, have a geodesic deviation vector which obeys the differential equations

$$\frac{\partial^2}{\partial t^2}\,\xi^x \;=\; \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2} h^{TT}_{xx} \tag{5.81}$$

and

$$\frac{\partial^2}{\partial t^2}\,\xi^y \;=\; \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2} h^{TT}_{xy} \tag{5.82}$$

Similarly, it is straightforward to show (see Examples sheet II.4) that two particles initially separated by ϵ in the y-direction, have a geodesic deviation vector which obeys the differential equations

$$\frac{\partial^2}{\partial t^2}\,\xi^x \;=\; \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2} h^{TT}_{xy} \tag{5.83}$$

and

$$\frac{\partial^2}{\partial t^2}\,\xi^y \;=\; -\frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2} h^{TT}_{xx} \tag{5.84}$$

5.5.2 Ring of test particles: polarisation of gravitational waves

We can further generalise equations (5.81) – (5.84) to consider the geodesic deviation of two particles – one at the origin and the other initially at coordinates $x = \epsilon\cos\theta$, $y = \epsilon\sin\theta$ and $z = 0$ – as a gravitational wave propagates in the z-direction. We can show (see Examples sheet II.4) that ξ^x and ξ^y obey the differential equations

$$\frac{\partial^2}{\partial t^2}\,\xi^x \;=\; \frac{1}{2}\epsilon\cos\theta\frac{\partial^2}{\partial t^2} h^{TT}_{xx} \;+\; \frac{1}{2}\epsilon\sin\theta\frac{\partial^2}{\partial t^2} h^{TT}_{xy} \tag{5.85}$$

and

$$\frac{\partial^2}{\partial t^2}\,\xi^y \;=\; \frac{1}{2}\epsilon\cos\theta\frac{\partial^2}{\partial t^2}h^{TT}_{xy} \;-\; \frac{1}{2}\epsilon\sin\theta\frac{\partial^2}{\partial t^2}h^{TT}_{xx} \tag{5.86}$$

Substituting from equations (5.61) and (5.64), we see that equations (5.85) and (5.86) have solution

$$\xi^x \;=\; \epsilon\cos\theta + \frac{1}{2}\epsilon\cos\theta\,B^{TT}_{xx}\cos\omega t \;+\; \frac{1}{2}\epsilon\sin\theta\,B^{TT}_{xy}\cos\omega t \tag{5.87}$$

and

$$\xi^y \;=\; \epsilon\sin\theta + \frac{1}{2}\epsilon\cos\theta\,B^{TT}_{xy}\cos\omega t \;-\; \frac{1}{2}\epsilon\sin\theta\,B^{TT}_{xx}\cos\omega t \tag{5.88}$$

Suppose we now vary θ between 0 and 2π, so that we are considering a circular ring of test particles, initially equidistant from the origin. Figure 10 (from Green Schutz) shows the effect of the passage of a plane gravitational wave on this ring of test particles. Panel (a) shows the particles before the wave reaches them.

Panel (b) shows the distortions produced by a wave for which $h^{TT}_{xx} \neq 0$ and $h^{TT}_{xy} = 0$. The upper and lower figures show two phases of the wave separated by π, at $t = t_1$ and $t = t_2$. When $t = t_1$, test particles on the x-axis are moved inwards; the gravitational wave reduces their proper distance from the origin. Particles on the y-axis, on the other hand, are moved outwards; the gravitational wave increases their distance from the origin. At time t_2 the behaviour is reversed as the oscillating component, h^{TT}_{xx}, has changed sign. Now particles on the x-axis are moving outwards in proper distance from the origin while particles on the y-axis are moving inwards.

Panel (c) shows the distortions produced by a wave for which $h^{TT}_{xx} = 0$ and $h^{TT}_{xy} \neq 0$. The upper and lower figures again show two phases of the wave separated by π, at $t = t_1$ and $t = t_2$. The behaviour of the ring of test particles in this case is the same as in panel (b), but simply rotated at $45°$ relative to that case.

Panels (b) and (c) of Figure 10 represent two distinct **polarisation states** for the gravitational wave; these states are rotated $45°$ with respect to one another. Contrast this with the two polarisation states of an electromagnetic wave, which are rotated by $90°$ relative to one another.

We can also see from Figure 10 that, at any instant, a gravitational wave is invariant under a rotation of $180°$ about its direction of propagation (in this case, the z-axis). By contrast, an electromagnetic wave is invariant under a rotation of $360°$, and a *neutrino* wave is invariant under a rotation of $720°$. We can understand this behaviour in terms of the *spin states* of the corresponding **gauge bosons**: the particles which are associated with the quantum mechanical versions of these waves.

