Chapter 4

Einstein's Equations for Static Spherically Symmetric Stars

In Chapter 3 we derived the Schwarzschild metric to describe the spacetime exterior to a star of mass, M by setting the components of the Ricci tensor identically equal to zero. We now turn our attention to the *interior* of a star, and derive within the framework of General Relativity differential equations to describe its structure.

In Newtonian theory one can derive the equation of **hydrostatic equilibrium** to describe the internal structure of a static, spherically symmetric star. We will, therefore, seek a GR solution which is also static and spherically symmetric, and will investigate how GR effects change this internal structure.

4.1 Components of the Einstein tensor

We begin by noting that the metric describing the stellar interior will again take the form of equation (2.23), namely:-

$$ds^2 = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{4.1}$$

where ν and λ are functions of the radial coordinate, r. This means that we can make use of the results which we already derived in Chapter 2 for the components of the Ricci tensor for a static, spherically symmetric metric. These were

$$R_{tt} = \frac{1}{2}e^{\nu-\lambda}\left(\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\nu'\lambda' + \frac{2}{r}\nu'\right) \tag{4.2}$$

$$R_{rr} = -\frac{1}{2}\left(\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\nu'\lambda' - \frac{2}{r}\lambda'\right) \tag{4.3}$$

$$R_{\theta\theta} = 1 - e^{-\lambda}\left[1 + \frac{r}{2}\left(\nu' - \lambda'\right)\right] \tag{4.4}$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta \tag{4.5}$$

and all other terms of the Ricci tensor are identically zero.

Moreover, since the metric of equation (4.1) is orthogonal, it follows that the contravariant components of the metric are

$$g^{tt} = -e^{-\nu} \tag{4.6}$$

$$g^{rr} = e^{-\lambda} \tag{4.7}$$

$$g^{\theta\theta} = \frac{1}{r^2} \tag{4.8}$$

$$g^{\phi\phi} = \frac{1}{r^2\sin^2\theta} \tag{4.9}$$

and all other components are zero.

Due to the orthogonality of the Ricci and metric tensors, the **curvature scalar**, R, is given by

$$R = g^{\mu\nu} R_{\mu\nu} = g^{tt} R_{tt} + g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} \tag{4.10}$$

Substituting from equations (4.2) – (4.9), and after some algebraic reduction, this gives

$$R = -e^{-\lambda} \left[\left(\nu'' + \frac{1}{2} \upsilon'^2 - \frac{1}{2} \nu' \lambda' \right) + \frac{\nu' - \lambda'}{r} \right] + \frac{2}{r^2} \left[1 - e^{-\lambda} \left(1 + \frac{(\nu' - \lambda') r}{2} \right) \right] \tag{4.11}$$

The Einstein tensor, in its fully covariant form, is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{4.12}$$

Substituting from equations (4.1) – (4.5) and equation (4.11) one finds that

$$G_{tt} = \frac{e^{\nu}}{r^2} \left[1 + e^{-\lambda} \left(r\lambda' - 1 \right) \right] \tag{4.13}$$

$$G_{rr} = \frac{\nu'}{r} - \frac{e^{\lambda}}{r^2} \left(1 - e^{-\lambda} \right) \tag{4.14}$$

$$G_{\theta\theta} = r^2 e^{-\lambda} \left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu' \lambda'}{4} + \frac{\nu' - \lambda'}{2r} \right] \tag{4.15}$$

$$G_{\phi\phi} = \sin^2\theta \, G_{\theta\theta} \tag{4.16}$$

and all other components are zero.

4.2 Components of the energy-momentum tensor

Recall from equation (1.32) that, for a perfect fluid, the components of the energy-momentum tensor in its fully contravariant form are given by

$$T^{\mu\nu} = (\rho + P) u^{\mu} u^{\nu} + P g^{\mu\nu} \tag{4.17}$$

and in fully covariant form

$$T_{\mu\nu} = (\rho + P) u_{\mu} u_{\nu} + P g_{\mu\nu} \tag{4.18}$$

where ρ and P are the mass-energy density and pressure respectively, and u^μ and u_μ are the contravariant and covariant components respectively of the four velocity of a fluid element. Since we are seeking a static solution, we have that $u^r = u^\theta = u^\phi = 0$ and $u_r = u_\theta = u_\phi = 0$. From equation (1.28), it then follows that

