# 10 Einstein's equations

The Riemann Christoffel tensor, $R^{\mu}{}_{\alpha\beta\gamma}$, describes the curvature of spacetime. Einstein's equations relate this curvature to the matter and energy content of the Universe, as described by the energy momentum tensor, $\mathbf{T}$. But in Section 8 we introduced $\mathbf{T}$ as a $(2,0)$ tensor. Thus, Einstein's equations involve various *contractions* of the Riemann Christoffel tensor.

## 10.1 Ricci tensor

We can contract the Riemann Christoffel tensor to form a $(0,2)$ tensor, which we call the **Ricci tensor** defined by

$$R_{\alpha\gamma} = R^{\mu}_{\alpha\mu\gamma} \tag{253}$$

i.e. contracting on the *second* of the lower indices. (N.B. some authors choose to define $R_{\alpha\gamma}$ as minus this value). From equation (247) it follows that we can also write the components of the Ricci tensor as

$$R_{\alpha\gamma} = g^{\sigma\delta} R_{\sigma\alpha\delta\gamma} \tag{254}$$

It is easy to show (see Tutorial 6) that $R_{\alpha\beta} = R_{\beta\alpha}$, i.e. the Ricci tensor is symmetric.

## 10.2 Curvature Scalar, $R$, and contravariant Ricci tensor

By further contracting the Ricci tensor with the contravariant components of the metric, one obtains the **curvature scalar**, viz:

$$R = g^{\alpha\beta} R_{\alpha\beta} \tag{255}$$

One may also use the metric to raise the indices of the Ricci tensor, and thus express it in contravariant form, viz:

$$R^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta} \tag{256}$$

$R^{\mu\nu}$ is also symmetric.

## 10.3 The Einstein tensor

Using the contravariant form of the Ricci tensor, we define the **Einstein tensor**, $\mathbf{G}$, viz:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{257}$$

where $R$ is the curvature scalar. Note that since $R^{\mu\nu}$ is symmetric, so too is $G^{\mu\nu}$.

The Einstein tensor is of crucial physical significance in general relativity, since it can be shown from the Bianchi identities that

$$G^{\mu\nu}_{\;\;;\nu} = 0 \qquad (258)$$

### 10.4 Proof of $G^{\mu\nu}_{\;\;;\nu} = 0$

Consider the Bianchi identities

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0 \qquad (259)$$

From the symmetry properties of the Riemann Christoffel tensor, equation (248), we have

$$R_{\alpha\beta\lambda\mu;\nu} = -R_{\alpha\beta\mu\lambda;\nu} \qquad (260)$$

so that the Bianchi identities can be re-written as

$$R_{\alpha\beta\mu\nu;\lambda} - R_{\alpha\beta\mu\lambda;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0 \qquad (261)$$

Contracting equation (261) with $g^{\alpha\mu}$ we obtain

$$g^{\alpha\mu}R_{\alpha\beta\mu\nu;\lambda} - g^{\alpha\mu}R_{\alpha\beta\mu\lambda;\nu} + g^{\alpha\mu}R_{\alpha\beta\nu\lambda;\mu} = 0 \qquad (262)$$

Using the product rule for covariant differentiation, and the fact that $g^{\alpha\mu}_{\;\;;\beta} = 0$, equation (262) can be written as

$$(g^{\alpha\mu}R_{\alpha\beta\mu\nu})_{;\lambda} - (g^{\alpha\mu}R_{\alpha\beta\mu\lambda})_{;\nu} + (g^{\alpha\mu}R_{\alpha\beta\nu\lambda})_{;\mu} = 0 \qquad (263)$$

This simplifies to

$$R^{\mu}{}_{\beta\mu\nu;\lambda} - R^{\mu}{}_{\beta\mu\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu} = 0 \qquad (264)$$

i.e.

$$R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu} = 0 \qquad (265)$$

We can further contract this equation by multiplying by $g^{\beta\nu}$ to obtain

$$g^{\beta\nu}R_{\beta\nu;\lambda} - g^{\beta\nu}R_{\beta\lambda;\nu} + g^{\beta\nu}R^{\mu}{}_{\beta\nu\lambda;\mu} \qquad (266)$$