In general, the classical radiation field of a particle of spin, S, is invariant under a rotation of $360°/S$. Moreover, a radiation field of spin S has precisely two independent polarisation states, which are inclined to each other at an angle of $90°/S$. Thus, for an electromagnetic wave, corresponding to a photon of spin $S = 1$, the independent polarisation modes are inclined at $90°$ to each other.

We can, therefore, deduce from the inclination of the gravitational wave polarisation states, that the **graviton** (which is, as yet undiscovered, since we do not yet have a fully developed theory of quantum gravity!) must be a spin $S = 2$ particle. The fact that electromagnetic waves correspond to a spin $S = 1$ field and gravitational waves correspond to a spin $S = 2$ field is also intimately connected to their mathematical description in terms of geometrical objects: spin $S = 1$ fields are **vector fields**, which is why we require only a vector description for the electromagnetic field; spin $S = 2$ fields, on the other hand, are **tensor fields**, which is why we required to introduce tensors to describe the properties of the gravitational field.

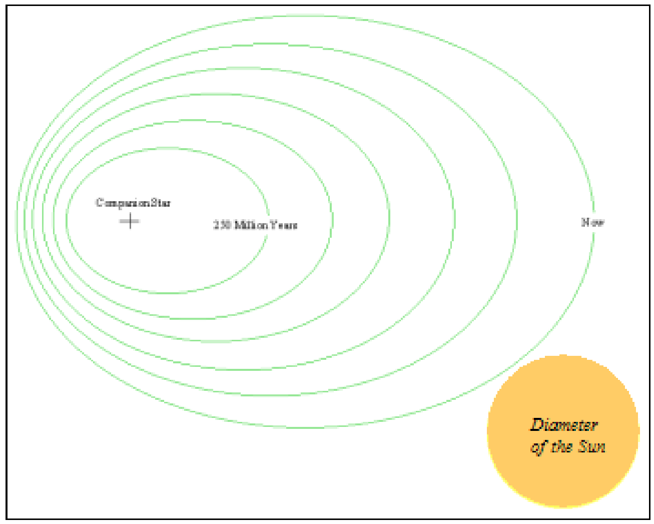

Figure 9

Change in the semi-major axis of the orbit of the binary pulsar, PSR 1913+16, resulting from energy loss due to gravitational radiation.

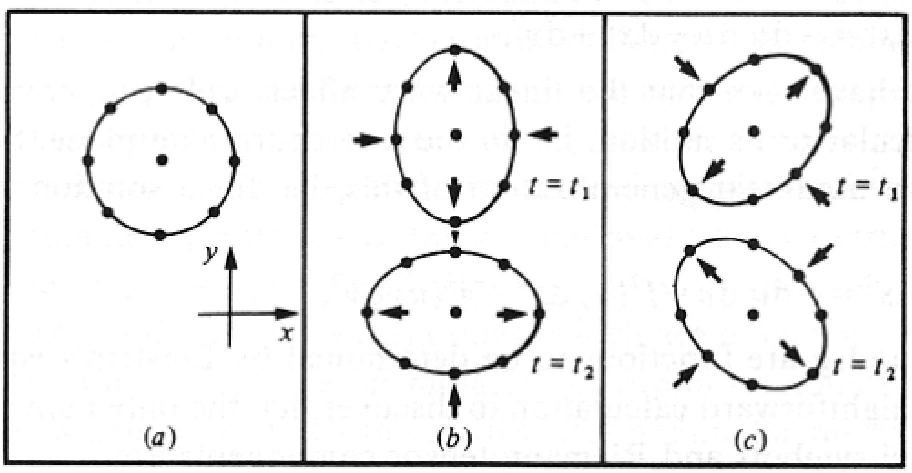

Figure 10

The two 'polarisation modes' of plane gravitational waves. See notes for an explanation of the different panels.

5.6 The amplitude of gravitational waves

How strong is gravitational radiation? In the previous sections we have developed a theoretical understanding of plane gravitational waves within the framework of a 'weak' gravitational field. This would tend to suggest that the effects of gravitational radiation should also be weak, but how can we quantify what we mean by 'weak' in this context?

A proper, quantitative treatment of this question lies well beyond the scope of this course, as it would require us to model in detail the astrophysical processes which produce gravitational waves. We will only address one particular aspect of this topic: in the next section we will show that gravitational waves are *quadrupolar* in nature.

Nevertheless, regardless of exactly *how* gravitational waves are produced, a very simplistic argument shows us that – even for 'strong' gravitational metric perturbations produced by by a distant source – one would expect the amplitude of gravitational waves incident at the Earth to be very small. Essentially, the argument requires us only to invoke a very general property of wave phenomena: that the amplitude of the wave is inversely proportional to the distance from the source.