$$g_{tt}\left(u^t\right)^2 = -1 \;\Rightarrow\; u^t = e^{-\nu/2} \tag{4.19}$$

and

$$u_t = g_{t\mu}u^\mu = g_{tt}u^t = -e^{\nu/2} \tag{4.20}$$

Substituting in equations (4.18) then yields the following non-zero components of the energy-momentum tensor in its fully covariant form

$$T_{tt} = \rho e^\nu \tag{4.21}$$

$$T_{rr} = P e^\lambda \tag{4.22}$$

$$T_{\theta\theta} = P r^2 \tag{4.23}$$

$$T_{\phi\phi} = P r^2 \sin^2\theta \tag{4.24}$$

and all other components are zero.

4.3 Einstein's equations

Recall from GRG-I that Einstein's equations have solution, in fully covariant form

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{4.25}$$

where the constant 8π is derived by requiring that we recover the laws of Newtonian gravity (with $G = 1$) in the limit of a weak gravitational field and non-relativistic motion.

Thus, it follows that

$$\frac{e^{\nu}}{r^2}\left[1+e^{-\lambda}\left(r\lambda'-1\right)\right]=8\pi\rho e^{\nu} \tag{4.26}$$

$$\frac{\nu'}{r}-\frac{e^{\lambda}}{r^2}\left(1-e^{-\lambda}\right)=8\pi P e^{\lambda} \tag{4.27}$$

$$r^2 e^{-\lambda}\left[\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'\lambda'}{4}+\frac{\nu'-\lambda'}{2r}\right]=8\pi P r^2 \tag{4.28}$$

The '$\phi\phi$' Einstein equation is indentically equal to the '$\theta\theta$' equation, multiplied by $\sin^2\theta$.

4.4 Solution of the first Einstein equation

Cancelling e^{ν} from either side of equation (4.26) and re-arranging, it is easy to show that this equation may be re-written as

$$\frac{d}{dr}\left[r(1-e^{-\lambda})\right]=8\pi\rho r^2 \tag{4.29}$$

To solve this equation it is convenient to introduce a new function, $m(r)$, defined by the differential equation

$$\frac{dm}{dr}=4\pi\rho r^2 \tag{4.30}$$

from which it follows that

$$\frac{d}{dr}\left[r(1-e^{-\lambda})\right]=2\frac{dm}{dr} \tag{4.31}$$

Equation (4.30) has a Newtonian analogue: dm is equal to the mass contained within a thin shell of radius r and thickness dr. Hence, in GR $m(r)$ is usually referred to as the **mass function**, although its simple Newtonian interpretation no longer holds in GR, since one cannot describe the mass-energy in a frame-independent way solely in terms of a single scalar function. Nevertheless, the introduction of the mass function leads to some useful simplification.

Integrating equations (4.31)

$$r(1 - e^{-\lambda}) = 2m + C \tag{4.32}$$

where C is a constant, which is equal to zero unless the star is singular at $r = 0$. We will consider such a case later in the chapter on Black Holes. Thus

$$e^{-\lambda} = 1 - \frac{2m}{r} \tag{4.33}$$

This is, of course, reminiscent of the equation (2.38) which we derived for the Schwarzschild metric exterior to the star. In that case, however, we identified the constant, M, as the total mass of the star, from considering the Newtonian far-field limit in which the Schwarzschild metric reduces to that of Minkowski spacetime. In the case of the interior metric, we must remember that m is *not* a constant but an (as yet unspecified) function of the coordinate radius, r.

4.5 The Oppenheimer-Volkoff equation

Re-arranging equation (4.27) we obtain, after some straightforward algebra

$$\frac{d\nu}{dr} = e^{\lambda}\left[8\pi Pr + \frac{1}{r}\left(1 - e^{-\lambda}\right)\right] \tag{4.34}$$

Substituting from equation (4.33) this reduces to

$$\frac{d\nu}{dr} = \left(1 - \frac{2m}{r}\right)^{-1}\left[8\pi Pr + \frac{2m}{r^2}\right] = 2\left[\frac{4\pi Pr^3 + m}{r(r - 2m)}\right] \tag{4.35}$$

Rather than attempting to re-arrange the third Einstein equation (4.28), which is rather messy in form, we can use the conservation of mass-energy to obtain a differential equation for the function ν.

