9 The Riemann Christoffel tensor

GR explains gravitational effects as consequences of the curvature of spacetime, which arises from the presence and motion of gravitating matter. In the previous section we outlined how the matter content of spacetime could be described by the energy momentum tensor, $\mathbf{T}$. Thus our remaining objective is to relate $\mathbf{T}$ to the curvature of spacetime.

The curvature of a metric space is described by the **Riemann Christoffel tensor** (often also referred to simply as the **Riemann tensor**). One would expect any tensorial quantity that describes the curvature at least to depend on the second order derivatives of the metric tensor, as for a suitable choice of coordinates (geodesic coordinates) we saw previously that the first order derivatives can always be made zero. Thus we should expect the Riemann Christoffel tensor to depend on $g_{\mu\nu}$, $g_{\mu\nu,\alpha}$ and $g_{\mu\nu,\alpha,\beta}$. (N.B. in future we shall simply write $g_{\mu\nu,\alpha\beta}$ for $g_{\mu\nu,\alpha,\beta}$).

The form of the Riemann Christoffel tensor can be derived in several different ways, although we shall discuss here only one derivation – which follows on from our preliminary discussion of spacetime curvature in Section 1.

9.1 Riemann Christoffel tensor from the geodesic deviation

We consider two free falling test particles above the Earth and measure their separation in a frame free-falling with the particles (i.e. a LIF). In the free falling frame the particles will accelerate towards one another, albeit almost imperceptibly. Their worldlines are geodesics in spacetime, and the geodesic deviation is the separation of these geodesics. The **acceleration of the geodesic deviation** tells us that spacetime is curved.

Consider then two test particles (labelled 1 and 2) moving along nearby geodesics (see Figure 16). Let $\xi^\mu(\tau)$ denote the (infinitesimal) separation of the particles at proper time τ, so that

$$x_2^\mu(\tau) = x_1^\mu(\tau) + \xi^\mu(\tau) \tag{227}$$

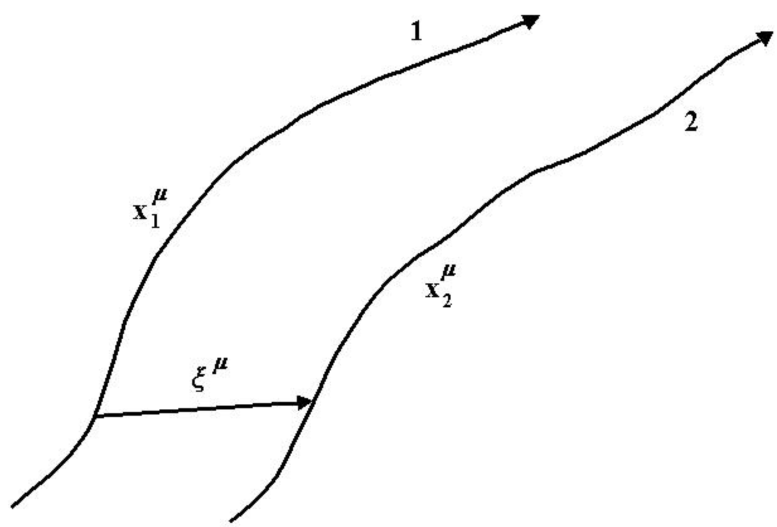
Now the worldlines of the two particles are described by the Geodesic equation, viz:-

$$\frac{d^2 x_1^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}(x_1)\frac{dx_1^\alpha}{d\tau}\frac{dx_1^\beta}{d\tau} = 0 \tag{228}$$

and

$$\frac{d^2 x_2^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}(x_2)\frac{dx_2^\alpha}{d\tau}\frac{dx_2^\beta}{d\tau} = 0 \tag{229}$$

Figure 16



Note that we may write

$$\Gamma^\mu_{\alpha\beta}(x_2) = \Gamma^\mu_{\alpha\beta}(x_1 + \xi) = \Gamma^\mu_{\alpha\beta}(x_1) + \Gamma^\mu_{\alpha\beta,\gamma}\xi^\gamma \tag{230}$$

Subtracting equation (229) from equation (228) and keeping only up to first order terms in ξ yields the following equation for the acceleration of ξ^μ (dropping the subscript 1)

$$\frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}v^\alpha\frac{d\xi^\beta}{d\tau} + \Gamma^\mu_{\alpha\beta}v^\beta\frac{d\xi^\alpha}{d\tau} + \Gamma^\mu_{\alpha\beta,\gamma}\xi^\gamma v^\alpha v^\beta = 0 \tag{231}$$

Equation (231) can be put into covariant form by introducing the covariant operator $D/D\tau$.