Again using the product rule and the fact that $g^{\alpha\mu}_{\;\;;\beta} = 0$, this further simplifies to (after contracting the first two terms and writing the third in terms of the fully covariant form of the Riemann Christoffel tensor)

$$R_{;\lambda} - R^{\nu}_{\lambda;\nu} + g^{\beta\nu}g^{\alpha\mu}R_{\alpha\beta\nu\lambda;\mu} = 0 \qquad (267)$$

Using the fact that

$$R_{\alpha\beta\nu\lambda} = -R_{\beta\alpha\nu\lambda} \qquad (268)$$

the third term on the left hand side of equation (267) can therefore be re-written as

$$g^{\beta\nu}g^{\alpha\mu}R_{\alpha\beta\nu\lambda;\mu} = -(g^{\beta\nu}g^{\alpha\mu}R_{\beta\alpha\nu\lambda})_{;\mu} = -(g^{\alpha\mu}R_{\alpha\lambda})_{;\mu} = -R^{\mu}{}_{\lambda;\mu} \equiv -R^{\nu}{}_{\lambda;\nu} \quad (269)$$

Thus, from equation (267) we have

$$R_{;\lambda} - 2R^{\nu}{}_{\lambda;\nu} = 0 \quad (270)$$

Multiplying each term by $-\frac{1}{2}g^{\mu\lambda}$ we obtain

$$g^{\mu\lambda}R^{\nu}{}_{\lambda;\nu} - \frac{1}{2}g^{\mu\lambda}R_{;\lambda} \quad (271)$$

Once again using the product rule and the fact that the first covariant derivatives of the metric vanish this yields

$$(g^{\mu\lambda}R^{\nu}{}_{\lambda})_{;\nu} - (\frac{1}{2}g^{\mu\nu}R)_{;\nu} = 0 \quad (272)$$

which of course is equivalent to

$$G^{\mu\nu}_{;\nu} = 0 \quad (273)$$

### 10.5 Einstein's equations

We saw in Section 8 that conservation of energy and momentum implied that

$$T^{\mu\nu}_{;\nu} = 0 \quad (274)$$

We now see that from the Bianchi identities

$$G^{\mu\nu}_{;\nu} = 0 \quad (275)$$

Thus we have automatically that

$$T^{\mu\nu}_{;\nu} = G^{\mu\nu}_{;\nu} \quad (276)$$

These are Einstein's equations, and Einstein took as their solution

$$G^{\mu\nu} = kT^{\mu\nu} \quad (277)$$

i.e. the Einstein tensor is equal to the energy momentum tensor, multiplied by a constant, $k$.

**Thus, the geometry of spacetime (described by the Einstein tensor) and the mass-energy content of spacetime (described by the energy momentum tensor) are fundamentally interlinked.**

Note that there are 10 independent equations incorporated in equation (277), since $G^{\mu\nu}$ and $T^{\mu\nu}$ are symmetric.

We determine the constant $k$ by requiring that we should recover the laws of Newtonian gravity and dynamics in the limit of a weak gravitational field and non-relativistic motion.

## 10.6 Non-relativistic weak field limit and Newtonian gravity

If the deviation from flatness in our curved spacetime is small we can find a coordinate system for which

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{278}$$

where $\eta_{\alpha\beta}$ is the Lorentz metric $(-1, 1, 1, 1)$ and $h_{\alpha\beta} << 1$ for $\alpha,\ \beta = 0, 1, 2, 3$. If we also write

$$g^{\alpha\gamma} = \eta^{\alpha\gamma} + h^{\alpha\gamma} \tag{279}$$

then

$$\delta^{\gamma}_{\beta} = g_{\alpha\beta}g^{\alpha\gamma} = (\eta_{\alpha\beta} + h_{\alpha\beta})(\eta^{\alpha\gamma} + h^{\alpha\gamma}) \tag{280}$$

from which it follows that, to first order in $h_{\alpha\beta}$

$$h^{\nu\gamma} = -\eta^{\alpha\gamma}\eta^{\nu\beta}h_{\alpha\beta} \tag{281}$$

Consider now the motion of a 'free-falling' test particle. The wordline of the particle is a geodesic and so satisfies

$$\frac{d^2x^{\mu}}{d\tau^2} = -\Gamma^{\mu}_{\sigma\delta}v^{\sigma}v^{\delta} \tag{282}$$