Recall from GRG-I that

$$T^{\alpha\beta}{}_{;\beta} = 0 \tag{4.36}$$

i.e.

$$\Big[(\rho + P)u^\alpha u^\beta + P g^{\alpha\beta}\Big]_{;\beta} = 0 \tag{4.37}$$

We apply the product rule for covariant differentation (see Example Sheet I.5, question 4) to obtain

$$(\rho + P)_{,\beta}\, u^\alpha u^\beta + (\rho + P)(u^\alpha)_{;\beta} u^\beta + (\rho + P)u^\alpha (u^\beta)_{;\beta} + P_{,\beta} g^{\alpha\beta} + P g^{\alpha\beta}{}_{;\beta} = 0 \tag{4.38}$$

Note that here we have used the fact that ρ and P are scalar functions, and the covariant derivative of a scalar is simply its usual partial derivative. Moreover, in this case partial derivatives with respect to r are, in fact, total derivatives. This is because ρ and P depend only on r, as we are considering a static, spherically symmetric solution.

Equation (4.38) is, in fact, four equations, since α is a free index. Let us consider only the $\alpha \equiv r$ term, so that

$$(\rho + P)_{,\beta}\, u^r u^\beta + (\rho + P)(u^r)_{;\beta} u^\beta + (\rho + P)u^r (u^\beta)_{;\beta} + P_{,\beta} g^{r\beta} + P g^{r\beta}{}_{;\beta} = 0 \tag{4.39}$$

The first and third terms on the left hand side of equation (4.39) vanish because $u^r = 0$ for a static solution. The fifth term also vanishes because the covariant derivatives of the metric tensor are always identically zero (see Example Sheet I.5, questions 5 and 6). In the second term only the t component is non-zero because $u^r = u^\theta = u^\phi = 0$. (Note that, even though $u^r = 0$ it does not necessarily follow – and indeed we will see later is *not* the case here – that its covariant derivative is zero). Finally, in the fourth term only the r component is non-zero, because of the orthogonality of the metric.

Hence, equation (4.39) simplifies to

$$(\rho + P)(u^r)_{;t}u^t + \frac{dP}{dr}g^{rr} = 0 \tag{4.40}$$

The covariant derivative of u^r is given by

$$u^r_{;t} = u^r_{,t} + \Gamma^r_{tj}u^j \tag{4.41}$$

The first term is zero and only the $j \equiv t$ contravariant component of the four velocity is non-zero, so that

$$u^r_{;t} = \Gamma^r_{tt}u^t = \frac{1}{2}\nu' e^{\nu-\lambda}e^{-\nu/2} = \frac{1}{2}e^{-\lambda}\nu' e^{\nu/2} \tag{4.42}$$

Substituting in equation (4.40) finally yields the differential equation

$$\frac{1}{2}(\rho + P)e^{-\lambda}\frac{d\nu}{dr} + e^{-\lambda}\frac{dP}{dr} = 0 \tag{4.43}$$

or

$$\frac{d\nu}{dr} = -\frac{2}{(\rho + P)}\frac{dP}{dr} \tag{4.44}$$

We can use equation (4.44) to eliminate ν' from equation (4.35), giving

$$\frac{dP}{dr} = -\frac{(\rho + P)(4\pi P r^3 + m)}{r(r - 2m)} \tag{4.45}$$

Equation (4.45) is known as the **Oppenheimer-Volkoff equation** and is the General Relativistic equivalent of the Newtonian equation of hydrostatic equilibrium. We can see this by considering the weak-field limit of the Oppenheimer-Volkoff equation, taking $P << \rho$, which in turn implies that $4\pi P r^3 << m$. Moreover, since the weak-field metric must be nearly flat, it follows that $m << r$, and equation (4.45) simplifies to

$$\frac{dP}{dr} = -\frac{\rho m}{r^2} \tag{4.46}$$

which is identical in form to the Newtonian hydrostatic equilibrium equation.