9.1.1 Definition of $D/D\tau$

Consider an arbitrary vector field, A^μ, and a worldline defined by $x^\alpha = x^\alpha(\tau)$. We define

$$\frac{DA^\mu}{D\tau} = \frac{dA^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}A^\alpha\frac{dx^\beta}{d\tau} \tag{232}$$

$D/D\tau$ is essentially the covariant derivative along the worldline. To see how equation (232) is obtained, we displace $A(\tau)$ through parallel transport from the event at proper time τ on the worldline of a particle to the event at $\tau + d\tau$. Thus

$$\frac{DA^\mu}{D\tau} = \lim_{d\tau\to 0}\frac{A^\mu(\tau + d\tau) - DA^\mu(\tau)}{d\tau} \tag{233}$$

Since

$$DA^{\mu} = A^{\mu} - \Gamma^{\mu}_{\alpha\beta} A^{\alpha} dx^{\beta} \tag{234}$$

substitution into equation (233) yields

$$\frac{DA^{\mu}}{D\tau} = \frac{dA^{\mu}}{d\tau} + \Gamma^{\mu}_{\alpha\beta} A^{\alpha} \frac{dx^{\beta}}{d\tau} \tag{235}$$

Writing

$$v^{\beta} = \frac{dx^{\beta}}{d\tau} \tag{236}$$

we have

$$\frac{DA^{\mu}}{D\tau} = \frac{dA^{\mu}}{d\tau} + \Gamma^{\mu}_{\alpha\beta}\ A^{\alpha} v^{\beta} \tag{237}$$

or in operator form

$$\frac{D}{D\tau} = \frac{d}{d\tau} + \Gamma^{\mu}_{\alpha\beta} v^{\beta} \tag{238}$$

Exercise: Show that the geodesic equation may be written as

$$\frac{Dv^{\mu}}{D\tau} = 0$$

9.1.2 Covariant differential equation for the Riemann Christoffel tensor

From equation (238) it follows that

$$\frac{D^2 \xi^{\mu}}{D\tau^2} = \frac{D}{D\tau}(\frac{D\xi^{\mu}}{D\tau}) = \frac{d}{d\tau}(\frac{D\xi^{\mu}}{D\tau}) + \Gamma^{\mu}_{\sigma\delta} \frac{D\xi^{\sigma}}{D\tau} v^{\delta} \tag{239}$$

Substituting for $D\xi^{\mu}/D\tau$ in equation (239) we obtain

$$\frac{D^2 \xi^{\mu}}{D\tau^2} = \frac{d}{d\tau}(\frac{d\xi^{\mu}}{d\tau} + \Gamma^{\mu}_{\alpha\beta} \xi^{\alpha} v^{\beta}) + \Gamma^{\mu}_{\sigma\delta}(\frac{d\xi^{\sigma}}{d\tau} + \Gamma^{\sigma}_{\alpha\beta}\ \xi^{\alpha} v^{\beta}) v^{\delta} \tag{240}$$

Now

$$\frac{d}{d\tau}(\Gamma^{\mu}_{\alpha\beta} \xi^{\alpha} v^{\beta}) = \Gamma^{\mu}_{\alpha\beta,\gamma} \frac{dx^{\gamma}}{d\tau} \xi^{\alpha} v^{\beta} + \Gamma^{\mu}_{\alpha\beta} \frac{d\xi^{\alpha}}{d\tau} v^{\beta} + \Gamma^{\mu}_{\alpha\beta} \xi^{\alpha} \frac{dv^{\beta}}{d\tau} \tag{241}$$

Since each particle's worldline is a geodesic we have

$$\frac{dv^{\beta}}{d\tau} = \frac{d^2 x^{\beta}}{d\tau^2} = -\Gamma^{\beta}_{\sigma\delta} v^{\sigma} v^{\delta} \tag{242}$$

where we have written

$$v^{\beta} = \frac{dx^{\beta}}{d\tau}$$

Substituting equations (241) and (242) into (240) and reordering we obtain

$$\frac{D^2\xi^\mu}{D\tau^2} = \frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta,\gamma}v^\gamma\xi^\alpha v^\beta + \Gamma^\mu_{\alpha\beta}\frac{d\xi^\alpha}{d\tau}v^\beta + \Gamma^\mu_{\sigma\delta}\frac{d\xi^\sigma}{d\tau}v^\delta$$
$$+(\Gamma^\mu_{\beta\delta}\Gamma^\beta_{\alpha\sigma} - \Gamma^\mu_{\alpha\beta}\Gamma^\beta_{\alpha\delta})v^\sigma v^\delta\xi^\alpha \tag{243}$$

