Astronomy A3/A4M, Physics P4M

Gravitation and Relativity II: Example Sheet 4

1. Following Section 5.5.1 of your GR-II notes, show that two test particles initially separated by ϵ in the y-direction, have a geodesic deviation vector which obeys the differential equations

$$\frac{\partial^2}{\partial t^2}\,\xi^x = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xy}$$

and

$$\frac{\partial^2}{\partial t^2}\,\xi^y = -\frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xx}$$

as they are disturbed by a plane gravitational wave propagating along the z-axis.

2. More generally, show that the geodesic deviation vector of two particles – one at the origin and the other initially at coordinates $x = \epsilon\cos\theta$, $y = \epsilon\sin\theta$ and $z = 0$ – as a gravitational wave propagates in the z-direction, has components ξ^x and ξ^y which obey the differential equations

$$\frac{\partial^2}{\partial t^2}\,\xi^x = \frac{1}{2}\epsilon\cos\theta\frac{\partial^2}{\partial t^2}h^{TT}_{xx} + \frac{1}{2}\epsilon\sin\theta\frac{\partial^2}{\partial t^2}h^{TT}_{xy}$$

and

$$\frac{\partial^2}{\partial t^2}\,\xi^y = \frac{1}{2}\epsilon\cos\theta\frac{\partial^2}{\partial t^2}h^{TT}_{xy} - \frac{1}{2}\epsilon\sin\theta\frac{\partial^2}{\partial t^2}h^{TT}_{xx}$$

3. Show further that

$$\xi^x = \epsilon\cos\theta + \frac{1}{2}\epsilon\cos\theta\,B^{TT}_{xx}\cos\omega t + \frac{1}{2}\epsilon\sin\theta\,B^{TT}_{xy}\cos\omega t$$

and

$$\xi^y = \epsilon\sin\theta + \frac{1}{2}\epsilon\cos\theta\,B^{TT}_{xy}\cos\omega t - \frac{1}{2}\epsilon\sin\theta\,B^{TT}_{xx}\cos\omega t$$

are solutions to the differential equations of Q.2

4. Does a similar analysis to that considered in Q.1 and Q.2, but now for two test particles – one at the origin and the other at $z = \epsilon$ – lead to the conclusion that the particles will be *unaffected* by the passage of the gravitational wave along the z-axis?

5. A particle is released from rest at coordinate radius $r = R_0$ and falls radially inwards towards a Schwarzchild black hole. Show that equation (6.8) of your lecture notes may be re-written in the form

$$\Delta\tau = \sqrt{\frac{R_0}{R_S}}\int_{R_S}^{R_0}\frac{r^{1/2}\,dr}{(R_0 - r)^{1/2}}$$

Hence, show that the proper time taken for the particle to reach the Event Horizon is given by

$$\Delta\tau = \left(\frac{{R_0}^3}{R_S}\right)^{1/2}\left(\frac{1}{2}\pi - \theta_0 + \sin\theta_0\cos\theta_0\right)$$

where

$$\theta_0 = \sin^{-1}\sqrt{\frac{R_S}{R_0}}$$

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