For non-relativistic particles the proper time $\tau$ may be replaced by $t$. Furthermore $v^i << 1$ for $i = 1,\ 2,\ 3$ and $v^0 = 1$. Thus the space components of equation (282) reduce to

$$\frac{d^2x^i}{dt^2} = -\Gamma^{i}_{00} \tag{283}$$

Evaluating $\Gamma^{i}_{00}$ using equation (173) we have

$$\begin{aligned}\Gamma^{i}_{00} &= \frac{1}{2}g^{i\sigma}(g_{0\sigma,0} + g_{\sigma i,0} - g_{00,\sigma}) \\ &= \frac{1}{2}(\eta^{i\sigma} + h^{i\sigma})(h_{0\sigma,0} + h_{\sigma i,0} - h_{00,\sigma})\end{aligned} \tag{284}$$

Keeping only first order terms in $h_{\alpha\beta}$, and noting equation (281) we have

$$\Gamma^{i}_{00} = \frac{1}{2}(h_{0i,0} + h_{ii,0} - h_{00,i}) \tag{285}$$

Assuming time derivatives are small compared with space derivatives this implies[4]

$$\Gamma^{i}_{00} = -\frac{1}{2}h_{00,i} \tag{286}$$

Thus the equation of motion of the particle is simply

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}h_{00,i} \tag{287}$$

[4]If time derivatives are *not* small then one generates gravitational waves – see G&R II for more details

In Newtonian theory the equation of motion is

$$\frac{d^2x^i}{dt^2} = -\frac{\partial \phi}{\partial x^i} \tag{288}$$

where $\phi$ is the Newtonian gravitational potential. This agrees with the weak field GR limit of equation (288) if we put

$$h_{00,i} = -2\frac{\partial \phi}{\partial x^i} \tag{289}$$

Equation (289) simply shows how the metric (in GR) can be chosen to ensure that geodesics (worldlines of free test particles) correspond to the classical non-relativistic trajectories of particles accelerating in a gravitational field. But GR also tells us, through Einstein's equations, how matter and energy curve spacetime. The classical analogue to this is how mass density produces the gravitational field. This is most easily expressed through Poisson's equation, viz

$$\nabla^2 \phi = 4\pi G \rho \tag{290}$$

where $G$ is the Newtonian gravitational constant (not to be confused with the Einstein tensor!). General relativity must agree with equation (290) in the non-relativistic weak field limit.

### 10.6.1 Geodesic deviation in Newtonian theory

In a gravitational field, $\phi$, the Newtonian acceleration of a test particle will be given by equation (288), i.e.

$$\frac{d^2x^i}{dt^2} = -\frac{\partial \phi}{\partial x^i}$$

Suppose a neighbouring particle, with coordinates $x^i + \xi^i$, accelerates in the same gravitational field. Evidently

$$\frac{d^2x^i + \xi^i}{dt^2} = -\frac{\partial \phi(x+\xi)}{\partial x^i} = -(\frac{\partial \phi}{\partial x^i} + \frac{\partial^2 \phi}{\partial x^i \partial x^j}\xi^j) \tag{291}$$

(we have used a first order Taylor's expansion in the last step).