However from equation (231)

$$\frac{d^2\xi^\mu}{d\tau^2} = -(\Gamma^\mu_{\alpha\beta}v^\alpha\frac{d\xi^\beta}{d\tau} + \Gamma^\mu_{\alpha\beta}v^\beta\frac{d\xi^\alpha}{d\tau} + \Gamma^\mu_{\alpha\beta,\gamma}\xi^\gamma v^\alpha v^\beta) \tag{244}$$

and substitution into equation (243) yields

$$\frac{D^2\xi^\mu}{D\tau^2} = -R^\mu{}_{\alpha\beta\gamma}v^\alpha\xi^\beta v^\gamma \tag{245}$$

where

$$R^\mu{}_{\alpha\beta\gamma} = \Gamma^\sigma_{\alpha\beta}\Gamma^\mu_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma}\Gamma^\mu_{\sigma\beta} + \Gamma^\mu_{\alpha\beta,\gamma} - \Gamma^\mu_{\alpha\gamma,\beta} \tag{246}$$

The $(1,3)$ tensor, $\mathbf{R}$, is the Riemann Christoffel tensor. Notice that if the spacetime is flat then

$$R^\mu{}_{\alpha\beta\gamma} = 0$$

i.e. all components of the Riemann Christoffel tensor are identically zero.

9.1.3 Fully covariant form of the Riemann Christoffel tensor and its symmetries

We may obtain the components of the $(0,4)$ tensor corresponding to $R^\mu{}_{\alpha\beta\gamma}$ by using the covariant components of the metric tensor to lower the index, viz.

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\sigma}R^\sigma{}_{\beta\gamma\delta} \tag{247}$$

$R_{\alpha\beta\gamma\delta}$ has the following important symmetries, which reduce the number of independent components in $4d$ to 20.

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \tag{248}$$

$$R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\gamma\delta} \tag{249}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0 \tag{250}$$

In equation (250) we have simply taken a cyclic permutation on the last three indices.

It is fairly straightforward, though tedious, to derive the above symmetry relations if we work in a geodesic coordinate system, in which it follows from equation (173) that

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta} + g_{\beta\gamma,\alpha\delta} - g_{\beta\delta,\alpha\gamma}) \tag{251}$$

Since the symmetries hold in geodesic coordinates, we argue that they hence must generally be valid. Note also that equation (251) shows the Riemann Christoffel tensor to depend on second derivatives of the metric tensor, as predicted.

9.2 The Bianchi identities

The Bianchi identities, for the derivatives of the Riemann Christoffel tensor, state that

$$R_{\alpha\beta\gamma\delta;\lambda} + R_{\alpha\beta\lambda\gamma;\delta} + R_{\alpha\beta\delta\lambda;\gamma} = 0 \tag{252}$$

Note that the three terms cyclically permute the final three indices. We will not discuss in any detail the proof of the Bianchi identities in this course, although the proof is again quite straightforward if one chooses first a geodesic coordinate system and then generalises to any frame by appealing to the tensorial nature of equation (252). For completeness, a summary of the proof is appended at the end of this section.

9.3 Riemann tensor via parallel displacement

Another geometrically instructive method for deriving the form of the Riemann Christoffel tensor involves parallel transporting a vector around a closed curve in spacetime. If the spacetime is flat, then the vector is mapped back onto itself when the loop is completed – i.e. when we return to the starting point (see Figure 17a). In a curved spacetime, however, the parallel transported vector will not in general be mapped onto itself when we return to the starting point (see Figure 17b). We can express the net change in the components of the vector, after transport around the closed loop, in terms of the Riemann Christoffel tensor. We will not discuss this approach any further here, but the interested reader is referred to e.g. Schutz, Chap. 6, for more details. Obviously, one obtains the same form for the Riemann Christoffel tensor when one derives it by considering parallel displacement in this way.