A 'strong' metric perturbation would have $|h_{\alpha\beta}| \simeq 1$, in equation (5.1); we could expect perturbations of this amplitude only very close to the source, where the Newtonian potential (in geometrised units – see e.g. equation 3.89) is of order unity. For a source of mass, M (again, in geometrised units), this would occur at distances of order M from the source.

Hence, even if $|h_{\mu\nu}| \sim 1$ when $r \sim M$, it follows that, at distance, R, from a source of metric perturbations

$$|h_{\mu\nu}| \sim \frac{M}{R} \tag{5.89}$$

Consider, for example, the metric perturbations generated by the formation of a 8 solar mass black hole in e.g. the Andromeda galaxy – at a distance of $R \simeq 2 \times 10^{22}$ metres. The mass of the black hole, expressed in metres, is

$$M = 1.6 \times 10^{31}\,\mathrm{kg} \quad \simeq \quad 1.2 \times 10^{4}\,\mathrm{m} \tag{5.90}$$

Hence, even if the black hole produced strong metric perturbations, with $|h_{\mu\nu}| \sim 1$ at distances of order M, upon reaching the Earth these metric perturbations would have reduced in amplitude to

$$h_{\mu\nu} \sim 6 \times 10^{-19} \tag{5.91}$$

This number would also be of order the size of the fractional change in proper distance experienced by our ring of test particles as the metric perturbation passes. Moreover, this very simplistic calculation is a gross *over*-estimate, since it was based on assuming that the metric pertubations close to the source were of order unity. More realistic calculations – including detailed physical modelling (e.g. core collapse of a massive star, or coalescence of binary neutron stars) of the process which produces the metric perturbations – will lead to very much smaller amplitude perturbations close to the source, and hence correspondingly smaller perturbations at the Earth.

Thus, we see that the detection of gravitational waves from even relatively nearby (in cosmological terms) sources presents enormous technological challenges.

5.7 The quadrupolar nature of gravitational waves

We can understand something important about the nature of gravitational radiation by drawing analogies with the formulae that describe electromagnetic radiation. This approach only provides a crude, order-of-magnitude, estimate of the power of gravitational radaition, since the electromagnetic field is a vector field while the gravitational field is a tensor field, but it is good enough for our purposes in this course. Essentially, we will take familiar electromagnetic radiation formulae and simply replace the terms which involve the Coulomb force by their gravitational analogues from Newtonian theory.

5.7.1 Electric and magnetic dipoles

In electromagnetic theory, the dominant form of radiation from a moving charge or charges is **electric dipole radiation**. For a single particle (e.g. an electron) of charge, e, with acceleration, $\mathbf{a}$, and dipole moment changing as $\ddot{\mathbf{d}} = e\,\ddot{\mathbf{x}} = e\,\mathbf{a}$, the power output, or luminosity, is given by

$$L_{\text{electric dipole}} \propto e^2\,\mathbf{a}^2 \tag{5.92}$$

For a general distribution of charges, with net dipole moment, $\mathbf{d}$, the luminosity is

$$L_{\text{electric dipole}} \propto e^2\,\ddot{\mathbf{d}}^2 \tag{5.93}$$

The next strongest types of electromagnetic radiation are **magnetic dipole** and **electric quadrupole radiation**. For a general distribution of charges, the luminosity arising from magnetic dipole radiation is proportional to the second time derivative of the magnetic dipole moment, i.e.

$$L_{\text{magnetic dipole}} \propto \ddot{\mu} \tag{5.94}$$

where μ is given by a sum (or integral) over a distribution of charges:-

$$\mu \;=\; \sum_{q_i} \,(\text{position of } q_i) \times (\text{current due to } q_i) \tag{5.95}$$

5.7.2 Gravitational analogues

The gravitational analogue of the electric dipole moment is the **mass dipole moment**, $\mathbf{d}$, summed over a distribution of particles, $\{A_i\}$

$$\mathbf{d} \;=\; \sum_{A_i} m_i \mathbf{x}_i \tag{5.96}$$

where m_i is the rest mass and $\mathbf{x}_i$ is the position of particle A_i.

By analogy with equation (5.94), the luminosity of gravitational 'mass dipole' radiation should be proportional to the second time derivative of $\mathbf{d}$. However, the *first* time derivative of $\mathbf{d}$ is

$$\dot{\mathbf{d}} \;=\; \sum_{A_i} m_i \dot{\mathbf{x}}_i \quad \equiv \quad \mathbf{p} \tag{5.97}$$

where p is the **total linear momentum** of the system. Since the total momentum is conserved, it then must follow that the gravitational 'mass dipole' luminosity is zero – i.e **there can be no mass dipole radiation from any source**.