The relative acceleration of the neighbouring particle to the original test particle is found by subtracting equation (291) from equation (290) to give

$$\frac{d^2\xi^i}{dt^2} = -\frac{\partial^2 \phi}{\partial x^i \partial x^j}\xi^j \tag{292}$$

### 10.6.2 Geodesic deviation in GR

Consider the same two neighbouring test particles, but now from the point of view of GR. In GR the geodesic deviation of the two particles is given by (c.f. equation 245)

$$\frac{D^2\xi^\mu}{D\tau^2} = -R^\mu_{\alpha\beta\gamma} v^\alpha \xi^\beta v^\gamma \tag{293}$$

Take a geodesic coordinate system in which the first test particle is instantaneously at rest. Thus $v^1 = v^2 = v^3 = 0$ and $v^0 = 1$. In this coordinate system the covariant derivative operator $D^2/D\tau^2$ can be approximated simply by $d^2/dt^2$. Thus, taking the spatial components of equation (293) yields

$$\frac{d^2\xi^i}{dt^2} = -R^i_{0\beta 0}v^0\xi^\beta v^0 = -R^i_{0\beta 0}\xi^\beta \tag{294}$$

If we take the initial separation such that $\xi^0 = 0$, (i.e. the test particles are released simultaneously in the chosen coordinate frame), equation (294) becomes

$$\frac{d^2\xi^i}{dt^2} = -R^i{}_{0j0}\xi^j \tag{295}$$

Comparing this with the Newtonian expression (292) we see immediately that

$$R^i{}_{0j0}\xi^j = \frac{\partial^2\phi}{\partial x^i \partial x^j}\xi^j \tag{296}$$

but since the components $\xi^j$ were arbitrary we must have

$$R^i{}_{0j0} = \frac{\partial^2\phi}{\partial x^i \partial x^j} \tag{297}$$

### 10.6.3 Relating the Einstein and energy momentum tensors

It follows from equation (297) that

$$R_{00} = R^\alpha{}_{0\alpha 0} = R^i{}_{0i0} + R^0{}_{000} = R^i{}_{0i0} = \frac{\partial^2\phi}{\partial x^i \partial x^i} \tag{298}$$

(Note that $R^0{}_{000} = 0$ from its skew-symmetry; see Tutorial 6, Q.1)

$R_{00}$ can also be related to the mass-energy density through Einstein's equations, this time written in *covariant* form, i.e.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = kT_{\mu\nu} \tag{299}$$

Taking the '00' component we have

$$R_{00} = kT_{00} + \frac{1}{2}g_{00}R \tag{300}$$

However, in our geodesic coordinates $g_{00} = -1$ and $g_{ii} = 1$. Thus

$$R_{00} = kT_{00} - \frac{1}{2}R \tag{301}$$

Suppose we contract the energy momentum tensor, defining the scalar, $T$, as

$$T = g^{\mu\nu}T_{\mu\nu} \tag{302}$$

In geodesic coordinates this is simply

$$T = -T_{00} + T_{11} + T_{22} + T_{33} \tag{303}$$

but for non-relativistic matter, the energy density is much greater than the momentum density (i.e. for a non-relativistic fluid, the pressure $P << \rho$). Hence, $T \simeq -T_{00} = -\rho$. Contracting Einstein's equations (299) with $g^{\mu\nu}$ we, therefore, obtain

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = k g^{\mu\nu} T_{\mu\nu} = kT \tag{304}$$

i.e.

$$R - \frac{1}{2}\delta^{\mu}{}_{\mu} R = R - 2R = -R = -kT_{00} = -k\rho \tag{305}$$

Substituting into equation (301),

$$R_{00} = kT_{00} - \frac{1}{2}R = k\rho - \frac{k}{2}\rho = \frac{k}{2}\rho \tag{306}$$

Finally, combining equations (305) and (306) we see that in the non-relativistic limit

$$R_{00} = \nabla^2 \phi = \frac{k}{2}\rho \tag{307}$$

This is precisely the Newtonian equation of gravity, expressed by Poisson's equation (290), provided we choose the as yet arbitrary constant $k$ to be given in terms of the Newtonian gravitation constant, $G$, by

$$k = 8\pi G \tag{308}$$

Thus the final form of Einstein's field equations is

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu} \tag{309}$$

or in units where $G = 1$, (i.e. mass in metres)

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi T^{\mu\nu} \tag{310}$$