4.6 Solving the Oppenheimer-Volkoff equation

The Oppenheimer-Volkoff equation involves three unknown functions: $P(r)$, $\rho(r)$ and $m(r)$. We also have equation (4.30), which relates $m(r)$ and $\rho(r)$. To solve for the internal structure of the star requires a third equation linking the three functions, however. This usually comes from the **Equation of state**, which is a relation between the pressure and density, i.e.

$$P(r) = P(\rho(r)) \tag{4.47}$$

For a fluid in local thermodynamic equilibrium, there always exists a relation between the pressure, density and *entropy*, S, such that

$$P = P(\rho, S) \tag{4.48}$$

Equation (4.47) is the particular case where the entropy can be considered constant. This case is a valid approximation for most astrophysical situations (and, indeed, the adoption of an equation of state is the usual method for solving the non-relativistic equations of stellar structure).

How would these three equations for P, ρ and m be solved in practice? Equations (4.30) and (4.45) are first order differential equations; hence we must adopt boundary conditions in order to solve them.

Suppose we take $P = P_0$ and $m = 0$ (i.e. no singularity) at the centre of the star. We can then integrate the equations outwards, taking small increments in radial coordinate distance, r, until we reach $P = 0$, which defines the surface of the star – at, say, $r = R$ and $m = M$. We then identify M as the constant in the Schwarzschild metric for the spacetime exterior to the star; this ensures that the metric coefficients are *continuous* across the surface of the star.

Having thus integrated to obtain m, P and ρ as functions of r, we can use e.g. equations (4.33) and (4.44) to obtain $\nu(r)$ and $\lambda(r)$, which specifies completely the form of the interior metric.

We will consider the exact solution of the Oppenheimer-Volkoff equation for a particular case in the next section. Before we do so, however, we can see from equation (4.45) that – for normal matter with ρ, P and $m \geq 0$, the effect of GR must always be to *steepen* the pressure gradient inside the star, compared with the pressure gradient required to maintain a static solution in a purely Newtonian treatment.

4.7 Exact solution for a constant density star

Suppose we assume that $\rho = \rho_0 =$ constant. This assumption is certainly not appropriate for a star like the Sun (since, for one thing, it implies an infinite sound speed, $c_S \propto dP/d\rho$) but theoretical modelling of the internal structure of neutron stars suggests that it is a reasonably good approximation in their case.

Under this assumption we can integrate equation (4.30) immediately to give

$$m(r) = \frac{4}{3}\pi\rho_0 r^3 \tag{4.49}$$

Substituting into the Oppenheimer-Volkoff equation, it is easy to see that this gives us

$$\frac{dP}{dr} = -\frac{4}{3}\pi r \frac{(\rho_0 + P)(\rho_0 + 3P)}{\left(1 - \frac{8\pi\rho_0 r^2}{3}\right)} \tag{4.50}$$

which we can re-arrange as

$$\frac{dP}{(\rho_0 + P)(\rho_0 + 3P)} = -\frac{4\pi r dr}{3\left(1 - \frac{8\pi\rho_0 r^2}{3}\right)} \tag{4.51}$$

We can simplify the left hand side using the method of partial fractions, which leads to

$$\frac{1}{2\rho_0}\left[\frac{3dP}{(\rho_0+3P)}-\frac{dP}{(\rho_0+P)}\right]=-\frac{4\pi}{3}\frac{rdr}{\left(1-\frac{8\pi\rho_0 r^2}{3}\right)} \tag{4.52}$$

We can now integrate both sides, yielding

$$\ln(\rho_0+3P)-\ln(\rho_0+P)=\frac{1}{2}\ln\left(1-\frac{8\pi\rho_0 r^2}{3}\right)+\text{constant} \tag{4.53}$$

or

$$\frac{\rho_0+3P}{\rho_0+P}=A\left(1-\frac{8\pi\rho_0 r^2}{3}\right)^{1/2} \tag{4.54}$$