Similarly, the gravitational analogue of the magnetic dipole moment is

$$\mu \;=\; \sum_{A_i} \,(\mathbf{x}_i) \times (m_i \mathbf{v}_i) \quad \equiv \quad \mathbf{J} \tag{5.98}$$

where $\mathbf{J}$ is the **total angular momentum** of the system. Since the total angular momentum is *also* conserved, again it follows that the gravitational analogue of magnetic dipole radiation must have zero luminosity. Hence **there can be no dipole radiation of any sort from a gravitational source**.

Thus, the simplest form of gravitational radiation which has non-zero luminosity is **quadrupolar** in nature. We do not consider the mathematical details of quadrupolar radiation here, save to point out that it can be shown that the quadrupole from a **spherically symmetric mass distribution** is identically zero. This suggests an important result: that, at least up to quadrupole order, **metric perturbations which are spherically symmetric do not produce gravitational radiation**. Thus, if e.g. the collapse of a massive star is spherically symmetric, it will generate *no* gravitational waves.

In fact, it is possible to prove that this result is *also* true for higher order radiation (e.g. octupole etc.), although the proof is very technical and is not discussed further. Interested readers are referred to Chapters 9 and 10 of Green Schutz.

5.8 Example: a binary neutron star system

To close this chapter we consider the example of the gravitational wave signature of a particular (and rather extreme!) astrophysical system: a binary neutron star.

In general it can be shown (see, e.g. Green Schutz) that in the so-called *slow motion approximation* for a weak metric perturbation $h_{\mu\nu} << 1$ then for a source at distance r

$$h_{\mu\nu} = \frac{2G}{c^4 r} \ddot{I}_{\mu\nu} \tag{5.99}$$

where $I_{\mu\nu}$ is the **reduced quadrupole moment** defined as

$$I_{\mu\nu} = \int \rho(\vec{r}) \left(x_\mu x_\nu - \frac{1}{3}\delta_{\mu\nu} r^2 \right) dV \tag{5.100}$$

Consider a binary neutron star system consisting of two stars both of Schwarzschild mass M, in a circular orbit of coordinate radius R and orbital frequency f. For simplicity we define our coordinate system so that the orbital plane of the pulsars lies in the $x - y$ plane, and at coordinate time $t = 0$ the two pulsars lie along the $x-$axis. Substituting into equation (5.100)[1] it is then straightforward to show that

$$I_{xx} = 2MR^2 \left[\cos^2(2\pi f t) - \frac{1}{3} \right] \tag{5.101}$$

$$I_{yy} = 2MR^2 \left[\sin^2(2\pi f t) - \frac{1}{3} \right] \tag{5.102}$$

$$I_{xy} = I_{yx} = 2MR^2 \left[\cos(2\pi f t) \, \sin(2\pi f t) \right] \tag{5.103}$$

From equations (5.99) and (5.101) – (5.103) it then follows that (see examples sheet 3)

$$h_{xx} = -h_{yy} = h \cos\left(4\pi f t\right) \tag{5.104}$$

and

$$h_{xy} = h_{yx} = -h \sin\left(4\pi f t\right) \tag{5.105}$$

where the amplitude term h is given by

$$h = \frac{32\pi^2 G M R^2 f^2}{c^4 r} \tag{5.106}$$

We see from equation (5.106) that the binary system emits gravitational waves at twice the orbital frequency of the neutron stars. It is easy to verify that h in this equation is a dimensionless quantity; it is known as the **dimensionless strain**, and

[1] taking the mass density distribution to be a sum of dirac delta functions – i.e. treating the pulsars as point masses

is related directly to the fractional change in arm length of e.g. a laser interferometric gravitational wave detector.

How large is h for a typical source? Suppose we take M equal to the Chandrasekhar mass, $M \sim 1.4 M_{\mathrm{solar}} = 2.78 \times 10^{30}$kg. We can then evaluate the constants in equation (5.106) and express h in more convenient units as

$$h = 2.3 \times 10^{-28} \, \frac{R^2[\mathrm{km}] f^2[\mathrm{Hz}]}{r[\mathrm{Mpc}]} \tag{5.107}$$

If we take $R = 20$km, say, $f = 1000$Hz (which is approx. the frequency that Newtonian gravity would predict) and $r = 15$Mpc (corresponding to a binary system in e.g. the Virgo cluster), then we find that $h \sim 6 \times 10^{-21}$. Thus we see – just as we found in Section 5.6 from a more simplistic estimation – that the strain produced by a typical gravitational wave source places extreme demands upon detector technology.