Figure 17a

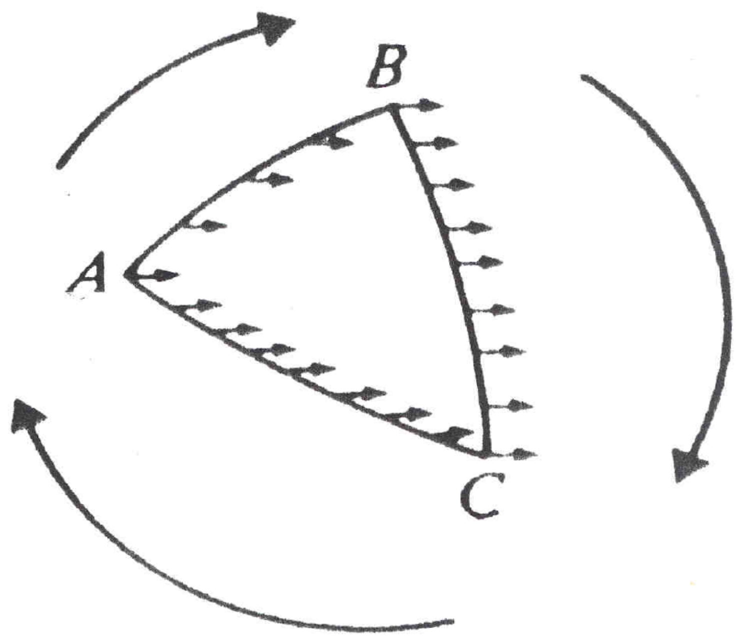
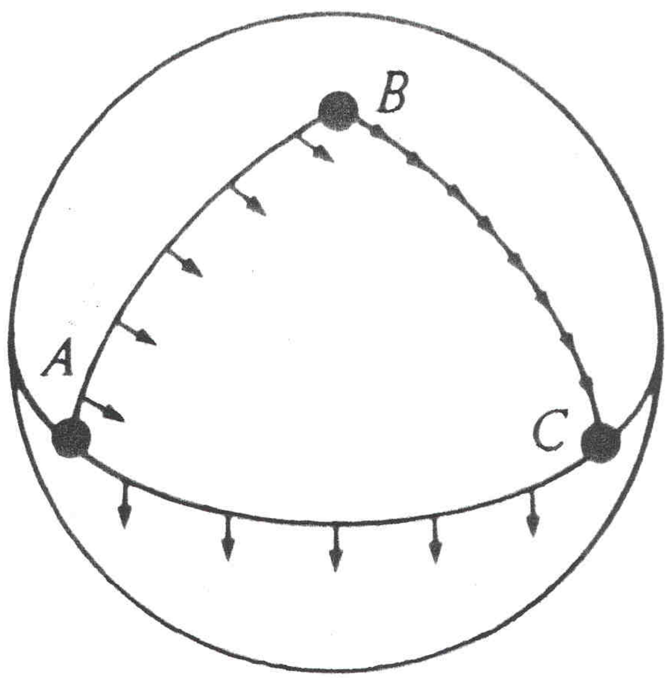


Figure 17b



9.4 Proof of the Bianchi identities

The left hand side of equation (252) is a $(0, 5)$ tensor. Hence if we can show that its components are zero at an arbitrary point on the manifold in a given coordinate system, then they must be zero in all coordinate systems. We choose geodesic coordinates at arbitrary point, P. From the definition of the Riemann Christoffel tensor

$$R^{\mu}{}_{\beta\gamma\delta} = \Gamma^{\sigma}_{\beta\gamma}\Gamma^{\mu}_{\sigma\delta} - \Gamma^{\sigma}_{\beta\delta}\Gamma^{\mu}_{\sigma\gamma} + \Gamma^{\mu}_{\beta\gamma,\delta} - \Gamma^{\mu}_{\beta\delta,\gamma}$$

Now, since $g_{\alpha\mu;\lambda} = 0$ (see tutorial 5, Qs. 5,6), we can write

$$R_{\alpha\beta\gamma\delta;\lambda} = (g_{\alpha\mu}R^{\mu}{}_{\beta\gamma\delta})_{;\lambda} = g_{\alpha\mu}R^{\mu}{}_{\beta\gamma\delta;\lambda}$$

Now all the Christoffel symbols are zero at P in geodesic coordinates, so from the above two equations (changing covariant back to partial derivatives, which are interchangeable at P in geodesic coordinates)

$$R_{\alpha\beta\gamma\delta;\lambda} = g_{\alpha\mu}(\Gamma^{\mu}_{\beta\gamma,\delta\lambda} - \Gamma^{\mu}_{\beta\delta,\gamma\lambda})$$

Substituting for the Christoffel symbols using

$$\Gamma^{\sigma}_{\alpha\beta} = g^{\sigma\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$$

and also using the fact that in geodesic coordinates $g_{\alpha\beta,\gamma} = g^{\alpha\beta}_{\ ,\gamma} = 0$ one obtains

$$R_{\alpha\beta\gamma\delta;\lambda} = \frac{1}{2}(g_{\alpha\gamma,\beta\delta\lambda} + g_{\alpha\delta,\beta\gamma\lambda} - g_{\beta\gamma,\alpha\delta\lambda} - g_{\beta\delta,\alpha\gamma\lambda})$$

Writing out the corresponding expressions for the other two terms on the left hand side of equation (252) and adding gives the required result, i.e.

$$R_{\alpha\beta\gamma\delta;\lambda} + R_{\alpha\beta\lambda\gamma;\delta} + R_{\alpha\beta\delta\lambda;\gamma} = 0$$

in geodesic coordinates at P. Hence the Bianchi identities must hold in all coordinate systems because of their tensorial nature.