Astronomy A3/A4M, Physics P4M

Gravitation and Relativity II: Example Sheet 5

1. Show that, under the coordinate transformation defined by

$$t \to \tilde{t} = t + 2M \ln \left| \frac{r}{2M} - 1 \right|$$

the expression for the interval of the Schwarzschild metric, with $d\theta = d\phi = 0$, reduces to

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 + \frac{4M}{r} dr\, d\tilde{t} + \left(1 + \frac{2M}{r}\right) dr^2$$

2. Show that the null cones for the metric of Q.1 satisfy the equations

$$\frac{d\tilde{t}}{dr} = -1 \qquad \text{and} \qquad \frac{d\tilde{t}}{dr} = \frac{1 + 2M/r}{1 - 2M/r}$$

and sketch the behaviour of the null cones in the vicinity of the Event Horizon

3. Verify that the right hand side of equation (6.19) of your lecture notes is an indefinite integral of the integrand on the right hand side of equation (6.18), and hence show that

$$\int_{r_e}^{r_o} \frac{dr}{1 - R_S/r} = r_o - r_e + R_S \ln \left(\frac{r_o - R_S}{r_e - R_S} \right)$$

4. Show that the geodesic equation for a material particle in the form

$$v^\beta \, v^\alpha{}_{;\beta} = 0$$

where v^α is a tangent vector to a geodesic, can be re-written in the form

$$v^\alpha \, v_{\beta;\alpha} = 0$$

as stated in equation (6.35) of your lecture notes. Show, further, that this leads to equation (6.38)

$$p^\alpha \, p_{\beta,\alpha} = \Gamma^\gamma_{\beta\alpha} p^\alpha p_\gamma = \frac{1}{2} g^{\gamma\nu} \left(g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu} \right) p^\alpha p_\gamma$$

which in turn can be reduced to

$$m \frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta} \, p^\nu p^\alpha$$

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