When $r=0$ we take $P=P_0$ (i.e. the central pressure), so that we can express the constant A in terms of the density and central pressure

$$A=\frac{\rho_0+3P_0}{\rho_0+P_0} \tag{4.55}$$

Hence we may write

$$\frac{\rho_0+3P}{\rho_0+P}=\frac{\rho_0+3P_0}{\rho_0+P_0}\left(1-\frac{8\pi\rho_0 r^2}{3}\right)^{1/2} \tag{4.56}$$

Using equation (4.49) we see that we may re-write equation (4.56) as

$$\frac{\rho_0+3P}{\rho_0+P}=\frac{\rho_0+3P_0}{\rho_0+P_0}\left(1-\frac{2m}{r}\right)^{1/2} \tag{4.57}$$

At the surface of the star, $P=0$ and the left hand side of equation (4.56) reduces to unity, so that

$$\frac{\rho_0+3P_0}{\rho_0+P_0}\left(1-\frac{2M}{R}\right)^{1/2}=1 \tag{4.58}$$

where M is the Schwarszchild mass and R is the coordinate radius of the star.

By re-arranging equation (4.57) we can obtain an expression for P as a function of r, in terms of ρ, $m(r)$ – which is in turn given by equation (4.49) – and the central pressure, P_0. Moreover, from equation (4.58) we can also obtain an expression for the central pressure, P_0, in terms of the Schwarzschild mass and radius of the star, namely

$$P_0 = \frac{\rho_0 \left[1 - \left(1 - \frac{2M}{R}\right)^{1/2}\right]}{3\left(1 - \frac{2M}{R}\right)^{1/2} - 1} \tag{4.59}$$

Having obtained expressions for $P(r)$ and $m(r)$, and given that we are assuming ρ is constant, we can plug our expressions for these three functions into equations (4.33) and (4.44) to obtain $\nu(r)$ and $\lambda(r)$, which specifies completely the form of the interior metric. This is left as an exercise. (See Examples sheet II.2).

4.8 Buchdahl's theorem and limits on the radius of static stars

We can see from equation (4.59) that

$$P_0 \to \infty \quad \text{when} \quad 3\left(1 - \frac{2M}{R}\right)^{1/2} \to 1, \quad \text{i.e. when} \quad \frac{M}{R} \to \frac{4}{9} \tag{4.60}$$

Hence, this tells us that there can be *no* static stars of uniform density which have radius, $R < 9M/4$, since these would require pressures greater than infinite!

Note that, if we require the exterior metric to be 'well-behaved', then we could already exclude static stars of radius $R < 2M$. We can see this by considering the exterior metric at the surface of the star (i.e. for $r = R$) which takes the Schwarzschild form

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{R}\right)} + R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 \tag{4.61}$$

From this equation it follows that when $R < 2M$ the metric 'misbehaves' because:

- intervals for which $dr \neq 0$ and $dt = d\theta = d\phi = 0$ (which should be **spacelike** for a well-behaved spacetime) give $ds^2 < 0$, i.e. the interval is **timelike**.

- intervals for which $dt \neq 0$ and $dr = d\theta = d\phi = 0$ (which should be **timelike** for a well-behaved spacetime) give $ds^2 > 0$, i.e. the interval is **spacelike**.

(In Chapter 6 we will consider the implications of *allowing* these misbehaving solutions when we investigate black holes).

Hence, equation (4.59) means that we can also rule out the existence of a static star of uniform density for $R < 9M/4$. A theorem due to **Buchdahl** (1959), which we do not prove here, extends this result to stars of non-uniform density: no static, spherically symmetric solutions exist for $R < 9M/4$.

We note, finally, that this result is not merely of academic interest. Suppose, for example, we take a neutron star of mass equal to 2 solar masses. Expressed in units of length then

$$M_{\rm NS} \simeq 3\,{\rm km} \tag{4.62}$$

Hence, Buchdahl's theorem implies that static neutron stars cannot exist with radii less than about 7km. Since neutron stars are actually thought to have radii of about 10km, then we see that these stars approach rather closely the GR limit implied by Buchdahl's theorem.

Of course many neutron stars are rapidly spinning pulsars, which means that their internal structure is not static. To determine a limit on the allowed radius of a spinning pulsar, one must go beyond the simple assumption of a static solution to Einstein's equations. We will not attempt such an extension in this course, although (time permitting) we will consider the metric of a rotating black hole in Chapter 6.